Relaxed Utility Maximization

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Abstract

For utility functions $U : \mathbb{R}_+ \to \mathbb{R}$ Kramkov and Schachermayer [KS99] showed that under a condition on U, the well- known condition on the Asymptotic Elasticity

$$AE(U) := \limsup_{x \to +\infty} \frac{xU'(x)}{U(x)} < 1$$

the associated utility maximization problem has a (unique) optimal solution, independently of the probabilistic model $(\Omega, \mathcal{F}, P)$.

What can we say about the *relaxed* investor, i.e. the case $AE(U) = 1$? This was also treated in [KS99, Theo 2.0], in the complete market case, but the optimal solution is characterized only for sufficiently small initial endowments. Under an extremely weak joint condition on the probabilistic model and the utility, we show by relaxation and duality techniques that the maximization problem admits solution for *any* initial endowment. However, a singular part may pop up, that is the optimal investment may have a component which is concentrated on a set of probability zero. This singular part may fail to be unique.

1 Introduction

Consider a set Ω, endowed with a topology $\mathcal{T}$, which makes it a Polish (separable, completely metrizable topological) space. On Ω we put the Borel σ-algebra, denoted by $\mathcal{B}(\Omega, \mathcal{T})$, which is the smallest σ-algebra that contains the open sets of $\mathcal{T}$. A Borel set is thus *any* element of $\mathcal{B}(\Omega, \mathcal{T})$. Finally, let P and Q be two equivalent probabilities on $(\Omega, \mathcal{B}(\Omega, \mathcal{T}))$. It is also assumed that P, Q give strictly positive probability to all the open sets of $\mathcal{T}$, that is their support is the whole Ω. As usual, the notations $E_P[\cdot]$ and $E_Q[\cdot]$ indicate the expected value under P and Q respectively.

Given a concave function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ with $\mathrm{Dom}(U) = \{U > -\infty\} \subseteq [0, +\infty)$, consider an optimization problem of the type

$$\sup_{X \in C} E_P [U(X)]$$

where C is a bounded subset of $L^1_+(Q)$. A key point is that while the expectation is taken with respect to P, the set of constraints is in $L^1(Q)$ which in general is different from $L^1(P)$.

If U is interpreted as the utility function of an agent, this optimization is the abstract version of the utility maximization problem in the complete market case, when the unique pricing measure is Q. Given such a context, one is concerned about domains of the specific form

$$C(x) = \left\{ X \in L^1_+(Q) : E_Q[X] \leq x \right\}$$

Thus the domain $C(x)$ is the set of claims that can be financed with initial endowment x. For a shorthand, I_U will denote the integral functional to be optimized, that is

$$I_U(X) = E_P[U(X)] = E_Q[\frac{dP}{dQ}(\omega)U(X(\omega))]$$

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Since the set of constraints lies in $L^1(Q)$, mathematically it would be natural to write the integral functional as expectation under Q. We will indeed do this in many occasions. However, we state the main theorem and the main results with the P-expectation form $E_P[U(X)]$. In short, the specific maximization problem faced is thus

$$\sup_{X \in C(x)} I_U(X) \tag{1}$$

The above problem (the *primal* problem) may fail to have a solution, because maximizing sequences are merely bounded sets of $L^1(Q)$ which in general are not relatively compact.

To recover existence of a solution, the first step is to look at a larger space where to embed $L^1(Q)$, in such a way that $L^1(Q)$-bounded sets become relatively compact (so any $L^1(Q)$-bounded sequence finds a cluster point). There are a few possible choices for the embedding of $L^1(Q)$. Since any $X \in L^1(Q)$ can be identified with the bounded measure $\mu = XdQ$, this larger space will be a space of bounded additive set functions on $\mathcal{B}(\Omega, \mathcal{T})$. The most natural choice would probably be the space

$$ba(\Omega, \mathcal{B}(\Omega, \mathcal{T}), Q)$$

(simply $ba(\Omega, Q)$ in the following) i.e. the Borel bounded charges ν which vanish on the Q-null sets. In this paper, while "charge" indicates an additive set function ν with the property that $\nu(\emptyset) = 0$, "measure" keeps its classical meaning. In other words, a measure is a *countably additive* charge (see [AB05, Section 10.1] or [DS67, Section III.1]). In fact, $ba(\Omega, Q)$ is a Banach space with the variation norm

$$\|\nu\| = \sup \left\{ \sum_{i=1}^{n} |\nu(E_i)| \mid (E_i)_i \text{ Borel partition of } \Omega \right\}$$

and it is the dual of $L^\infty(Q)$. So by the well-known Banach-Alaoglu Theorem norm bounded sets in $ba(\Omega, Q)$ are weak* relatively compact.

This approach mathematically would work well, but it has a heavy drawback in terms of financial interpretation. A sketch of what goes wrong is the following. $ba(\Omega, Q)$ admits the decomposition $L^1(Q) \oplus (L^1(Q))^\perp$, so that any $\nu \in ba(\Omega, Q)$ admits a (generalized) Lebesgue decomposition

$$\nu = \nu_a + \nu_p$$

into a measure absolutely continuous with respect to Q, ν_a, and a "pure charge" part ν_p, which has a very unpleasant behavior as it is purely finitely additive. If an optimizer ν^* of problem (1) belonged to $ba(\Omega, Q)$ what would the "pure charge" component of the optimal investment mean? The lack of countable additivity and consequent vanishing of part of the mass really would make it impossible to interpret an element ν of $ba(\Omega, Q)$ with $\nu_p \neq 0$ as an "investment".

Therefore our philosophy is that when possible one should avoid the use of charges and stick to measures. In apparent contrast with this message, for our purposes the best selection of a larger space where to embed $L^1(Q)$ turns out to be

$$rba(\Omega)$$

which is defined as the space of all *bounded Borel-regular charges* on $\mathcal{B}(\Omega, \mathcal{T})$. Here the topology $\mathcal{T}$ comes into play. In fact, recall that a charge μ is (Borel-)regular if it has the inner-outer approximation property

$$E \in \mathcal{B}(\Omega, \mathcal{T}) \Rightarrow \sup_{\{F \text{ closed}, F \subseteq E\}} \mu(F) = \mu(E) = \inf_{\{A \text{ open}, A \supseteq E\}} \mu(A)$$

Like $ba(\Omega, Q)$, the space $rba(\Omega)$ is a Banach space with the variation norm. However, the fact that $rba(\Omega)$ does not depend on the reference probability Q is *crucial*, as will be clear in a moment.

Since $(\Omega, \mathcal{T})$ is a Polish space, $rba(\Omega)$ can be identified with a dual space [DS67, Section IV.6]

$$rba(\Omega) = (\mathcal{C}_b(\Omega))^*$$

where $\mathcal{C}_b(\Omega)$ is the space of continuous bounded functions on Ω. $\mathcal{C}_b(\Omega)$ is a Banach space if endowed with the sup-norm. Therefore, the norm-bounded subset of $rba(\Omega)$, like $C(x)$, are relatively compact for the weak* topology $\sigma(rba(\Omega), \mathcal{C}_b(\Omega))$ by the Banach-Alaoglu Theorem. Recall also the lattice (Yosida-Hewitt) decomposition of the space $rba(\Omega)$ as a direct sum of the bounded regular measures $\mathcal{M}$ plus the purely finitely additive regular charges $\mathcal{N}$

$$rba(\Omega) = \mathcal{M} \oplus \mathcal{N} \tag{2}$$

that is any $\nu \in rba(\Omega)$ can be uniquely decomposed as a sum $\nu = \nu_c + \nu_p$ where ν_c is the measure component, countably additive, and ν_p is the purely finitely additive component. As any ν_c is a bounded measure on a Polish space, it is also compact-inner regular (see [AB05, Theo 12.7]):

$$E \in \mathcal{B} \Rightarrow \nu_c(E) = \sup_{K \text{ compact }, K \subseteq E} \nu_c(K) \tag{3}$$

Therefore $\mathcal{M}$ is nothing but the familiar space of the Radon bounded measures on Ω. On the contrary, any $\nu_p \in \mathcal{N}$ has the fundamental property

$$\nu_p(K) = 0 \text{ for any compact } K \tag{4}$$

which follows from [AB05, Theorem 12.4].

If the Lebesgue decomposition with respect to Q inside $\mathcal{M}$ is taken into account one has

$$rba(\Omega) = \mathcal{M}_a \oplus \mathcal{M}_s \oplus \mathcal{N} \tag{5}$$

that is for any $\nu \in rba(\Omega)$, the measure component ν_c can further be decomposed into an absolutely continuous part with respect to Q, ν_a, and singular part ν_s, supported by a Q -null Borel set:

$$\nu = \nu_c + \nu_p = \nu_a + \nu_s + \nu_p = \frac{d\nu_a}{dQ} dQ + \nu_s + \nu_p$$

Remark 1.1. If $(\Omega, \mathcal{T})$ is compact, then the dual of $\mathcal{C}_b(\Omega)$ is $\mathcal{M}$.

Now that the right space and a good duality have been identified, let's go back to the optimization, which is solved through a mixed relaxation-duality technique.

Let

$$\overline{C}(x) \subseteq \{\mu \in rba(\Omega)_+ : \mu(\Omega) \leq x\}$$

indicate the weak* closure of the set $C(x)$.

The problem is to find a new functional $\overline{I_U} : rba(\Omega) \to \mathbb{R} \cup \{-\infty\}$, defined on the whole $rba(\Omega)$, such that the problem

$$\max_{\mu \in \overline{C}(x)} \overline{I_U}(\mu) \tag{6}$$

is equivalent to (1), in the following sense:

i) $\max_{\mu \in \overline{C}(x)} \overline{I_U}(\mu) = \sup_{X \in C(x)} I_U(X)$

ii) if $X_n \in C(x)$ is a maximizing sequence for (1), converging weakly* to some $\mu \in \overline{C}(x)$, then μ is a maximizer for (6).

Such $\overline{I_U}$ is the *relaxation* of the functional I_U and its explicit calculation is the main result of the paper. As far as the authors know, the result is an extension of well- known relaxation results over $\mathcal{M}$ in Convex Analysis (see the next Section and the reference there cited). The relaxation of functionals depends on the selected topology. And as the one considered is the weak* topology on $rba(\Omega)$, $\overline{I_U}$ is defined as:

$$\overline{I_U}(\mu) = \inf\{G(\mu) \mid G : rba(\Omega) \to \mathbb{R} \cup \{-\infty\}, G \text{ } weak^*u.s.c., G \geq I_U \text{ on } L^1(Q)\} \tag{7}$$

We will also prove that under a rather mild condition (A), the restriction of $\overline{I_U}$ to $\mathcal{M}$ is simply the sum of two integrals, the expected utility from the a.c. part plus a singular term:

$$\overline{I_U}(\mu) = E_P\left[U\left(\frac{d\mu_a}{dQ}\right)\right] + \int \varphi \, d\mu_s, \quad \mu = \mu_a + \mu_s \tag{8}$$

where φ is a suitably defined nonnegative u.s.c. function.

Under an additional condition (B), (both A and B are implied by the ones assumed in the current literature on utility maximization to recover a primal optimal solution), we show that problem (6) finds its maximum on $\overline{C}(x) \cap \mathcal{M}$. That is, any (possibly non unique) optimizer μ^* is a *measure*, even if not necessarily absolutely continuous with respect to Q. Therefore, the optimal investments may have a "singular component". Nevertheless, the financial interpretation is preserved because any μ^* is a measure and no mass disappears. The singular part μ_s^* can be seen as a very risky bet, as the agent invests the amount of money $\mu_s^*(\Omega)$ on the support of μ_s^* which is a Q (and P)-negligible set.

Assumptions on the utility function. $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ verifies $(0, +\infty) \subseteq \mathrm{Dom}(U) \subseteq [0, +\infty)$. We assume that on $(0, +\infty)$ U is: i) strictly increasing; ii) strictly concave; iii) continuously differentiable and iv) satisfies the Inada conditions

$$\lim_{x \downarrow 0} U'(x) = +\infty \ , \quad \lim_{x \uparrow +\infty} U'(x) = 0$$

Recall that the convex conjugate of U is defined as $V : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$

$$V(y) = \sup_x \{U(x) - yx\}$$

V verifies: $(0, +\infty) \subseteq \mathrm{Dom}(V) \subseteq [0, +\infty)$; on $(0, +\infty)$ it is monotone decreasing and continuously differentiable with derivative V' given by

$$V'(y) = -(U')^{-1}(y)$$

The following assumption is a first (rather weak) joint condition on the utility U (through V) and on the probabilistic model.

Assumption 0. There exists an $y_0 > 0$ such that

$$E_P\left[V\left(y_0 \frac{dQ}{dP}\right)\right] < +\infty \tag{9}$$

In other words, the *relative generalized entropy of* Q with respect to P *must be finite for some scaling* y_0.

Remark 1.2. Since V is monotone decreasing on $\mathbb{R}_+$, this assumption is equivalent to the existence of a (positive) bounded, continuous $g_0 : \Omega \to \mathbb{R}_+$ such that $E_P\left[V\left(g_0 \frac{dQ}{dP}\right)\right] < +\infty$ (take $y_0 = \sup_\omega g_0(\omega)$). Note also that from convexity of V, Assumption 0 amounts to asking $E_P\left[V\left(y_0 \frac{dQ}{dP}\right)\right] \in \mathbb{R}$.

Remark 1.3. Assumption 0 is equivalent to that used in [KS99, Theorem 2.0], that is the problem (1) has finite value

$$\sup_{X \in C(x)} I_U(X) < +\infty$$

It is much milder than condition (10) in KS2-nec&suff, Theo 2, where it is required $E_P\left[V\left(y \frac{dQ}{dP}\right)\right] < +\infty$ for all $y > 0$. We will see that Assumption 0 already permits to recover an integral representation for $\overline{I_U}$ over $\mathcal{M}$.

2 The integral representation of the relaxed functional $\overline{I_U}$

2.1 Some preliminary Lemmata

In what follows, we assume I_U is defined on all $rba(\Omega)$, with $I_U(\mu) = -\infty$ if μ is not a.c. with respect to Q.

Lemma 2.1. *The concave functional I_U is proper, i.e. it never assumes the value $+\infty$.*

Proof. We must only check what happens for $X \in L^1(Q)$. Fix an y_0 satisfying Assumption 0. By Fenchel inequality:

$$U(X) - Xy_0\frac{dQ}{dP} \leq V\left(y_0\frac{dQ}{dP}\right)$$

Taking P-expectations, we obtain $I_U(X) \leq y_0 E_Q[X] + E_P\left[V\left(y_0\frac{dQ}{dP}\right)\right] < +\infty$. □

The proof above shows also that the definition of $\overline{I_U}$ in (7) is well- posed. In fact I_U is dominated by the (weak*) continuous, affine functional $G : rba(\Omega) \to \mathbb{R}$

$$G(\mu) \triangleq y_0\mu(\Omega) + E_P\left[V\left(y_0\frac{dQ}{dP}\right)\right]$$

Also, the infimum in (7) is attained, as the infimum of a family of u.s.c. functionals is u.s.c. The minimum is

$$\overline{I_U}(\mu) = \max_{\mu^\alpha \xrightarrow{*} \mu} \limsup_\alpha I_U(\mu^\alpha) \tag{10}$$

where the supremum is taken over all the nets $(\mu_\alpha)_\alpha$ that weak* converge to μ (see e.g. [Buttazzo]).

The major difficulty in the computation of $\overline{I_U}$ over the whole $rba(\Omega)$ is that $\mathcal{C}_b(\Omega)$ is not necessarily separable under our conditions (Ω is not assumed to be compact), so that the weak* topology on norm-bounded subsets of $rba(\Omega)$ is not necessarily metrizable. That's why nets must be used instead of sequences in (10).

Luckily, some help comes from Duality Theory. The convex conjugate $J_V : \mathcal{C}_b(\Omega) \to \mathbb{R} \cup \{+\infty\}$ of I_U is defined as

$$J_V(g) = \sup_{\mu \in rba(\Omega)} \{I_U(\mu) - \mu(g)\} = \sup_{X \in L^1(Q)} \{E_P[U(X)] - E_Q[Xg]\}$$

Note that $J_V(g)$ is nothing but the usual convex conjugate of the convex functional $-I_U$ at $-g$, i.e.

$$J_V(g) = \sup_{X \in L^1(Q)} \{E_Q[X(-g)] - E_P[-U(X)]\} = (-I_U)^*(-g)$$

Lemma 2.2. *We have:*

1. $J_V(g) = E_P\left[V\left(g\frac{dQ}{dP}\right)\right]$;
2. *the proper domain of J_V, which is defined as*

$$\mathrm{Dom}(J_V) = \left\{g \in \mathcal{C}_b(\Omega) \mid E_P\left[V\left(g\frac{dQ}{dP}\right)\right] < +\infty\right\}$$

   is contained in $\mathcal{C}_b(\Omega)_+$ and it is directed downward;

3. *The variable $\varphi : \Omega \to \mathbb{R}$*

$$\varphi(\omega) = \inf_{g \in \mathrm{Dom}(J_V)} g(\omega)$$

   is nonnegative, bounded, upper semicontinuous and can be monotonically approximated by a sequence $(g_k)_k$ in $\mathrm{Dom}(J_V)$

$$g_k(\omega) \downarrow \varphi(\omega) \text{ for all } \omega \tag{11}$$

Proof. $L^1(Q)$ is decomposable [1], so Theorem 21 in [Roc74], part a) gives the formula for $J_V(g)$.

Since $\mathrm{Dom}(V) \subseteq \mathbb{R}_+$, then clearly $\mathrm{Dom}(J_V) \subset \mathcal{C}_b(\Omega)_+$ (and thus it is a set bounded from below).

[1] A space L of random variables on $(\Omega, \mathcal{F}, P)$ is decomposable if, whenever $A \in \mathcal{F}$ and f is a bounded random variable on A, then, for every $g \in L$, $\widetilde{g} = fI_A + gI_{A^c}$ also belongs to L.

Therefore the pointwise infimum φ is well-defined, it is nonnegative, bounded and upper semicontinuous as the inf of a family of continuous functions. Also, since the space $\mathcal{C}_b(\Omega)$ has the countable sup property (see Aliprantis-Border Th 8.22), there exists a sequence $(g_k)_k$ in $\mathrm{Dom}(J_V)$ such that $g_k \geq \varphi$, $g_k \to \varphi$ pointwise.

To prove that $\mathrm{Dom}(J_V)$ is directed downward, note that if $g, f \in \mathrm{Dom}(J_V)$ then $g \wedge f \in \mathrm{Dom}(J_V)$. In fact

$$E_P\left[V\left(g \wedge f\frac{dQ}{dP}\right)\right] = E_P\left[V\left(g\frac{dQ}{dP}\right)I_{\{g\leq f\}}\right] + E_P\left[V\left(f\frac{dQ}{dP}\right)I_{\{f<g\}}\right] < +\infty$$

This property yields the selection of a monotone approximating sequence $(g_k)_k$. □

Remark 2.3. When $AE(U) < 1$ holds in addition to Assumption 0, it is known (see [KS99]) that $E[V(y\frac{dQ}{dP})] < +\infty$ for **all** $y > 0$. Therefore, $\varphi \equiv 0$.

Lemma 2.4. *Let $(I_U)^{**}$ be the biconjugate functional*

$$(I_U)^{**}(\mu) = \inf_{g\in\mathcal{C}_b(\Omega)} \{\mu(g) + J_V(g)\}$$

Then,

$$\overline{I_U} = (I_U)^{**} \tag{12}$$

Proof. By definition (7), the relaxed functional $\overline{I_U}$ is the smallest concave u.s.c. functional bigger than I_U. Now, the biconjugate of I_U verifies

$$(I_U)^{**}(\mu) = -\sup_{g\in\mathcal{C}_b(\Omega)} \{-\mu(g) - J_V(g)\} = -(J_V)^*(-\mu) = -(-I_U)^{**}(-\mu) \tag{13}$$

where $(-I_U)^{**}$ is the usual (convex) biconjugate of $-I_U$, that is $(-I_U)^{**}(\mu) = \sup_{g\in\mathcal{C}_b(\Omega)}\{\mu(g) - (-I_U)^*(g)\}$. Classic convex duality gives that $(-I_U)^{**}$ is the greatest convex, l.s.c. functional which is smaller that $-I_U$ see e.g. [Bre83]. Hence, $(I_U)^{**}$ is the smallest u.s.c. concave functional which is bigger than I_U, thus it coincides with $\overline{I_U}$. □

2.2 The computation of $\overline{I_U}$

The line of the proof consists of the following three steps:

1. we show that in the relaxation $\overline{I_U}(\mu)$ the contribution of the measure component μ_c and of the pure charge component μ_p can be separated
2. then, the relaxation over $\mathcal{M}$ is explicitly computed
3. Assumption A that enables the financial interpretation is introduced and we finally get to the formula that will be used to prove our main Theorem.

Due to the regularity of Ω (Polish space), $\overline{I_U}(\mu)$ is equal to $-\infty$ if μ doesn't belong to $rba(\Omega)_+$. This is the reason why in what follows the focus is on $rba(\Omega)_+$ only.

2.2.1 Step 1: $\overline{I_U}$ on $rba(\Omega)$

Proposition 2.5. *Let $\mu \in rba(\Omega)_+$. Then,*

$$\overline{I_U}(\mu) = \overline{I_U}(\mu_c) + \inf_{f\in\mathrm{Dom}(J_V)} \mu_p(f) \tag{14}$$

Proof. The inequality $\overline{I_U}(\mu) \geq \overline{I_U}(\mu_c) + \inf_{g\in\mathrm{Dom}(J_V)} \mu_p(g)$ follows immediately from $\overline{I_U} = (I_U)^{**}$ and from the inequality

$$E[V(g\frac{dQ}{dP})] + \mu(g) \geq E[V(g\frac{dQ}{dP})] + \mu_c(g) + \inf_{f\in\mathrm{Dom}(J_V)} \mu_p(f) \geq (I_U)^{**}(\mu_c) + \inf_{f\in\mathrm{Dom}(J_V)} \mu_p(f)$$

To prove the opposite inequality and thus (14), recall that μ_c is also compact-inner regular. In particular, there exists an increasing sequence of compact sets K^n such that $(P+\mu_c)(\Omega \setminus K^n) < \frac{1}{n}$. Now, from (4) $\mu_p(K^n)$ is zero for all n. Fix $n > 0$. Borel-regularity of μ_p guarantees that there exists a closed set $C^n \subseteq \Omega \setminus K^n$ such that

$$\mu_p(C^n) > \mu_p(\Omega) - \frac{1}{n}$$

As Ω is Polish (thus normal) there exists a continuous function $\alpha^n : \Omega \to [0,1]$ which is equal to 1 on K^n and 0 over C^n. Up to a subsequence, α^n converges to 1 $(P+\mu_c)$-a.s. Fix a pair $f, g \in \mathrm{Dom}(J_V)$. Then, set

$$h^n = \alpha^n g + (1-\alpha^n) f$$

which still belongs to $\mathrm{Dom}(J_V)$. Now by mere convexity,

$$E[V(h^n \frac{dQ}{dP})] + \mu(h^n) \le E[\alpha^n V(g \frac{dQ}{dP})] + E[(1-\alpha^n) V(f \frac{dQ}{dP})] + \mu(h^n)$$

and

$$\mu(h^n) = \mu_c(h^n) + \mu_p(h^n) \le \mu_c(h^n) + \frac{1}{n}\|g+f\|_\infty + \mu_p(f)$$

so that

$$E[V(h^n \frac{dQ}{dP})] + \mu(h^n) \le E[\alpha^n V(g \frac{dQ}{dP})] + E[(1-\alpha^n) V(f \frac{dQ}{dP})] + \mu_c(h^n) + \frac{1}{n}\|g+f\|_\infty + \mu_p(f)$$

Passing to the liminf,

$$\liminf_n E[V(h^n \tfrac{dQ}{dP})] + \mu(h^n) \le$$
$$\liminf_n \{E[\alpha^n V(g\tfrac{dQ}{dP})] + E[(1-\alpha^n)V(f\tfrac{dQ}{dP})] + \mu_c(h^n) + \tfrac{1}{n}\|g+f\|_\infty + \mu_p(f)\} = E[V(g\tfrac{dQ}{dP})] + \mu_c(g) + \mu_p(f)$$

as the liminf is in fact a limit everywhere in the second line above since the Dominated Convergence Theorem can be applied as α^n converges to 1 $(P+\mu_c)$-a.s. Therefore,

$$(I_U)^{**}(\mu_c) + \inf_{f \in \mathrm{Dom}(J_V)} \mu_p(f) = \inf_{f,g \in \mathrm{Dom}(J_V)} \{E[V(g\frac{dQ}{dP})] + \mu_c(g) + \mu_p(f)\} \ge (I_U)^{**}(\mu)$$

□

Remark 2.6. Even if $\inf_{f \in \mathrm{Dom}(J_V)} \mu_p(f) = \inf_k \mu_p(g_k)$, where the g_k are the approximations for φ in (11), the inf and the expected value cannot be exchanged to conclude $\inf_{f \in \mathrm{Dom}(J_V)} \mu_p(f) = \mu_p(\varphi)$ because μ_p is **not** countably additive.

2.2.2 Step 2: $\overline{I_U}$ over $\mathcal{M}$

The great advantage of working on $\mathcal{M}$ only is that the trace of the weak* topology on norm bounded subsets of $\mathcal{M}_+$ can be *metrized* (e.g. by the Dudley distance). So, the computation of the relaxed functional is much easier on $\mathcal{M}_+$, as sequences can be used instead of nets to characterize convergence and all the arguments from standard integration theory with respect to measures do apply. Therefore, not too surprisingly, the following Lemma proves the same result as that in [BV88], which was stated for Ω compact - or locally compact. We give a slightly different and self-contained proof.

Define the function

$$W(\omega, x) := \sup_{z \le x} \left\{ U(z) + (x-z)\varphi(\omega)\frac{dQ}{dP}(\omega) \right\} \tag{15}$$

Note that $W(\omega, \cdot)$ is the so-called **sup-convolution** of the utility function U and of the linear, ω-dependent function $x \mapsto x\varphi(\omega)\frac{dQ}{dP}(\omega)$.

Lemma 2.7. *Let $\mu \in \mathcal{M}_+$. Then*

$$\overline{I_U}(\mu) = E[W(\cdot, \frac{d\mu_a}{dQ})] + \int \varphi d\mu_s \tag{16}$$

Proof. In the proof, the operational definition (10) of $\overline{I_U}$ is used, but with nets replaced by sequences (which is a consequence of the metrizability of the weak* topology on bounded subsets of $\mathcal{M}_+$) together with the identity $\overline{I_U} = (I_U)^{**}$:

$$\overline{I_U}(\mu) = \max_{X_n \xrightarrow{*} \mu} \limsup_n I_U(X_n) = (I_U)^{**}(\mu) = \inf_{g \in \mathcal{C}_b(\Omega)} \left\{ \mu(g) + E[V(g\frac{dQ}{dP})] \right\}$$

Let us pick a maximizing sequence $(X_n)_n$ for the value $\overline{I_U}(\mu)$ and which is weak* converging to μ. As $(X_n)_n$ is weak* convergent, it is bounded in $L^1(Q)$. Then by Komlos Theorem there exists a convex combinations of tails of the $(X_n)_n$, which we denote by $(Y_n)_n$ which converges Q-a.s. (and thus P-a.s) to a certain nonnegative Z (and clearly still weak* converge to μ). From Lemma 2.8 below, necessarily $Z \leq \frac{d\mu_a}{dQ}$, with possibly strict inequality. Now, since convex combinations may only improve the concave functional, $(Y_n)_n$ continues to be a maximizing sequence for $\overline{I_U}(\mu)$. From Fenchel inequality, for any fixed $g \in \mathrm{Dom}(J_V)$

$$U(Y_n) - Y_n g\frac{dQ}{dP} \leq V(g\frac{dQ}{dP})$$

Integrating, taking the limsup over n and applying Fatou Lemma,

$$\limsup_n E[U(Y_n) - Y_n g\frac{dQ}{dP}] \leq E[U(Z) - Zg\frac{dQ}{dP}] \leq E[V(g\frac{dQ}{dP})]$$

As the Y_n still maximize the relaxed functional and $E[Y_n g\frac{dQ}{dP}] = E_Q[Y_n g]$ converges to $\mu(g)$, the above relation amounts to

$$\overline{I_U}(\mu) - \mu(g) \leq E[U(Z) - Zg\frac{dQ}{dP}] \leq E[V(g\frac{dQ}{dP})]$$

or equivalently, adding $\mu(g)$:

$$\overline{I_U}(\mu) \leq E[U(Z) - Zg\frac{dQ}{dP}] + \mu(g) \leq E[V(g\frac{dQ}{dP})] + \mu(g)$$

Splitting μ as $\mu_a + \mu_s$,

$$\overline{I_U}(\mu) \leq E[U(Z) + (\frac{d\mu_a}{dQ} - Z)g\frac{dQ}{dP}] + \mu_s(g) \leq E[V(g\frac{dQ}{dP})] + \mu(g)$$

for all $g \in \mathrm{Dom}(J_V)$. Take now the inf over $g \in \mathrm{Dom}(J_V)$ and use (11) to get

$$\overline{I_U}(\mu) \leq E[U(Z) + (\frac{d\mu_a}{dQ} - Z)\varphi\frac{dQ}{dP}] + \mu_s(\varphi) \leq (I_U)^{**}(\mu)$$

so that equality must hold in the above formula. Getting rid of Z is not difficult, albeit a bit tedious - the main tool being the Fenchel inequality. Fix $x \geq 0$. For any $z \leq x$,

$$U(z) + (x-z)\varphi\frac{dQ}{dP} \leq U(z) + (x-z)g\frac{dQ}{dP} \leq V(g\frac{dQ}{dP}) + xg\frac{dQ}{dP}$$

for all $g \in \mathrm{Dom}(J_V)$. Therefore, for any $g \in \mathrm{Dom}(J_V)$

$$U(z) + (x-z)\varphi\frac{dQ}{dP} \leq W(\omega, x) := \sup_{z \leq x}\{U(z) + (x-z)\varphi\frac{dQ}{dP}\} \leq V(g\frac{dQ}{dP}) + xg\frac{dQ}{dP}$$

Substituting x with $\frac{d\mu_a}{dQ}$, z with Z (which is smaller than $\frac{d\mu_a}{dQ}$) and integrating give

$$E[U(Z) + (\frac{d\mu_a}{dQ} - Z)\varphi\frac{dQ}{dP}] \leq E[W(\cdot, \frac{d\mu_a}{dQ})] \leq E[V(g\frac{dQ}{dP})] + \mu_a(g)$$

and therefore

$$\overline{I_U}(\mu) = E[U(Z) + (\frac{d\mu_a}{dQ} - Z)\varphi\frac{dQ}{dP}] + \mu_s(\varphi) \leq E[W(\cdot, \frac{d\mu_a}{dQ})] + \mu_s(\varphi) \leq E[V(g\frac{dQ}{dP})] + \mu_a(g) + \mu_s(g)$$

and taking the inf over the $g \in \mathrm{Dom}(J_V)$, the conclusion (16) follows. □

Lemma 2.8. *Let $(X_n)_{n\geq 1}$ be a bounded sequence in $L^1_+(Q)$, such that X_n converges to X almost surely, and weak* to $\mu \in rba(\Omega)$. Then*

$$X \leq \frac{d\mu_a}{dQ}\ Q - a.s.$$

Proof. Note first that $\mu \geq 0$, $X \geq 0$ and an application Fatou's Lemma to $(X_n)_n$ gives $X \in L^1(Q)$. Thanks to the compact-inner regularity of $\mu_a + \mu_s$ (see (3)), it is enough to show that

$$E_Q[I_K X] \leq (\mu_a + \mu_s)(K) \text{ for all compacts } K$$

In fact, the inequality above for all the compacts implies the inequality for all the Borel sets B, $E_Q[I_B X] \leq (\mu_a + \mu_s)(B)$ and clearly this gives $X \leq \frac{d\mu_a}{dQ}$ Q-a.s.

Fix then a compact K. As $\mu_p \in \mathcal{N}$, with an argument similar to that used in Proposition 2.5, for any $h \in \mathbb{N}_+$ there exists a closed set $C_h \subseteq K^c$ with $\mu_p(C_h) \geq \mu_p(\Omega) - \frac{1}{h}$. And there is a continuous function g_h^K such that $0 \leq g_h^K \leq 1$, $g_h^K = 1$ on K, $g_h^K = 0$ on C_h and consequently $g_h^K \to I_K$ pointwisely. Then for all h

$$E_Q[I_K X] \leq E_Q[g_h^K X] \leq \lim_n E_Q[g_h^K X_n] = \mu(g_h^K)$$

where the first inequality is trivial, the second is a consequence of Fatou, the equality follows from weak* convergence of $X_n dQ$ to μ. By construction, $\mu_p(g_h^K) \leq \frac{1}{h}\mu_p(\Omega)$ and then

$$E_Q[I_K X] \leq (\mu_a + \mu_s)(g_h^K) + \frac{1}{h}\mu_p(\Omega)$$

and the conclusion follows passing to the limit on h.

□

Remark 2.9. The inequality $X \leq \frac{d\mu_a}{dQ}$ can be strict. See [BM89, Example 2] where $X = 0$ while $\frac{d\mu_a}{dQ} = 1$.

2.2.3 Step 3: Assumption A and the final formula

Summing up the results from Step1 and Step2, we have shown the following general formula

$$\overline{I_U}(\mu) = E[W(\cdot, \frac{d\mu_a}{dQ})] + \int \varphi d\mu_s + \inf_{f \in \mathrm{Dom}(J_V)} \mu_p(f)$$

However, our problem stems from a clear application - namely, utility maximization. In other terms, inside the expectation one would like to see *the utility function only*, i.e. it should be for any $x > 0$, $W(\omega, x) = U(x)$ P-a.s. The following Lemma characterizes the situations where this financial interpretation of the relaxation is preserved.

Lemma 2.10. *For any fixed $x > 0$ $W(\omega, x) = U(x)$ P-a.s. if and only if $\varphi = 0$ P-a.s.*

Proof. The implication: if $\varphi = 0$ P-a.s. then $W(\omega, x) = U(x)$ a.s. for any x is straightforward from the definition of W. To show the converse, from $W(\omega, x) = U(x)$ a.s. then there exists a Borel set B with $P(B) = 1$ such that $W(\omega, q) = U(q)$ for all $q \in \mathbb{Q}_+$ and $\omega \in B$. But both $W(\omega, \cdot)$ and U are continuous on $\mathbb{R}_+$, so that $W(\omega, x) = U(x)$ on B for all $x > 0$. But from the very definition of W, this happens only if

$$U'(z) - \varphi(\omega)\frac{dQ}{dP}(\omega) \geq 0 \text{ for all } z > 0\ , \omega \in B$$

and using the Inada condition at $+\infty$ on U, for $\omega \in B$ it holds

$$0 = \lim_{z \to +\infty} U'(z) \geq \varphi(\omega) \frac{dQ}{dP}(\omega)$$

and given that $Q \sim P$, $\varphi = 0$ on B.

$\square$

Assumption A $\varphi = 0$ P-a.s.

Corollary 2.11. *Under Assumption A, the relaxation $\overline{I_U}$ over $\mathcal{M}_+$ is given by*

$$\overline{I_U}(\mu) = E[U(\frac{d\mu_a}{dQ})] + \int \varphi d\mu_s + \inf_{f \in \mathrm{Dom}(J_V)} \mu_p(f) \tag{17}$$

The following Subsection contains some consideration on φ, given its importance for the integral representation of $\overline{I_U}$. It can be skipped if one is interested in the utility maximization only (Section 3).

2.3 On the random variable φ

The notation $B(\omega, r), \omega \in \Omega, r > 0$ indicates as usual the ball of center ω and radius r.

Definition 2.12. *F is the set of points which admit a neighborhood where $\frac{dP}{dQ}$ is a.s.-bounded, i.e.*

$$F = \left\{ \omega \mid \exists K > 0, \exists V \text{ open s.t. } V \ni \omega, \frac{dP}{dQ} \leq K \text{ a.s. on } V \right\}$$

and the set of the **poles** *of $\frac{dP}{dQ}$ is $D = F^c$.*

Remark 2.13. F is open (and thus D is closed) and it depends only on the class of $\frac{dP}{dQ}$ in $L^1(Q)$. The set of the poles D can be described as

$$D = \{\omega \mid \exists (\epsilon_n)_n, (k_n)_n \text{ with } \epsilon_n \downarrow 0, k_n \uparrow +\infty \text{ s.t. for all } n \, \exists A_n \subseteq B(\omega, \epsilon_n), Q(A_n) > 0, \frac{dP}{dQ} \geq k_n \text{ a.s. on } A_n\}$$

Lemma 2.14. *φ is identically null on F.*

Proof. Suppose $\omega^* \in F$. Then, there exists an open ball $B(\omega^*, \epsilon)$ all contained in F such that $\frac{dP}{dQ} \leq K$ a.s. on $B(\omega^*, \epsilon)$. Take y_0 as in (9), for any $\delta > 0, \delta < y_0$ consider the continuous function $g_\delta = \delta\alpha + y_0(1 - \alpha)$ where

$$\alpha(\omega) = \frac{d(\omega, \Omega \setminus B(\omega^*, \epsilon))}{d(\omega, \overline{B}(\omega^*, \frac{\epsilon}{2})) + d(\omega, \Omega \setminus B(\omega^*, \epsilon))}$$

By construction, g_δ is such that

1. its range is the interval $[\delta, y_0]$;
2. $g_\delta^{-1}(\delta) = \overline{B}(\omega^*, \frac{\epsilon}{2})$;
3. $g_\delta^{-1}(y_0) = \Omega \setminus B(\omega^*, \epsilon)$.

To see that $g_\delta \in \mathrm{Dom}(J_V)$, note

$$E\left[V\left(g_\delta \frac{dQ}{dP}\right)\right] = E\left[V\left(g_\delta \frac{dQ}{dP}\right) I_{\{\frac{dQ}{dP} \geq \frac{1}{K}\}}\right] + E\left[V\left(g_\delta \frac{dQ}{dP}\right) I_{\{\frac{dQ}{dP} < \frac{1}{K}\}}\right]$$

and therefore

$$E\left[V\left(g_\delta \frac{dQ}{dP}\right)\right] \leq V\left(\frac{\delta}{K}\right) P\left(\{\frac{dQ}{dP} \geq \frac{1}{K}\}\right) + E\left[V\left(y_0 \frac{dQ}{dP}\right) I_{\{\frac{dQ}{dP} < \frac{1}{K}\}}\right] < +\infty$$

By definition of φ,

$$\varphi(\omega^*) = \inf_{g \in \mathrm{Dom}(J_V)} g(\omega^*)$$

and since $g_\delta(\omega^*) = \delta$ we get

$$0 \le \varphi(\omega^*) \le \inf_{\delta > 0} \delta = 0$$

□

Corollary 2.15. *$D \supseteq \{\varphi > 0\}$. Thus, if D is negligible, then Assumption A holds. In particular, this is the case when $\frac{dP}{dQ} \in L^\infty$ or, more generally, D is empty.*

3 Utility maximization

3.1 The classic duality

Consider the restriction of J_V to the constant functions, namely the function $v : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$,

$$v(y) = E_P \left[V \left(y \frac{dQ}{dP} \right) \right]$$

and let

$$L := \inf\{y \mid v(y) < +\infty\}$$

Note that $L \ge 0$. The next Lemma is a straightforward consequence of the definitions.

Lemma 3.1. *Properties of v.*

1. *v is l.s.c. and monotone decreasing;*
2. *for all $y < L$, $v(y) = +\infty$;*
3. *for $y > L$, v is differentiable and*

$$v'(y) = E_Q \left[V' \left(y \frac{dQ}{dP} \right) \right]$$

4. *if $v'_+(L) := \lim_{y \downarrow L} v'(y)$ is finite, then $v(L)$ is finite as well.*

Proof. Items 1, 2 are obvious as v is the restriction of J_V. In item 3, the formula for the derivative when $y > L$ can be proved by a classic argument based on convexity and Fatou Lemma. Item 4 is a consequence of the l.s.c. and convexity of v.

□

The next Lemma provides the link between J_V and the function v which is typically seen as dual function in the current literature.

Lemma 3.2. *1.*

$$\inf_{g \in \mathcal{C}_b(\Omega)} \{J_V(g) + \|g\|_\infty x\} = \inf_{y \ge L} \{v(y) + xy\}$$

2. *As in the lemma above, let $v'_+(L) = \inf_y v'(y) \in [-\infty, 0)$ and set $x_M = -v'_+(L)$.*

   a- *For any $x < x_M$, the above infimum is reached and it is given by the unique solution of*

$$v'(y) - x = 0, \tag{18}$$

   namely $y(x) = (v')^{-1}(-x)$.

   b- *If $x_M < +\infty$ and $x \ge x_M$ the infimum is reached by $y(x_M) = L$.*

Proof. 1. The inequality

$$\inf_{g\in\mathcal{C}_b(\Omega)}\{J_V(g)+x\|g\|_\infty\}\leq\inf_{y\geq L}\{J_V(y)+xy\} \tag{19}$$

is obvious. But V is decreasing, so for any g

$$J_V(g)+x\|g\|_\infty\geq v(\|g\|_\infty)+x\|g\|_\infty$$

From the previous Lemma, we can consider only $y\geq L$ whence the equality in (19) follows.

2. The assertions in case $x<x_M$ are immediate consequences of the previous Lemma. We are left with the case x_M finite and $x\geq x_M$. When $x_M=-v'_+(L)<+\infty$, then by the previous Lemma, item 4, necessarily $v(L)$ is also finite. As $v(y)$ is infinite for $y<L$ and when $x\geq x_M$ the derivative of $w(y)=v(y)+xy$ is strictly positive for $y>L$, the unique minimum of w is $y(x)=L$.

□

Here is the third main assumption of the paper.

Assumption B Take an approximating sequence g_k for φ as in (11), that is $g_k\in\mathrm{Dom}(J_V)$, $g_k\downarrow\varphi$. Let $h=\lim_k\|g_k\|_\infty$. If $h>0$, then we require that there exist a k^* and an $\epsilon^*>0$ such that the closed set

$$K^*=\{g_{k^*}\geq h-\epsilon^*\}$$

is compact.

This assumption is needed to ensure that any optimal μ^* in the relaxed maximization problem is a measure. See Section 4 for an easy counterexample.

Remark 3.3. It is not difficult to see that Assumption B does not depend on the particular approximating sequence $(g_k)_k$.

Remark 3.4. Assumption B is automatically satisfied when Ω is compact.

Lemma 3.5. *Under Assumption B,*

$$h=\max_\omega\varphi(\omega)=L \tag{20}$$

Proof. If $h=0$, the Lemma is obvious. Suppose then $h>0$.

First let us show $h=\max\varphi$. The inequality $g_k\geq\varphi$ implies $\|g_k\|_\infty\geq\sup_\omega\varphi(\omega)$, so that passing to the limit on k, $h\geq\sup_\omega\varphi(\omega)$. Outside K^*, $\varphi\leq h-\epsilon^*$ and as K^* is compact and φ is u.s.c., it attains its maximum on K^*. Since K^* contains all the non empty, closed sets with the finite intersection property: $V_{k,\epsilon}=\{g_k\geq h-\epsilon\}$ for all $k\geq k^*$, $\epsilon<\epsilon^*$, the intersection $Y:=\bigcap_{k,\epsilon}V_{k,\epsilon}$ is not empty and it consists of all the points ω^* where $\lim_k g_k(\omega^*)=h$. Therefore $h=\max\varphi$ and $Y=\operatorname{argmax}\varphi$.

Now, let us prove $h=L$. Like in the proof of the first part of Lemma 3.2, $L\leq\|g_k\|_\infty$, so $L\leq h$. But $\varphi=\inf\{g\mid g\in\mathrm{Dom}(J_V)\}$, in particular $\varphi\leq y$ for all $y\in\mathrm{Dom}(v)$, so $\varphi\leq L\leq h$ and then $\max\varphi\leq L\leq h$. From the first part of this Lemma, (20) must hold. □

3.2 The maximization via duality

The next result is the key step to our main result (standing Assumptions 0+A+B).

Proposition 3.6. *Let $D(x):=\{\mu\in rba_+\mid\mu(\Omega)\leq x\}$. Then,*

$$\max_{\mu\in D(x)}\overline{I_U}(\mu)=\min_{y\geq L}\{v(y)+xy\} \tag{21}$$

In addition, any primal optimal μ^* satisfies

1. $\mu^*(\Omega)=x$,

2. $\mu^*\in D(x)\cap\mathcal{M}_+$,

3. $\mathrm{supp}(\mu^*_s)\subseteq\operatorname{argmax}\varphi$.

Proof. Let us call γ the convex function (scaling of the norm) $\gamma(g) = \|g\|_\infty x$. Obviously: 1) γ is finite on $\mathcal{C}_b(\Omega)$ and 2) there exists a **norm- continuity** point (say, the constant $y_0 + \epsilon$) for J_V in $\mathcal{C}_b(\Omega)$. Fenchel duality Theorem (alias, the Minimax Theorem) ([Bre83, Chapter1]) can be applied to obtain

$$\inf_{g\in\mathcal{C}_b(\Omega)}\{J_V(g) + \gamma(g)\} = \max_{\mu\in rba(\Omega)}\{-(J_V)^*(-\mu) - \gamma^*(\mu)\}$$

From (13), $-(J_V)^*(-\mu) = \overline{I_U}(\mu)$. It is easy to see that

$$\gamma^*(\mu) = \sup_{g\in\mathcal{C}_b(\Omega)}\{\mu(g) - \|g\|_\infty x\} = \delta_{D(x)}(\mu)$$

where $\delta_{D(x)}$ is the function equal to 0 on $D(x)$ and $+\infty$ outside. So,

$$\inf_{g\in\mathcal{C}_b(\Omega)}\{J_V(g) + \|g\|_\infty x\} = \max_{\mu\in D(x)} \overline{I_U}(\mu)$$

Together with Lemma 3.2 (which also states that the inf is attained), this gives (21).

In the second part of the proposition, item 1, i.e. the fact that the constraint is binding, is a simple consequence of the monotonicity of $\overline{I_U}$. To show item 2, that is any μ^* is in fact a measure, let us rewrite the expression derived in (17) for the relaxation

$$\overline{I_U}(\mu^*) = E[U(\frac{d\mu_a^*}{dQ})] + \int \varphi d\mu_s^* + \inf_{f\in \mathrm{Dom}(J_V)} \mu_p^*(f)$$

Suppose that $\mu_p^* \neq 0$, say $0 < \mu_p^*(\Omega) = x' \leq x$. The contribution of μ_p^* to the (optimal) value $\overline{I_U}(\mu^*)$ can be easily majorized

$$\inf_{f\in \mathrm{Dom}(J_V)} \mu_p^*(f) \leq (L - \epsilon^*)x'$$

which is a consequence of (4), Assumption B and (20). Define $\widetilde{\mu} = \mu_a^* + \mu_s^* + x'\nu_s$, where ν_s is any probability with support contained in $\operatorname{argmax}\varphi$ (and therefore singular with respect to Q thanks to Assumption A). $\widetilde{\mu}$ is thus in $D(x) \cap \mathcal{M}_+$ and

$$\overline{I_U}(\widetilde{\mu}) = E[U(\frac{d\mu_a^*}{dQ})] + \int \varphi d(\mu_s^* + x'\nu_s) = E[U(\frac{d\mu_a^*}{dQ})] + \int \varphi d\mu_s^* + Lx' \geq \overline{I_U}(\mu^*) + \epsilon^* x' > \overline{I_U}(\mu^*)$$

which is a contradiction. A similar monotonicity argument shows that the support of any optimal μ_s^* is contained in $\operatorname{argmax}\varphi$. $\square$

Remark 3.7. $\overline{C}(x) \subseteq D(x)$, but it is not difficult to prove that (Ω Polish, Q with full support) that

$$\overline{C}(x) \cap \mathcal{M}_+ = D(x) \cap \mathcal{M}_+ = \{\mu \in \mathcal{M}_+ \mid \mu(\Omega) \leq x\}$$

Finally, here is our main result (standing Assumptions 0+A+B).

Theorem 3.8. *Define u to be the optimal value function $u(x) := \sup_{X\in C(x)} E[U(X)]$. Then*

a- *for $x > 0$, $u(x)$ is finite, concave and monotone non decreasing and*

$$u(x) = \max_{\mu\in\overline{C}(x)} \overline{I_U}(\mu) \tag{22}$$

b- *$u(x)$ is also equal to $\max_{\mu\in D(x)} \overline{I_U}(\mu)$, so that all the results in Prop. 3.6 apply. In particular,*

$$u(x) = \max_{\mu\in\mathcal{M}_+, \mu(\Omega)\leq x} \left\{E[U(\frac{d\mu_a}{dQ})] + \int \varphi d\mu_s\right\} = \min_{y\geq L}\{v(y) + xy\}$$

c- *Further results on the solutions structure.*

1. *case $x < x_M \le +\infty$.*
There exists a unique solution $X^*(x) \in L^1(Q)$ to the maximization problem (i.e. $\mu^* = X^*(x)dQ$ is unique) and it is equal to

$$X^*(x) = (U')^{-1}\left(y(x)\frac{dQ}{dP}\right) \tag{23}$$

where $y(x) = (v')^{-1}(-x)$. This implies that u is strictly concave on $(0, x_M)$;

2. *case $x_M < +\infty$ and $x \ge x_M$.*

If x coincides with x_M we still have a unique solution $X^*(x_M)$ given by (23) with $y(x) = L > 0$.

If $x > x_M$, a singular part is always present in the (possibly non unique) solution μ^*. The optimizers μ^* are characterized by:
- $\mu_a^* = X^*(x_M) = (U')^{-1}(L\frac{dQ}{dP})$, *unique;*
- $\mu_s^*(\Omega) = x - x_M$ *and* $\mathrm{supp}(\mu_s^*) \subseteq \mathrm{argmax}\{\varphi\}$.

As a consequence $u(x)$ becomes linear with slope L after x_M:

$$u(x) = \begin{cases} E[U(X^*(x))] & \text{if } 0 < x \le x_M \\ u(x_M) + L(x - x_M) & \text{otherwise} \end{cases}$$

Proof. **Item a).** The value function u is finite for all $x > 0$ thanks to Assumption 0. Monotonicity is clear. Since $C(tx_1 + (1-t)x_2) \supseteq tC(x_1) + (1-t)C(x_2)$ for all $x_1, x_2 > 0$ and $t \in [0,1]$ and I_U is concave, u is also concave. Therefore, it is continuous on $\mathbb{R}_+$.

Let us prove (22). The maximization domain $C(x)$ is weak* relatively compact and $\overline{C}(x)$ is its closure so the u.s.c. $\overline{I_U}$ attains its maximum on $\overline{C}(x)$, say $\overline{u}(x)$. If in turn $X^n \in C(x)$ is a maximizing sequence for $u(x)$, then by Banach-Alaoglu there exists a μ^* in $\overline{C}(x)$ which is a weak* cluster point of $(X^n)_n$. By u.s.c. of $\overline{I_U}$,

$$u(x) = \lim_n E_P[U(X^n)] \le \limsup_n \overline{I_U}(X^n) \le \overline{I_U}(\mu^*) \le \overline{u}(x)$$

On the other hand, if $\overline{\mu}$ is any optimizer of $\overline{u}(x)$, by definition of relaxed functional

$$\overline{u}(x) = \overline{I_U}(\overline{\mu}) = \max_{X_\alpha \overset{*}{\to} \overline{\mu}} \limsup_\alpha I_U(X_\alpha)$$

Select a maximizing net $(X_\alpha^*)_\alpha$ for $\overline{I_U}(\overline{\mu})$. By weak* convergence, $E_Q[X_\alpha^*] \to \overline{\mu}(\Omega)$ and $\overline{\mu}(\Omega) \le x$. Fix then $\epsilon > 0$: definitely $E_Q[X_\alpha^*] \le \overline{\mu}(\Omega) + \epsilon \le x + \epsilon$, so definitely $X_\alpha^* \in C(x+\epsilon)$. Clearly

$$\overline{u}(x) = \limsup_\alpha I_U(X_\alpha^*) \le u(x+\epsilon)$$

so that

$$u(x) \le \overline{u}(x) \le \lim_{\epsilon \downarrow 0} u(x+\epsilon)$$

By continuity of u, we get the equality $u(x) = \overline{u}(x)$ and that the cluster point μ^* of the original maximizing sequence X^n is also an optimizer for $\overline{I_U}$ over $\overline{C}(x)$. [2]

Item b). This follows from item a), Proposition 3.6 and Remark 3.7.

Item c). Characterization of the optimal solutions.

1. Case $x < x_M$. By Lemma 3.2 the equation $v'(y) = -x$ has a unique solution, $y(x) > L$. Writing it down,

$$v'(y(x)) = E_Q\left[V'\left(y(x)\frac{dQ}{dP}\right)\right] = -x$$

[2] Tutto questo e' per dire che il rilassato del funzionale + il vincolo: $I_U(X) - \delta_{C(x)}(X)$ e' esattamente $\overline{I_U}(\mu) - \delta_{\overline{C}(x)}(\mu)$

so the candidate primal optimum is $X^*(x) = -V'(y(x)\frac{dQ}{dP}) = (U')^{-1}(y(x)\frac{dQ}{dP})$ and $E_Q[X^*(x)] = x$. From the elementary Fenchel duality formula,

$$U(X^*(x)) - y(x)X^*(x)\frac{dQ}{dP} = V\left(y(x)\frac{dQ}{dP}\right)$$

Taking the expectations, from (21) we conclude that $X^*(x)$ is indeed a solution. It is also unique as shown below.

Uniqueness Suppose μ^0 is another solution. First we show that its a.c. part μ^0_a must be non zero. The reason is clear if $U(0) = -\infty$. To check it in case $U(0)$ is finite, let us suppose $\mu^0 = \mu^0_s$. We have

$$\overline{I_U}(\mu^0) = U(0) + \int \varphi d\mu^0_s = U(0) + Lx = u(x)$$

by optimality. But then a contradiction would follow:

$$u(x) = v(y(x)) + xy(x) \geq V(y(x)) + xy(x) > V(+\infty) + xL = U(0) + xL = u(x)$$

in which the first inequality is due to Jensen's and the second by strict monotonicity of V plus $y(x) \geq L$. Therefore $\mu^0_a \neq 0$ and taking the convex combination $\mu_t = tX^*(x)dQ + (1-t)\mu^0 \in \overline{C}(x)$ with $t \neq 0$, a contradiction follows again as a standard concavity argument shows $\overline{I_U}(\mu_t) > \overline{I_U}(X^*(x)dQ)$.

2. We are left with the case $x_M < +\infty$ and $x \geq x_M$. If $x = x_M$, by Lemma 3.2 $v(L)$ is finite and $y(x) = L$, with optimal $X^*(x_M) = (U')^{-1}(L\frac{dQ}{dP})$. This solution is also unique, as the argument above still applies.
If $x > x_M$, by Lemma 3.2 again the dual minimizer is still $y(x) = y(x_M) = L$ which is then strictly positive[3]. Indicating with μ^* any optimizer and with X^* its a.c. part, by duality

$$u(x) = \overline{I_U}(\mu^*) = E[U(X^*)] + \int \varphi d\mu^*_s = E\left[V\left(L\frac{dQ}{dP}\right)\right] + Lx \tag{24}$$

By Fenchel inequality,

$$U(X^*) - X^*L\frac{dQ}{dP} \leq V\left(L\frac{dQ}{dP}\right)$$

Taking expectations and rearranging,

$$E[U(X^*)] \leq E\left[V\left(L\frac{dQ}{dP}\right)\right] + LE_Q[X^*]$$

and equality holds iff $X^* = X^*(x_M)$.
On the other hand, by Proposition 3.6, $\int \varphi d\mu^*_s = L\mu^*_s(\Omega)$ so that equality (24) implies that $X^* = X^*(x_M)$. Since $E_Q[X^*(x_M)] = x_M$, the mass of the singular part μ^*_s is $(x - x_M) > 0$ and therefore its contribution to the optimal $u(x)$ is $L(x - x_M)$.

□

4 Examples

In all the examples below, the function V is

$$V(y) = \begin{cases} e^{\frac{1}{y}} & y > 0 \\ +\infty & \text{otherwise} \end{cases}$$

[3] Argomento: Now, we show that L must be strictly positive. Since $V'(0) = -\infty$, if $L = 0$ we would have by monotone convergence $v'_+(0) = \lim_{y \downarrow 0} E_Q[V'(y\frac{dQ}{dP})] = -\infty$ which is a contradiction as $x_M = -v'_+(0)$ is assumed finite.

The derivative $V'(y) = -\frac{1}{y^2} e^{\frac{1}{y}}, y > 0$ satisfies $V'_+(0) = -\infty, V'(+\infty) = 0$ and consequently the conjugate utility U

$$U(x) = \inf_y \{xy + V(y)\}$$

verifies the Inada conditions and $U(+\infty) = +\infty$. but U does not have an elementary expression, as $V'(y) = -x$ cannot be analytically inverted. This doesn't affect by any means the examples, since we will work on the dual side. Thanks to the dual characterization of AE(U) provided by [KS99, Cor 6.1(iii)]:

Let U be a utility function with the Inada conditions and $U(+\infty) > 0$. Then U has AE(U) iff there exists $y_0 > 0$, $\lambda < 1$ and $C < +\infty$ such that $V(\lambda y) < CV(y)$ for $y < y_0$

it is clear that with such a choice of V, U **does not satisfy AE(U)**.

Also, the spaces $(\Omega, \mathcal{T})$ considered are elementary, so that the technicalities are minimal and a full description of the involved quantities is nearly straightforward. In fact, in almost all the examples, $(\Omega, \mathcal{T})$ *is compact, so that the dual of $\mathcal{C}_b(\Omega)$ is just $\mathcal{M}$ and Assumption B is automatically satisfied.* However, in Example 4.4 we show why Assumption B is necessary in the general case to recover primal optima that are in $\mathcal{M}_+$.

Example 4.1. Pick a bounded sequence $\{\omega_n\}_{n\geq 1}$ in $\mathbb{R}$ monotonically decreasing to ω_∞. Let Ω be the set $\{\omega_1, \cdots, \omega_\infty\}$ endowed with the euclidean topology and thus compact. Define P as:

$$p_n = P(\omega_n) = \frac{1}{(e-1)e^n}, p_\infty = P(\omega_\infty) = 0$$

and Q as

$$\frac{dQ}{dP}(\omega_n) = \frac{c}{n}$$

where $c > 1$ is the normalizing constant (the value at ω_∞ is irrelevant). Assumption 0 is verified, since

$$E\left[V\left(y\frac{dQ}{dP}\right)\right] < +\infty \text{ iff } y > \frac{1}{c}$$

so that $L = \max_\omega \varphi(\omega) = \frac{1}{c}$. For any $y > \frac{1}{c}$, $v'(y)$ is finite and it is equal to

$$v'(y) = -\frac{1}{y^2} \sum_{n\geq 1} \frac{n}{c(e-1)} e^{(\frac{1}{yc}-1)n}$$

and then

$$x_M = -v'_+(L) = -v'_+(\frac{1}{c}) = +\infty$$

Note also that the set of the poles of $\frac{dP}{dQ}$ is the singleton $\{w_\infty\}$, which is negligible, so by Corollary 2.15 $\varphi = 0$ a.s. and thus Assumption A holds. Also, Assumption B is automatically satisfied, as Ω is compact. All the hypotheses in Theorem 3.8 are then fulfilled. Since $x_M = +\infty$, the expression of φ is irrelevant, since for any $x > 0$

$$u(x) = \sup_{C(x)} E[U(X)] = \max_{C(x)} E[U(X)]$$

with $X^*(x)$ given by (23). The optimal value $u(x)$ satisfies the Inada condition at 0, but $u'(+\infty) = \frac{1}{c}$.

For completeness' sake, note that it is evident that

$$\varphi(\omega_n) = 0 \text{ if } n < +\infty \text{ while } \varphi(\omega_\infty) = \frac{1}{c}$$

Example 4.2. Here we give an example with $x_M < +\infty$ and optimal μ_s^* non unique when the endowment $x > x_M$. The idea is the same of the example above, but with slightly modified Ω and the probabilities, so that there are **two** (optimal) poles of $\frac{dP}{dQ}$ and $v'_+(L)$ is finite.

Let $\mathbb{Z}_0$ be the set of non- null integers. Consider a monotone increasing mapping from $\mathbb{Z}_0$ to $\mathbb{R}$, say

$\{\omega_z\}_{z\in\mathbb{Z}_0}$ so that 1) $\omega_z \downarrow \omega_{-\infty} \in \mathbb{R}$ if $z \to -\infty$ and 2) $\omega_z \uparrow \omega_{+\infty} \in \mathbb{R}$ if $z \uparrow +\infty$. Ω is then the compact $\{\omega_z\}_{z\in\mathbb{Z}_0} \cup \{\omega_{-\infty}, \omega_{+\infty}\}$. Define P as:

$$p_z = P(\omega_z) = \frac{k}{|z|^3 e^{|z|}}, P(\{\omega_{-\infty}, \omega_{+\infty}\}) = 0$$

and Q as

$$\frac{dQ}{dP}(\omega_z) = \frac{c}{|z|}$$

where k, c are the normalizing constant. By straightforward computations, $v(y)$ is finite iff $y \geq \frac{1}{c} = L$, $v'_+(L)$ is finite as well and equal to

$$v'_+(L) = -2kc \sum_{n\geq 1} \frac{1}{n^2}$$

By construction, the set of the poles $D = \{\omega_{-\infty}, \omega_{+\infty}\}$ is negligible, so $\varphi = 0$ a.s. Of course, this implies $\varphi(\omega_z) = 0$ for all z. The maximum of φ is L and by symmetry $\varphi(\omega_{-\infty}) = \varphi(\omega_{+\infty}) = L$. We can apply Theorem 3.8, case $x_M = -v'_+(L)$ finite. So, for any $x \leq x_M$ the optimal solution is unique and

$$u^*(x) = \max_{C(x)} E_P[U(X)]$$

with $X^*(x)$ given by (23). If $x > x_M$, the optimal value

$$u(x) = \sup_{C(x)} E_P[U(X)] = \max_{\mu\in\mathcal{M}_+, \mu(\Omega)\leq x} \left\{ E_P\left[U\left(\frac{d\mu_a}{dQ}\right)\right] + \int \varphi d\mu_s \right\}$$

and the optimal solutions are of the form

$$\mu^*(x) = X^*(x_M)dQ + \mu^*_s = X^*(x_M)dQ + (x - x_M)(t\epsilon_{\omega_{-\infty}} + (1-t)\epsilon_{\omega_{+\infty}})$$

where the last expression indicates any convex combination of the two Dirac probabilities in $\omega_{-\infty}, \omega_{+\infty}$. The optimal value is thus

$$u(x) = E[U(X^*(x_M))] + \int \varphi d\mu^*_s = E[U(X^*(x_M))] + L(x - x_M)$$

As it will be useful in the next (counter)example, note also that φ can be approximated by the functions

$$g_k(\omega_z) = \begin{cases} \frac{1}{k} & \text{if } |z| \leq k \\ \frac{1}{c} + \frac{1}{k} & \text{if } |z| > k \\ \frac{1}{c} + \frac{1}{k} & \text{on } \omega_{-\infty}, \omega_{+\infty} \end{cases}$$

which are in $\mathrm{Dom}(J_V)$ for all $k \geq 1$.

Remark 4.3. It would be rather easy to modify Ω in order to have a number of poles, not all optimal.

Example 4.4. This is to show that Assumption B is necessary in the general case to obtain measures as primal optima. The setup is the quite the same of Example 2 above, only remove the points $\omega_{-\infty}, \omega_{+\infty}$ from Ω (which now is *not compact anymore*). As before,

$$g_k(\omega_z) = \begin{cases} \frac{1}{k} & \text{if } |z| \leq k \\ \frac{1}{c} + \frac{1}{k} & \text{if } |z| > k \end{cases}$$

are in $\mathrm{Dom}(J_V)$ and $g_k \downarrow 0$, so that φ is identically null. But Assumption B is not satisfied, as no nonempty superlevel of the g_k is compact. Note that (20) does not apply: $\varphi = 0$ but $h = \lim_k \|g_k\|_\infty = \frac{1}{c}$. The value function u is clearly the same of the example above, so in particular for $x > x_M$

$$u(x) = \sup_{C(x)} E_P[U(X)] = E[U(X^*(x_M))] + L(x - x_M)$$

and it is also evident that we have the same maximizing sequences in $C(x)$ for $u(x)$ as before. These do have a weak*-cluster point, but it is an element of $rba(\Omega)_+$, not in $\mathcal{M}_+$. A primal optimum is of the form $\mu^* = X^*(x_M)dQ + (x - x_M)\nu$ where $\nu \in \mathcal{N}_+$ is any normalized pure charge such that $\nu(g) = \lim_{z\to+\infty} g(w_z) + \lim_{z\to-\infty} g(w_z)$ for any $g \in \mathcal{C}_b(\Omega)$ which admits limit to the extrema of Ω.

We conclude this Section with an exemplification of the necessity of Assumption A. When Assumption A does not hold, that is $\varphi > 0$ has positive P and Q- probability, the relaxed integrand W in (15) is different from the utility U.

Example 4.5. Let Ω be identical to the one in Example 4.1. Only, we shift some weight on the pole ω_∞. Precisely, fix $0 < \delta < 1$ and define

$$p_n = P(\omega_n) = \frac{1-\delta}{(e-1)e^n}, p_\infty = P(\omega_\infty) = \delta$$

and Q as

$$\frac{dQ}{dP}(\omega_n) = \frac{1}{n}, \frac{dQ}{dP}(\omega_\infty) = \frac{1 - \sum_n p_n x_n}{p_\infty} > 0$$

$D = \{\omega_\infty\}$ has now positive probability. $v(y)$ is finite iff $y > 1$, so $L = 1$. Since the continuous function

$$g_k(\omega_n) = \begin{cases} \frac{1}{k} & \text{if } n \leq k \\ 1 + \frac{1}{k} & \text{if } +\infty \geq n > k \end{cases}$$

is in $\mathrm{Dom}(J_V)$ for all $k \geq 1$, φ is identical to that in Example 4.1 with $c = 1$. So the problem is only at the pole w_∞. Let's write down $W(\omega_\infty, \cdot)$:

$$W(\omega_\infty, x) = \max_{z \leq x} \left\{ U(z) + (x - z)\varphi(\omega_\infty)\frac{dQ}{dP}(\omega_\infty) \right\}$$

and consider the derivative

$$U'(x) - \varphi(\omega_\infty)\frac{dQ}{dP}(\omega_\infty)$$

If $x > x^* = (U')^{-1}(\varphi(\omega_\infty)\frac{dQ}{dP}(\omega_\infty))$, the max in the expression for W is attained at x^*, so that

$$W(\omega_\infty, x) = \begin{cases} U(x) & \text{if } x \leq x^* \\ U(x^*) + (x - x^*)\varphi(\omega_\infty)\frac{dQ}{dP}(\omega_\infty) & \text{if } x > x^* \end{cases}$$

References

[AB05] C. D. Aliprantis and K. C. Border. *Infinite dimensional analysis.* Springer, 2005 (third edition).

[BM89] J. M. Ball and F. Murat. *Remarks on Chacon's Biting Lemma.* Proc. of the A.M.S. 107/3 655-663, 1989.

[BV88] G. Bouchitté and M. Valadier. *Integral Representation of Convex Functionals on a Space of Measures.* J. Func. An. 80, 398-420, 1988.

[Bre83] H. Brezis. *Analyse fonctionnelle.* Masson, 1983.

[DS67] N. Dunford and J. Schwartz. *Linear Operators: Part I.* Wiley, 4th Edition, 1967.

[KS99] D. Kramkov and W. Schachermayer. *The asymptotic elasticity of utility function and optimal investment in incomplete markets.* Ann. Appl. Prob. 9/3, 904-950, 1999.

[KS03] D. Kramkov and W. Schachermayer. *Necessary and sufficient conditions in the problem of optimal investments in incomplete markets* Ann. Appl. Prob. 13/4, 1504-1516, 2003.

[Roc74] R. T. Rockafellar. *Conjugate Duality and Optimization.* Conference Board of Math. Sciences Series, SIAM Publications, No. 16, 1974.