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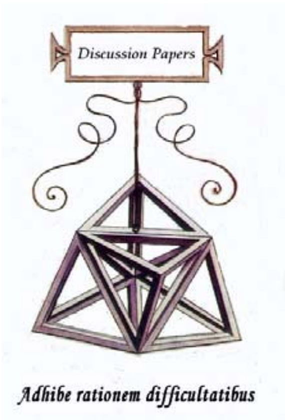

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Progressive Income Taxation and Economic Cycles: a Multiplier-Accelerator Model

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Progressive Income Taxation and Economic Cycles: a Multiplier-Accelerator Model

Abstract

Progressive Income Taxation and Economic Cycles: a Multiplier-Accelerator Model

The paper investigates the role of progressive income taxation in the frame of the basic multiplier-accelerator model in continuous time. It is shown that, while the proportional taxation is, as common wisdom believes, always stabilizing, in the case of non-linear progressive taxation, an increase of the degree of progression is always destabilizing. Moreover it is shown that in the presence of tax collection lags, the progressive taxation can have a twofold role, either stabilizing or destabilizing. Our results have a theoretical as well as a practical aspect. Firstly, they shed new light on the dynamic role of various economic behaviours such as the thriftiness of the agents, public expenditure, the maximal income tax rate and finally the degree of progression of income taxation. Secondly they show a further, so far neglected, factor causing economic fluctuations, through the complex effect of a progressive as well as lagged income tax. Thirdly, they provide the policy maker with clear stabilization rules and the warning of possible *complicated dynamic configurations (such as the "corridor" stability case).*

Classificazione JEL: E62, E3
Keywords: Progressive Taxation, Cycles

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1. Introduction

As known, since many decades tax structures are progressive in the most part of the countries. So far economic theory has been concerned with analysing the effects of progression of the tax structure on the microeconomic behaviour of the agents or on the macroeconomic aggregates, such as the tax revenue (e.g. the Laffer curve theory), rather than its effects on economic stability and cycles. As regards the traditional, and broadly investigated, proportional income taxation, the conventional wisdom attributes an "always stabilizing" role to income taxation. Though some authors have pointed out that in some cases proportional taxation can be destabilizing (Smyth (1974), (1978) and, among the others, the subsequent papers of Boyes (1975), Ip (1977), Delorme/Hayakawa (1977), Ozmucur (1979), Peel, (1979)), the above common-sense view has continued to prevail. For instance, with regard to an increase in the marginal tax rate, Smyth (1978, p.372) argues that "..it is still widely assumed that such a change is necessarily stabilizing."[1]

[1]Smyth himself however acknowledges that such a result may be model-dependent, particularly in relation with the adopted lag structure (all such models are indeed in discrete time) "...different lag structures may yield different results." (Smyth, 1978, p.372). A dramatic example that the stability properties of taxation are model-dependent is the dynamic IS-LM model of DeLorme/Hayakawa (1977), who show that when the accelerator part of the model is specified according to Hicks' (1950) - the investment is function of the change in income - taxation can be destabilizing,

Indeed, not only in the textbook static framework, but also in a fundamental model intrinsically unstable such as the multiplier-accelerator model in continuous time proportional taxation is necessarily stabilizing (as briefly sketched also in this paper), thus justifying the pervasiveness of the "stabilizing" belief. Even more surprisingly, not much attention has been devoted to the dynamic aspects of a non linear progressive tax structure[2] and to their consequences for stabilisation policies, with the exception of Peacock (1960) and Pohjola (1985). Peacock studies a growth model in discrete time with tax collection lags, showing that when the actual growth path is displaced from that of capacity output, an overshooting effect (rather than a truly destabilizing one) emerges as a consequence of the progressive tax during the adjustment of the growth path to the capacity output path. Pohjola combines the previous models of Peacock (1960) and Smyth (1974) in discrete-time framework with a logistic tax function. His final model obeys the well known logistic first-order difference equation, thus yielding complex dynamics, but as regards the role of progressive tax it simply confirms the results of the underlying previous models.

This paper investigates the dynamic effect of the progression of income taxation in a simple macroeconomic model, e.g. the classical multiplier-accelerator model in continuous time.[3] Subsequently, for purposes of realism, the effects of tax progression are investigated in presence of the realistic complication of a lag in tax collection.

Our choice allows for a better picking-up of the true role of the progressive income tax for three motives: 1) since the above cited models are in discrete time, we agree with the argument that the results are very sensitive to this type of modelling; 2) following Turnovsky (1977) we argue that, although both discrete and continuous models must be viewed as alternative approximations, the discrete time form suffers of three drawbacks: a) assuming perfectly synchronized decisions of different agents while such decisions are made at discrete time intervals, they are not coordinated and overlap in time in some stochastic manner (for this reason Invernizzi/Medio (1991) propose a continuous time modelling of the lag according to a continuous distributed lag function, which is used in the third section of this paper); b) an obvious time interval as the true or natural unit does not exist in most economic contexts; c) its underlying assumption of a fixed period length may lead to misleading conclusions;[4] 3) by using a macro-model which, being intrinsically unstable

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whereas when Samuelson's (1939) is followed - the investment is function of the change in consumption - taxation is only stabilizing.

[2]In fact usually a progressive tax structure has been represented by using a linear tax function with a negative intercept "proxing" the degree of progression.

[3]Although the mainstream literature, using mostly intertemporally optimising models, could classify this model as obsolete, we agree with Turnovsky (1996, p.3) that i) economic theory is something like clothing industry in that it is subject to fads and fashions, and ii) by keeping a historical perspective, one gains a better understanding of the current models and methods of analysis.

[4]In every case to avoid the risk to unwittingly yield misleading conclusions by using discrete time form, "it is desirable to allow the period length to vary and ultimately let it shrink to its limit."(Turnovsky, 1977, p.44).

and stabilised under a proportional tax function, is the natural framework in which to investigate the effects of progressive taxation.

We are able to show that while the proportional taxation is, as common wisdom believes, always stabilizing, in the case of non-linear progressive taxation, an increase in the degree of progression is always destabilizing. This main result may become more complicated when the collection lag is introduced. In the latter case the destabilizing effect of progression may surprisingly even be reversed depending on the relative magnitude of the collection lag versus the speed of adjustment of the output market.

The main focus is on how the taxation parameters affect stability and dynamics. Our analysis reveals somewhat surprising facts which so far have not received an adequate attention in the existing literature. Our results have a theoretical as well as a practical aspect. Firstly, they shed new light on the dynamic role of various economic behaviours as the thriftiness of the agents, public expenditure, the maximal income tax rate or the degree of progression of income taxation. Secondly they show a further, so far neglected, catalyst of the economic fluctuations, through the complex effect of a progressive as well as lagged income tax. Thirdly, they provide the policy maker with clear and unexpected stabilization rules and the warning of possible complicated dynamic configurations (such as the ”corridor” stability case).

The paper is organised as follows. Section II introduces our basic economic framework. Section III introduces a functional specification for the progressive tax rate function. Section IV analyses the dynamic effects of this progressive tax rate. Section V considers the more general model with collection lag. Numerical illustrations are dealt with in section VI. Conclusive remarks follow.

II. A basic model with progressive taxation

The basic model relies on the early work of Phillips (1954,1957) on the dynamic multiplier-accelerator model of Samuelson and Hicks (Turnovsky, 1977, ch.13). The dynamics of output is based on a product market disequilibrium relationship

(1) $$\dot{Y} = \sigma(D - Y), \qquad \sigma > 0$$

where Y and D denote the aggregate supply and aggregate demand of the economy and σ the speed of adjustment. Equation (1) shows that producers, when demand exceeds supply, increase supply at a rate proportional to the rate of excess demand (and conversely when supply exceeds demand). The aggregate demand is defined by

(2) $$\mathrm{D} = \mathrm{C} + \mathrm{I} + \mathrm{G}$$

where C, I and G respectively denote consumption, investment and the government expenditure. Consumption is assumed to obey the keynesian behavioural hypothesis

(3) $$C = \alpha_1 Y^D = \alpha_1 (Y - T), \qquad 0 < \alpha_1 < 1$$

where Y^D is disposable income and T is the total tax levied on income. The government expenditure G is taken as exogenous. The investment obeys the flexible accelerator model, in which the desired capital stock K^* is assumed proportional to income

(4) $$K^* = vY$$

and the investment $I = \dot{K}$ is defined as the lagged adjustment of capital stock to its desired level

(5) $$\dot{K} = \gamma(K^* - K) = \gamma(vY - K), \qquad \gamma > 0$$

Most standard macro-models assumed that either taxes are independent of current economic activity and exogenously given, or, when income tax is endogenous, are levied according to a proportional tax structure. We consider here the role played by an endogenous progressive income tax. Let total income taxation be defined as

(6) $$T = t(Y)Y$$

where the average tax rate $t = t(Y)$ is assumed to be a non decreasing (possibly strictly increasing) and saturating function of *Y*, with $t(0) = t_0$ (without loss of generality we take $t_0 = 0$), $t(\infty) < 1$.

III. A functional specification for the tax-rate function

As known, there are three definitions of progression of a tax structure. The latter is said to be progressive, proportional or regressive, if (Musgrave/Musgrave, 1984, p. 360): a) the elasticity of the tax (termed by Musgrave/Musgrave "liability progression"), defined as the ratio between the rate of change of the fiscal revenue and the rate of change of income, exceeds, equals or is less than unity; b) the average tax rate is increasing, constant or decreasing when income increases; c) the marginal tax rate exceeds, equals or is less than the average rate. A simple functional form satisfying all the above definitions is

(7) $$t = t(Y) = \frac{aY}{b+Y} \qquad 0 < a < 1\,,\; b > 0$$

This functional form is rather flexible. Moreover its two parameters *a*,*b* have simple "policy" interpretations. The first one, $a = t(\infty)$ is the maximal average tax rate. Notice that for *a*=0 then $t(Y) = 0$, whatever the value of *b*; this implies no taxation. Furthermore the first definition of a progressive taxation (e.g. elasticity *EL*>1) holds, given that

$$EL = \frac{\partial T}{\partial Y}\frac{Y}{T} = \frac{T'(Y)}{t(Y)} = \frac{2b+Y}{b+Y} > 1$$

Moreover, since: $t'(Y) = ab/(b+Y)^2 > 0$, also the second definition of tax progression, e.g. that the average tax rate is an increasing function of income, is fulfilled. Finally, as the total tax revenue is $T = aY^2/(b+Y)$, the marginal tax rate is

(8)
$$\frac{\partial T}{\partial Y} = \frac{2abY + aY^2}{(b+Y)^2} = t(Y)\frac{2b+Y}{b+Y}$$

showing that the marginal tax rate is always greater than the average tax rate, so that also the third definition of a progressive taxation is satisfied.

A problem which is rather controversial in the literature is the measurement of the degree of progression. As regards the latter, Musgrave/Thin (1948) and subsequently Musgrave/Musgrave (1984) propose the three definitions of progressive taxation above as three possible measures, but they also argue that "there is no single "correct" way by which to measure the degree of progression" (Musgrave/Musgrave, 1984, p.360). Atkinson/Stiglitz (1980, p.37), as regards the measures proposed by Musgrave/Thin, claim that "unfortunately these measures give different answers even with the simple linear tax". In this paper we adopt definition a) above, which embodies the other two, and in addition requires that the marginal tax rate is higher than average tax rate, as a standard criterion to evaluate the degree of progression: the larger this measure, the more progressive the tax.

Inspection of the elasticity *E* with respect to the parameters *a*,*b* and to income

(9)
$$EL = 1 + \frac{b}{b+Y}$$

shows that: a) progression is a steadily declining function of *Y* (a fundamental feature of all tax systems, as required by Musgrave/Musgrave, 1984, p.360: "progression tends to decline as we move up the income scale"; b) *EL* is independent of *a*; c) for each level of income, progression may be increased by raising *b*. Thus, under the present definition the *b* parameter is an unambiguous measure of the degree of tax progression (whereas *a* may be taken as a pure measure of the level of taxation), a fact that will be used in the dynamic analysis of the forthcoming sections. Notice finally that $t'(0) = a/b$. Thus, for constant *a*, the *b* parameter is correctly interpreted as a (negative) tuner of the level of fiscal pressure at "low" income levels.[5]

This implies that, by keeping *a* fixed and varying *b,* one has the possibility of setting different taxation policies characterised by the same maximal rate but by different levels of fiscal pressure on low incomes, as it is clear from Figure 1, which represents the average tax rate as a function of income.

[5] Notice moreover that further flexibility can be easily added by adding a nonzero intercept, e.g. considering $t = -c + \frac{aY}{b+Y},\quad c > 0$, but the analysis of this case is beyond the scope of the paper.

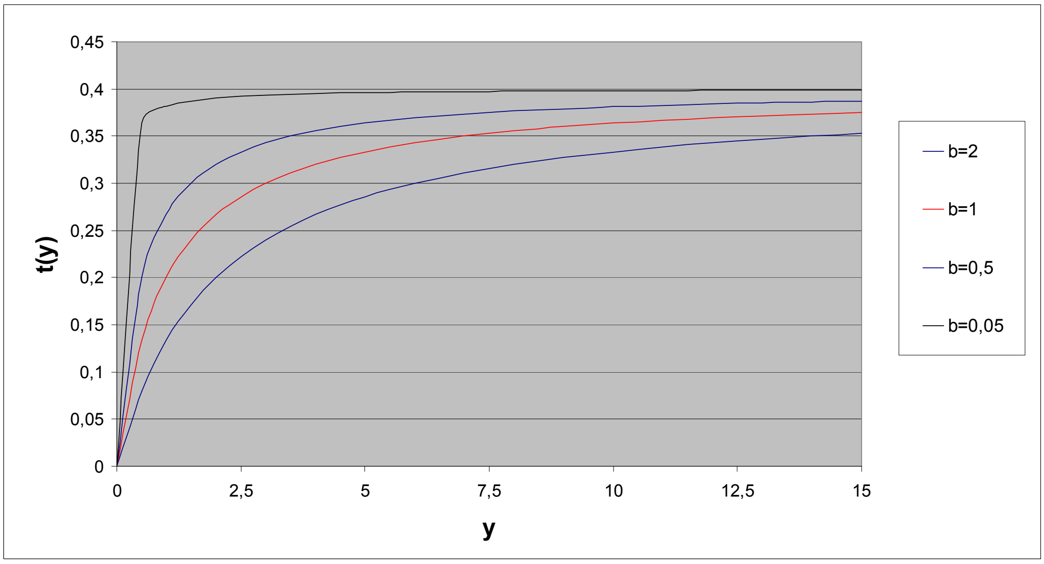

FIG. 1 (about here). Shape of the Saturating Tax Rate Function $t(Y) = aY/(b+Y)$ *for some values of the progression parameter b (a=0.4)*

In order to make some comparisons in the subsequent sections, we recall that a usual functional form for a progressive tax structure has been the following linear tax function (e.g. Turnovsky, 1977)

(10) $$T = t_p + t_1 Y$$

with $t_p<0$.

IV. Policy taxation parameters: their dynamic effects

We now investigate the dynamics of the multiplier-accelerator model when the tax structure is progressive according to the non-linear tax function (7). The consumption becomes $C = \alpha_1(1-t(Y))Y$, so that the aggregate demand (2) is: $D = \alpha_1(1-t(Y))Y + \gamma(vY - \mathrm{K}) + G$, leading to the following dynamic equation for Y

(11) $$\dot{Y} = \sigma[((\alpha_1 + \gamma v - 1) - \alpha_1 t(Y))Y - \gamma \mathrm{K} + G]$$

By coupling (11) and (5) we obtain the final model, given by the 2-dimensional ODE system

(12) $$\dot{Y} = \sigma[((\alpha_1 + \gamma v - 1) - \alpha_1 t(Y))Y - \gamma \mathrm{K} + G]$$
$$\dot{K} = \gamma(vY - K)$$

Remark. For $t(Y) = t_0$ (constant) the model reduces to the textbook multiplier-accelerator model, whose properties are well known.

IV.1 Equilibria and Static Relationships

The equilibria of the system are the solutions of the equations $\dot{Y}=0, \dot{K}=0$. From the $\dot{K}=0$ equation one has $K=vY$, leading thus to

$$[(\alpha_1+\gamma v-1)-\alpha_1 t(Y)]Y-\gamma vY+G=0$$

Rearranging and dividing both members by $Y>0$ we obtain

$$(\alpha_1-1)+\frac{G}{Y}=\alpha_1 t(Y) \tag{13}$$

A straightforward graphical analysis of the previous equation shows that a unique and strictly positive equilibrium E_1 always exists, as in the basic multiplier-accelerator model.

Proposition 1: The system (12) always admits one (and only one) positive (i.e. admissible) equilibrium $E_1=(Y_1, K_1)$ for whatever parameter constellation.

The following proposition shows how the equilibrium value of income (and therefore of capital, as $K=vY$) is affected by the basic economic parameters:

Proposition 2: The equilibrium value of Y is only affected by α_1, G and by the taxation parameters: neither the acceleration parameter v, nor the adjustment parameter γ affect Y_1. Moreover it holds

$$\frac{\partial Y_1}{\partial \alpha_1}>0;\quad \frac{\partial Y_1}{\partial a}<0;\quad \frac{\partial Y_1}{\partial b}>0;\quad \frac{\partial Y_1}{\partial G}>0 \tag{14}$$

Proof. The first part of proposition 2 is obvious by inspection of (13).
Moreover, the proof of (14) straightforwardly follows from an inspection of the curves $f_1(Y)=(\alpha_1-1)+\frac{G}{Y}$, $f_2(Y)=\alpha_1 t(Y)$, or by Dini's theorem. In the latter case define (we write for simplicity Y instead of Y_1)

$$F(Y,\alpha_1,a,b)=(\alpha_1-1)+\frac{G}{Y}-\alpha_1 t(Y)=0 \tag{15}$$

Then

$$\frac{\partial Y}{\partial a}=\frac{-\alpha_1\dfrac{\partial t(Y)}{\partial a}}{\Lambda};\quad \frac{\partial Y}{\partial b}=\frac{-\alpha_1\dfrac{\partial t(Y)}{\partial b}}{\Lambda};\quad \frac{\partial Y}{\partial G}=\frac{Y^{-1}}{\Lambda}, \tag{16}$$

where

$$\Lambda=\frac{G}{Y^2}+\alpha_1 t'(Y)$$

As

(17) $t'(Y) = \dfrac{ab}{(b+Y)^2} > 0$; $\dfrac{\partial t(Y)}{\partial a} = \dfrac{Y}{b+Y} > 0$; $\dfrac{\partial t(Y)}{\partial b} = \dfrac{-aY}{(b+Y)^2} < 0$

the proof follows.

IV.2 Local Stability Analysis and Bifurcations

Define now the quantity

(18) $$R(Y) = t(Y) + t'(Y)Y$$

where R is positive and monotonically increasing, with $R(0) = t(0)$. Moreover let $A = a_1 + \gamma v - 1$. The Jacobian $J(E_1) = J_1$ of system (12) evaluated at E_1 leads to the following expression for the Trace and the Determinant

(19) $$\begin{aligned} \mathrm{T}r(J_1) &= \sigma(\mathrm{A} - \alpha_1 R(Y)) - \gamma, \\ Det(J_1) &= (-1)\gamma\sigma((\mathrm{A} - \alpha_1 R(Y)) - \gamma v) \end{aligned}$$

Notice first that, since $0 < \alpha_1 < 1$, it holds

(20) $$(A - \alpha_1 R(Y)) - \gamma v = \alpha_1 - 1 - \alpha_1 R(Y) < 0$$

e.g. the determinant of the Jacobian at E_1 is always positive, so that any annihilation of the Trace implies the onset of local instability with the possible appearance of a Hopf bifurcation (Guckenheimer/Holmes, 1983). Let us then consider the condition $Tr(J) < 0$ (we suppress the subfix 1 for short).

As $R(Y) > 0$, there are two possibilities: a) if A<0 (i.e. $(\alpha_1 + \gamma v - 1) < 0$) then E_1 is always locally asymptotically stable (LAS). This condition is inherited from the basic multiplicator-accelerator model; b) if A>0 (i.e. $(\alpha_1 + \gamma v - 1) > 0$) then E_1 is not necessarily LAS, and we need to consider more in detail the stability condition $Tr(J) < 0$. We obtain

(21) $$\alpha_1 R(Y) > A - \frac{\gamma}{\sigma}$$

Therefore: a) if $A - \dfrac{\gamma}{\sigma} < 0$, i.e. $(\alpha_1 - 1) + \gamma(v - \dfrac{1}{\sigma}) < 0$ then stability continues to prevails. Conversely if $\alpha_1 - 1 + \gamma v - \dfrac{\gamma}{\sigma} > 0$, i.e.

(22) $$\alpha_1 - 1 + \gamma(v - \frac{1}{\sigma}) > 0$$

stability will be lost, through a Hopf bifurcation, in correspondence of the locus g=0, where

(23) $$g = R(Y) - \left[(\alpha_1 - 1) + \gamma(v - \frac{1}{\sigma}) \right]$$

IV.3 How Taxation Parameters affect Stability ?

The main target of this study however is not to investigate in depth the dynamic properties of (12). Rather, our aim is to investigate how the taxation parameters appearing in the formulation (7) affect stability. Therefore we now investigate the behaviour of the Trace $Tr\,(J)$ as a function of the taxation parameters *a*,*b*. We have

(24) $$Tr(J) = Tr(Y, a, b) = \sigma\left[A - \alpha_1 R(Y(a,b))\right] - \gamma$$

Notice that for a=0 (no taxation) then *t(Y)=0, R(Y)=0*, yielding $Tr(J) = \sigma A - \gamma$. Thus, if $\sigma A > \gamma$, then the absence of taxation implies instability, which a known feature of the basic multiplicator-accelerator model. The following fundamental inequalities hold (proof in the Appendix)

(25) $$\frac{\partial Tr(J)}{\partial a} < 0\,; \quad \frac{\partial Tr(J)}{\partial b} > 0$$

leading to the following

Proposition 3: The two parameters tuning the shape of the taxation function play opposite roles on stability: *a* always has a stabilizing effect, whereas the opposite holds for *b*. The latter means that the degree of progression is crucial for stability and in particular the more progressive the tax, the more likely is the instability.

The following remark holds: under the proportional tax function ($t(Y) = t_0$) increases in the average (and marginal) tax rate t_0 are always stabilizing. The proof is trivial in that in this case the Trace reduces to: $Tr(J) = Tr(Y, t_0) = \sigma(A - \alpha_1 t_0) - \gamma$, which negatively depends on t_0.

Moreover in the case of the linear tax function (10) with negative intercept t_p as "proxy" of the progression, it is easy to see that t_p does not affect the stability (the Trace is similar to the previous above: $Tr(J) = Tr(Y, t_1) = \sigma(A - \alpha_1 t_1) - \gamma$ [6] and the marginal tax rate t_1 is stabilizing.

As a consequence, we can argue that it is actually the degree of progression of the tax system rather than the average tax rate burdening the individuals, which can destabilise the economy. The message as regards stabilisation policy is clear: the tax progression, rather than the average tax rate, matters for stability.

[6] For economy of space we omit the easy proof.

V. Collection lags

A well established fact concerning income taxation is the existence, in most countries, of lags in tax collection. There is a wide empirical evidence in all countries on the existence of collection lags in major categories of government revenue. Furthermore the evidence indicates a wide variation in collection lags among the different categories of revenues (Choudry (1991)). The effects played by collection lags have been also recently extensively analysed, but mainly focusing on aspects such as 1) the estimation of revenue-eroding effects of inflation in presence of collection lags (Tanzi (1978); Choudry (1991)), 2) the analysis of the optimal rate of monetary expansion when there are collection lags in the tax system (Choudry (1992)), 3) the optimal design of taxation instruments by using inflationary finance and commodity taxation when there are collection lags (Dixit (1991), Mourmouras /Tijerina (1994)).

But so far, despite the broad aforementioned literature about the effects of collection lags, there is still a paucity of studies of the dynamic effects of such lags. Since collection lags can be considered as partially controllable by policy-makers, the usefulness of this investigation for a better understanding of stabilisation policies seems evident.

The investigation of the dynamic effects of collection lags requires to make some explicit assumption on the dynamic structure of the delay. We assume here that the time delay is exponentially distributed, as usual in economics. This assumption well mirrors the realistic case of (infinitely) many agents with different lags as regards the income tax collection (Invernizzi/Medio (1991)).[7]

The hypothesis of an exponentially distributed collection lag leads to replace the relation $T(Y)=t(Y)Y$ adopted in the previous section with the following one:

(26) $$T = t\left[\int_{-\infty}^{t} Y(u)G_1(t-u)du\right]\int_{-\infty}^{t} Y(u)G_1(t-u)du$$

The delaying kernel G_1 is assumed to follow an exponential distribution:

(27) $$G_1(\phi) = s_1 \exp(-s_1\phi), \quad \phi > 0, \quad s_1 > 0$$

The previous assumption thus implies, in particular, that the process of tax collection has a mean delay of $1/s_1$.

By explicit introducing the new variable

(28) $$S = \int_{-\infty}^{t} Y(u)G_1(t-u)du$$

[7]Indeed, given the wide empirical evidence in all countries on the existence of wide variation in collection lags concerning various categories of government tax revenue, it would be unrealistic to postulate the existence of a unique "period" of delay that would hold for the infinitely large number of agents. Therefore the case of an exponential distribution of the collection lag appears to be more realistic than the case of fixed delay.

and by a straightforward application of the linear trick (Mac Donald (1978), Fanti/Manfredi (1998)) we obtain

(29) $$T = t(S)S$$

and

(30) $$\dot{S} = \sigma_1 (Y - S)$$

The relations (29) and (30) therefore lead to the following 3-dimensional extension of the system (12) of the previous section

(31) $$\begin{aligned} \dot{\mathrm{Y}} &= \sigma(\mathrm{AY} - \alpha_1 \mathrm{t(S)S} - \gamma\mathrm{K} + \mathrm{G}) \\ \dot{\mathrm{K}} &= \gamma(\mathrm{vY} - \mathrm{K}) \\ \dot{S} &= s_1 (\mathrm{Y - S}) \end{aligned}$$

V.1 Dynamic Effects of Exponentially Distributed Collection Lags

It is an easy matter to prove that the equilibria of system (31) are the same of the basic 2-dimensional system (12), e.g. a unique positive (therefore economically meaningful) equilibrium E_1 always exists. Stability analysis leads to the 3-rd order characteristic polynomial $P_{E_1}(\lambda) = \lambda^3 + \alpha_1\lambda^2 + \alpha_2\lambda + \alpha_3$ with coefficients:

$$\begin{aligned} a_1 &= s_1 + \gamma - A\sigma \\ a_2 &= -\sigma(\mathrm{A} - v\gamma)\gamma + R\sigma s_1\alpha_1 + s_1(\gamma - \mathrm{A}\sigma) \\ a_3 &= \left(AR\sigma^2 s_1\alpha_1 + R\sigma s_1\alpha_1(\gamma - \mathrm{A}\sigma) + s_1(-\mathrm{A}\sigma\gamma + v\sigma\gamma^2)\right) \end{aligned}$$

The Routh-Hurwicz test for the stability of the equilibrium (Gandolfo, 1996) requires that $a_1>0$, and

(32) $$\Delta_2 = \alpha_1\alpha_2 - \alpha_3 = \mathrm{M}_1 + \mathrm{s}_1\alpha_1\sigma(\mathrm{s}_1 - \mathrm{A}\sigma)\mathrm{R} > 0$$

to hold, where

$$M_1 = v\sigma\gamma^3 - \mathrm{A}\sigma\gamma^2 - 2\mathrm{A}\sigma\gamma s_1 + \gamma s_1^{\,2} + \gamma^2 s_1 - \mathrm{A}\sigma s_1^{\,2} + \mathrm{A}^2\sigma^2\gamma - \mathrm{A}v\sigma^2\gamma^2 + \mathrm{A}^2\sigma^2 s_1$$

V.2 Taxation Parameters: their Effects on Stability

It may be shown that there are nice dynamic effects triggered by the introduction of the collection lag, in particular the onset of stable oscillations, as widely discussed in the next section. Indeed, it is easy to prove that if, still assuming $a_1>0$, it occurs $\Delta_2 = 0$, then instability arises through a Hopf bifurcation (Guckenheimer/Holmes, 1983). However, since our main purpose here, as already stated, is to investigate how the taxation parameters *a,b* (and *G*) affect stability, we focus on this topic first, and postpone the discussion of dynamics of system (31) to the simulative section.

The following proposition holds:

Proposition 4: It holds

$$\begin{aligned}\frac{\partial \Delta_2}{\partial a} &= \sigma\alpha_1 s_1 (s_1 - A\sigma)\frac{dR}{da}\\ \frac{\partial \Delta_2}{\partial b} &= \sigma\alpha_1 s_1 (s_1 - A\sigma)\frac{dR}{\partial b}\\ \frac{\partial \Delta_2}{\partial G} &= \sigma\alpha_1 s_1 (s_1 - A\sigma)\frac{dR}{\partial G}\end{aligned} \tag{34}$$

where $\frac{dR}{da} = \frac{dR(a,b,Y(a,b))}{da}$, $\frac{dR}{db} = \frac{dR(a,b,Y(a,b))}{db}$ and $\frac{dR}{dG} = \frac{dR(a,b,Y(a,b))}{dG}$ denote the total derivatives of the *R* quantity with respect to the taxation parameters *a,b* and the public expenditure *G*.

Proof. It straightforwardly follows by combining (32) with the results obtained on the 2-dimensional model, due to the fact that the equilibria of (12) are preserved when we pass to the extended model (31).

The following remark holds:

Remark: there is threshold effect in the sense that as the quantity $s_1 - A\sigma$ changes its sign, the action of the basic fiscal parameter on the stability of the system is reversed (notice that a_1 is not affected by the taxation parameters). Since s_1 represents the reciprocal of the average delay in the process of tax collection, the previous result has a clear interpretation. It says that when the average collection lag is "small" (in other words when is s_1 "large", e.g. $s_1 > A\sigma$), then *a* has a stabilising action on the equilibrium of the economy, and *b* a destabilising one, exactly as observed in the basic 2-dimensional model. This effect happens to be reversed for very large (values of the average) collection lag. The previous result appears to be of some importance for the management of fiscal policies.

VI. Some numerical investigation

The Hopf bifurcation theorem used in the previous sections is a local result which only permits to detect the existence of a (local) bifurcation of an equilibrium point in periodic orbits. Conversely, it nothing says about uniqueness and stability of the limit cycles emerging from the bifurcation. In particular, it nothing says on the question whether the bifurcation is supercritical or subcritical, i.e. whether the limit cycles which bifurcate from the stationary state are (at least) locally stable or not.[8] Moreover predictions of the theorem are local also for what concerns the parameter space: "the Hopf bifurcation theorem is local in character and only makes predictions for regions of parameter and phase space of unspecified size" (Medio (1992), ch. 2). This last fact makes it interesting to resort to numerical simulations to investigate matters such as i) the stability properties of the involved periodic orbits, ii) their uniqueness, iii) the size and the shape of its (or their) basin of attractions, iv) the size and the shape of the parameter region in which limit cycles exist, v) the global rather than local behaviour of the system.

We begin with simulations of the 2-dimensional model (12). We choose as bifurcation parameter the b parameter which measures the degree of tax progression of the chosen tax function. The following parameter values were chosen (just for illustrative purposes): $a=0.5$, $\alpha_1=0.8$, $\gamma=1$, $v=2$, $G=5$, $\sigma=0.667$.

As expected from our stability analysis, increases in tax progression tend to destabilise the equilibrium point. Simulations show that when, starting from a sufficiently low value of b ensuring stability, b is increased, a Hopf bifurcation emerges at $b=b_H=12.1$ and a periodic orbit appears. Such a periodic orbit surrounds the equilibrium point having coordinates $Y_1=12.3, K_1=24.7$ (that is a long run capital/product ratio about two).

This means that, given a maximal tax rate of 50%, the economy is stable when the progressive tax structure implies an average tax rate higher than about 25%. Unfortunately the bifurcation is of subcritical type, which is the dynamic scenario corresponding to the idea of "corridor stability" proposed by Leijonhufvud (1973) and Howitt (1978). As shown in Fig. 2, indeed, an unstable limit cycle emerges, e.g. for values of the state variables sufficiently close to the equilibrium (inside the corridor stability represented by the elliptic cycle) the economy converges to the stationary state, but possible shocks on the variables which bring them outside the corridor stability may ultimately

[8]This problem, which is of critical relevance from the point of view of a substantive economic analysis, is not tackled here analytically. As it is well known its analytical investigation needs a huge amount of cumbersome algebra (see for instance Guckenheimer/Holmes (1983)), whose interpretation is generally economically meaningless.

destabilise the economy: for instance in this case, assuming initial condition for *K* and *S* close to the long run equilibrium (*K*(0)=23 and *S*(0)=12) a negative shock at time t implying a percentage decrease about 25% of *Y*(*t*) may push the economy outside the corridor.

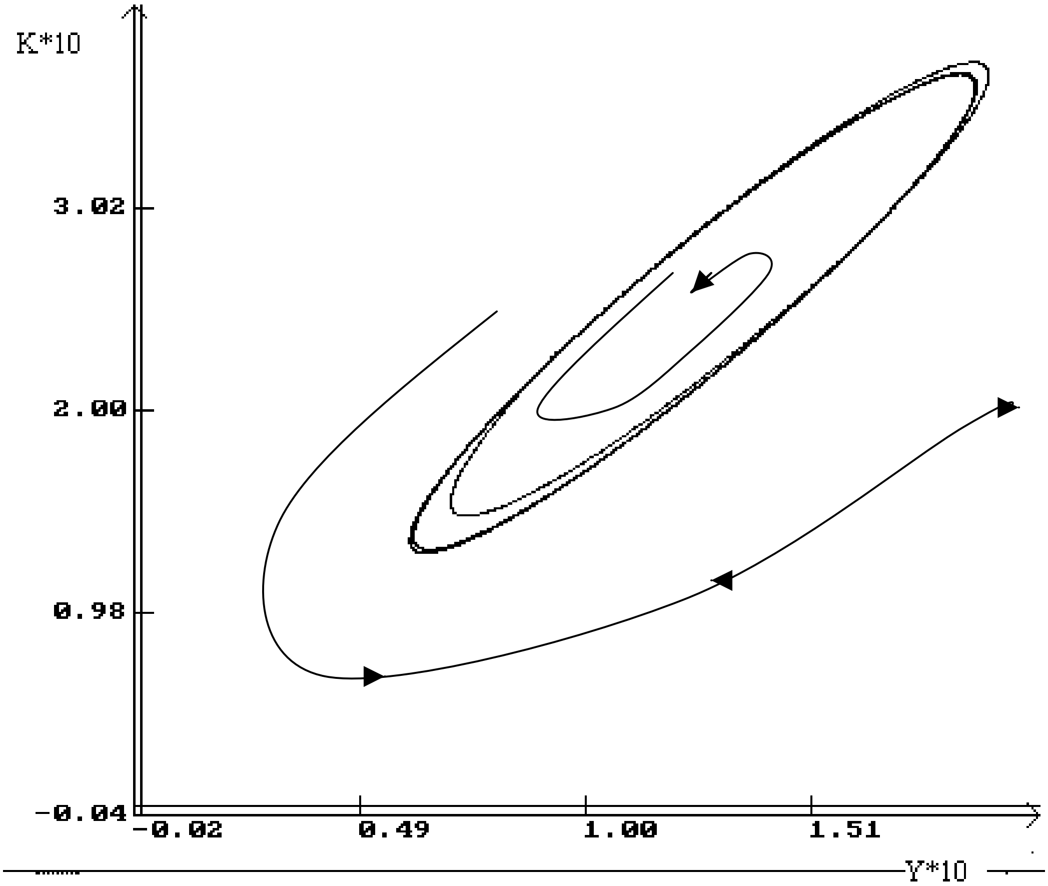

FIG. 2 (about here) An Unstable Limit Cycle Emerging in the 2-dimensional System (12)

Now we turn to the simulations of the three-dimensional model (31), using again *b* as bifurcation parameter. We start using a new parameter constellation with $\sigma=0.6441$ and $s_1 = 6$, all the remaining parameters being kept fixed at the above values. Since neither σ nor s_1 affect the steady state values of the system, this new parametric configuration has the same equilibrium (and thus the same capitalistic intensity) of the previous one.

As expected from our stability analysis, given that in this case $(s_1 - A\sigma) > 0$, a sufficiently high tax progression can generate instability. Indeed, when, starting from a sufficiently low value of *b* ensuring stability, *b* is increased, a Hopf bifurcation emerges again at $b = b_H = 12.1$, and a limit cycle surrounding the equilibrium point appears. This implies that the Hopf bifurcation has been observed in correspondence of the same bifurcation value of the average tax rate observed in the 2-dimensional model (in absence of collection lag) as above illustrated. Also this bifurcation is of subcritical type,

thus exhibiting the same qualitative pattern of Fig. 2, e.g. the emergence of an unstable limit cycle, representing the same phenomenon of ”corridor” stability observed in the 2-dimensional case, with a very similar size of the "corridor".
Now we let $\sigma=0.515$ and $s_1=0.06$, the remaining parameters being kept fixed. In this case, since $(s_1 - A\sigma) < 0$, the stability condition (31) tells that, in contrast with the previous case, a high tax progression tends to stabilise the economy. Indeed, starting from a small value of b for which the system would be unstable, we can show that when b increases a Hopf bifurcation emerges at the same previous value of $b = b_H = 12.1$, and a limit cycle appears surrounding the (same) equilibrium point. This bifurcation is of supercritical type. As Fig. 3 indeed shows, a stable limit cycle, having a nice elliptic shape on the 2-dimensional projection onto the (Y,K) plane, appears, to which all the orbits converge, both from outside and from inside the cycle. Thus in this case we can argue that when the progressive tax structure implies an average tax rate about 25% a business cycle is generated. Fig. 3 shows that shocks pushing the system out of the equilibrium will cause regular long run fluctuations.

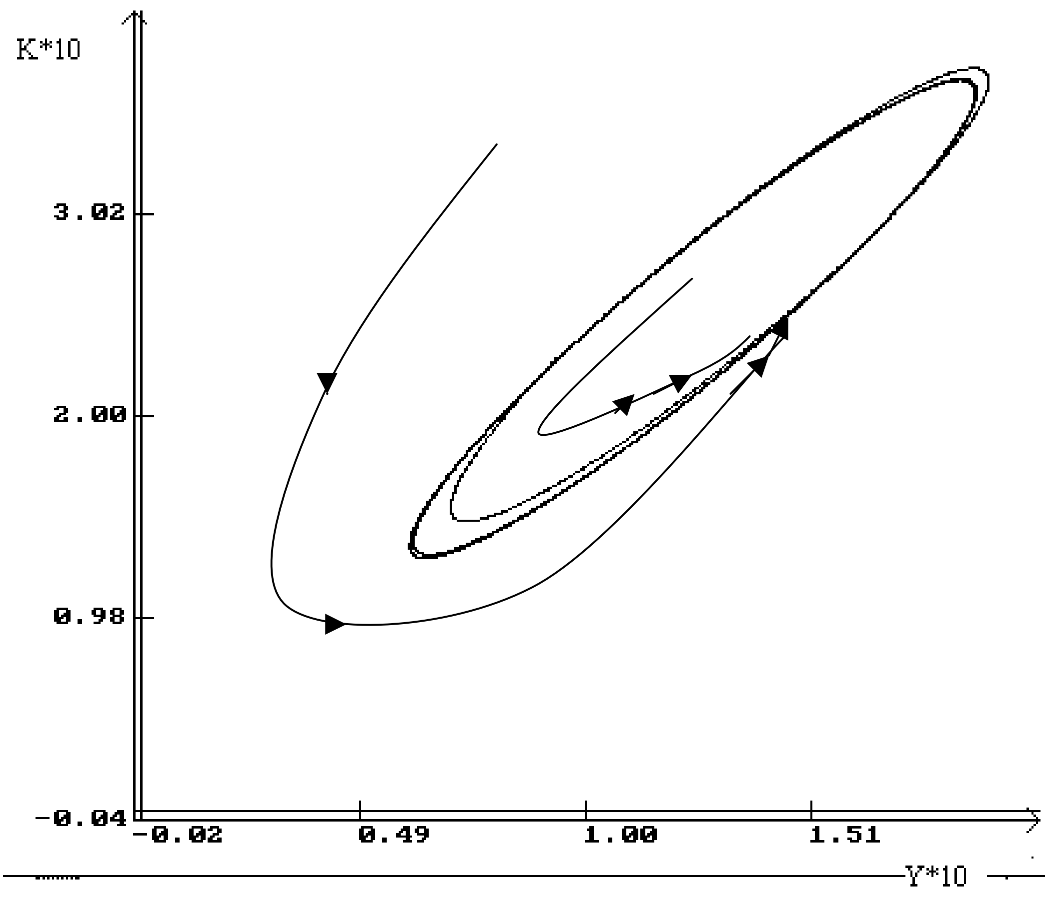

FIG. 3 (about here) A Stable Limit Cycle Appearing in the 3-dimensional Model (31)

A simple economic interpretation of the stability results shown in this section is the following: assume two countries with identical economic features and, initially, identical proportional tax system, but the second one has a tax code implying a relatively higher tax collection average lag compared to the first one (such as $(s_1 - A\sigma)$ is positive in the first country and negative in the second one). In this circumstance the introduction of a (equal) progressive tax system in both countries can be deleterious for the stability of the second country (whereas the first one shows long run stability) if the degree of progression is less than a prescribed threshold level; otherwise, if such a threshold is exceeded, the first country could experience a persistent business cycle whereas the other one shows long run stability.

Therefore we remark that, in economic situations characterised by a sufficiently low collection lag combined with a sufficiently slow adjustment in output market and/or a sufficiently low propensity to spend of the agents, a progressive income tax may trigger a stable business cycle. Conversely, in an opposite economic situation, the introduction or the raising of tax progression may stabilise the economy, but with the attention for the policy maker of the danger represented, in an economy subject to external shocks, by a possibly narrow corridor stability.

VII. Conclusive remarks

In this paper we have investigated the dynamic effect of the progression of income taxation in the frame of the basic multiplier-accelerator model in continuous time. As first main result, we have contrasted in this standard frame the conventional wisdom that income taxation is stabilizing: indeed, while increases in the average (and marginal) tax rate in a proportional tax income frame is always stabilizing, an increase in the coefficient of progression in a non-linear progressive income tax frame, is always destabilizing.[9] Moreover, when the realistic assumption of tax revenue collection lags is made, the role of taxation for stabilisation policy can be dramatically changed: in this case the progressive taxation can have a twofold role, either stabilizing or destabilizing, depending in a clear-cut way on the economic situation represented by the following parameters: the average lags both in the output market and in the tax collection and the marginal propensity to consume and invest.[10] These clear-cut results allow for a reconsideration of the stabilisation policies when there is a progressive income tax.

The other main message regards the emergence of a (deterministic) business cycle at all different from the manifestations of exogenous shocks to a

[9]As regards the maximal tax rate of the non-linear progressive tax system, it preserves the same stabilizing role of the proportional tax.

[10]Notice that the not only the coefficient of tax progression, but also the other fiscal parameters such as the maximal tax rate and public expenditure can have an ambiguous influence on stability.

fundamentally stable equilibrium proposed by the Real Business Cycle scholars; in particular the dynamics of the model shows that: 1) regular business cycles arises from a simple textbook macroeconomic model of multiplier-accelerator type depending on the tax progression of the income taxation; 2) on the contrary different economic situations, ceteris paribus as regards the taxation policy, can generate the dynamic state called (by Lejionhufvud and others) ”corridor stability”.

The rich spectrum of dynamic results found in this paper sheds new light on the macroeconomic dynamics of tax systems, and endows the policy maker with somewhat unexpected stabilisation rules of the fluctuations entirely predictable. Finally, as regards further research directions, the impact of tax progression on stability should be tested in an income growth and/or in an inflationary context, where the so called fiscal drag appears.

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Appendix: Proof of the Main Static Relationships

We prove the relations (25) which are fundamental for both the basic model (12) and the extended model with collection lag. Let us consider the basic model. From (24) we have:

$$\text{(35)}\qquad \begin{aligned} \frac{\partial Tr(J)}{\partial a} &= -\alpha_1 \frac{dR(a,b,Y(a,b))}{da} < 0 \\ \frac{\partial Tr(J)}{\partial b} &= -\alpha_1 \frac{dR(a,b,Y(a,b))}{db} > 0 \end{aligned}$$

where by $\frac{dR(a,b,Y(a,b))}{da}$ and $\frac{dR(a,b,Y(a,b))}{db}$ we have denoted the total derivatives of the *R* quantity with respect to the taxation parameters *a,b*. We have

$$\text{(36)}\qquad \begin{aligned} \frac{\partial R(a,b,Y(a,b))}{\partial a} &= \frac{\partial R}{\partial a} + \frac{\partial R}{\partial Y}\frac{\partial Y}{\partial a} \\ \frac{\partial R(a,b,Y(a,b))}{\partial b} &= \frac{\partial R}{\partial b} + \frac{\partial R}{\partial Y}\frac{\partial Y}{\partial b} \end{aligned}$$

Notice that

$$\text{(37)}\qquad R(a,b,Y) = \left(\frac{2b+Y}{b+Y}\right) t_{a,b}(Y) > 0$$

Therefore

$$\text{(38)}\qquad \frac{\partial R}{\partial a} = \frac{1}{a}\left(\frac{2b+Y}{b+Y}\right) t_{a,b}(Y) > 0$$

and

$$\text{(39)}\qquad \frac{\partial R}{\partial b} = \frac{-2b}{(b+Y)^2} t_{a,b}(Y) < 0$$

Moreover

$$\text{(40)}\qquad \frac{\partial R}{\partial Y} = \frac{2ab^2}{(b+Y)^3} > 0$$

Finally (these are the just computations used to prove proposition 2):

$$\text{(41)}\qquad \frac{\partial Y}{\partial a} = \frac{(-\alpha_1)Y^3(b+Y)}{\left[G(b+Y)^2 + \alpha_1 abY^2\right]} < 0$$

and

(42) $$\frac{\partial Y}{\partial b} = -\frac{\alpha_1 a Y^3}{\left[G(b+Y)^2 + \alpha_1 abY^2\right]} < 0$$

Therefore, going back to the total derivatives (35) we have first

$$\frac{dR(a,b,Y(a,b))}{da} = \frac{Y}{(b+Y)^2}\left[\frac{(2b+Y)\left[G(b+Y)^2 + \alpha_1 abY^2\right] - 2ab^2\alpha_1 Y^2}{\left[G(b+Y)^2 + \alpha_1 abY^2\right]}\right]$$

Let us consider only the numerator (*N*) of the bracketed expression; we quickly find

(43) $$N = Y\left(G(b+Y)^2 + a_1\alpha bY^2\right) + 2bG(b+Y)^2 > 0$$

implying

$$\frac{dR(a,b,Y(a,b))}{da} > 0$$

and therefore that $\partial Tr(J)/\partial a < 0$ as stated in the main text.

Moreover, by combining (39), (40), and (42) it is immediate to check that $\frac{dR(a,b,Y(a,b))}{db} < 0$, and therefore $\partial Tr(J)/\partial b > 0$, which completes the proof.