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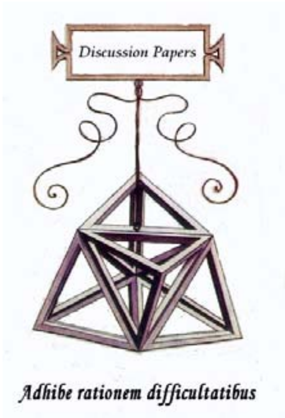

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Neo-classical labour market dynamics, chaos and the Phillips curve.

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Neo-classical labour market dynamics, chaos and the Phillips curve.

Abstract

The relationship between wage inflation and unemployment has been extensively investigated since the early work of Phillips (1958) and Lipsey (1960), and is still a matter of debate. In this paper we study the dynamics of a standard neoclassical labour market under Walrasian adjustment rules. We show that, when consumption and leisure are sufficiently low substitutes, the unique Walrasian equilibrium of the economy can be destabilised and regular or even chaotic fluctuations of wages and employment appear. This leads to an interesting resurrection of the Phillips curve as a long term phenomenon.

Classificazione JEL: J0, E3
Keywords: Phillips curve, chaos, economic cycle.

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1. Introduction

The relation between wage inflation and unemployment, extensively discussed since the early work of Phillips (1958) and Lipsey (1960), is a matter of renewed interest in the present days.[1]
In this paper we consider a model describing the dynamics of the labour market within the standard neo-classical approach. The labour market, demand and supply side, is described by a Cobb-Douglas technology and a CES individual utility function. The current labour demand and supply are assumed to adjust to their optimal counterparts, following a continuous-time adjustment, whereas the wage adjusts on the current excess demand for labour according to the usual "Walrasian" rule.
Our main finding is that robust chaotic behaviour of prices and quantities may occur even in a simple micro-founded economic

[1] See the Journal of Monetary Economics (1999, vol. 44, 2), special issue on: "The return of the Phillips curve".

model characterised by the simplest adjustment mechanism. In particular chaotic fluctuations occur when the substitution between consumption and leisure is sufficiently weak. Furthermore the following interesting phenomenon is observed: stickiness on the side of the firm, as well as flexibility in the supply of the worker, can destabilise the economy and lead it in a "trapping" chaotic region, in which: a) the Phillips curve necessarily re-emerges as a long term phenomenon; b) the Walrasian equilibrium (WE from now on)[2], viewed as the "center of mass" around which the system evolves, completely loose its "predictive ability"[3]. Thus a main finding of this paper is that the notion of Phillips curve appears to be a dynamical concept essentially related to some peculiar dynamical regime, namely the chaotic one, of a neoclassical labour market model with a Walrasian adjustment rule.[4] The latter fact implies that the points belonging to the Phillips' curve emerging in our model are just the realisation of a single trajectory of the underlying (fully deterministic) process, rather than a set of

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[2] A social welfare comparison (in a statistical sense) between the stable WE point and the stable chaotic set is beyond the scope of this paper. We notice however that the stable WE point does not seem necessarily better in terms of welfare.

[3] It is worth to note that in the chaotic regime, the Walrasian equilibrium not even preserves the role of the average value of the long term fluctuation.

[4] By definition of the Phillips curve, nor the stable Walrasian steady-state nor the cyclical regime seem to have this feature.

different equilibrium points among which the policy maker could choose - – as in the traditional view of the curve.
To sum up two results appear noteworthy: the emergence of the Phillips curve as a true long-run phenomenon on the one hand, and the fact that the curve is a useless policy tool on the other hand.

A related paper is Chichilnisky-Heal-Lin (1995), which found complex dynamics combining a discontinuous production function with a special discrete-time formulation. Such features make their framework less general compared with the one adopted here.
The plan of the paper is as follows. In the second section we present the model. Theoretical results and numerical simulations are reported in section three. Concluding remarks follow.

2. The model.

We consider a one-good economy, with a single representative firm and a single representative worker-consumer. Labour is the only input. The technology is represented by the following Cobb-Douglas production function:

$$Y = DL^a \qquad 0 < a < 1, D > 0 \qquad (1)$$

Let Π and w respectively define the total profit and the wage rate. By setting the price of the unit output equal to one, the profit function is defined as

$$\Pi = DL^{a} - wL \qquad (2)$$

A standard maximization of (2) gives the optimal demand for labour:

$$L^{D} = f_1(w) = \left(\frac{w}{aD}\right)^{\frac{1}{a-1}} \qquad (3)$$

The worker-consumer maximises the following CES utility function[5]:

$$U(C,L) = \left[C^{b} + (N-L)^{b}\right]^{1/b} \qquad b \in (-\infty,1] \qquad (4)$$

where N>0 is the maximal labour supply and C>0 the worker's consumption level. Let us assume N=1. The standard utility maximisation gives the optimal labor supply (see appendix a)):

$$L^{S} = f_2(w) = \frac{1}{1 + w^{\frac{b}{b-1}}} \qquad (5)$$

Let us now consider the Walrasian dynamics of this economy. Let $z(w) = L - S$ be the excess demand for labour, where L,S are the current demand and supply of labour. If the optimal demand and supply of labour instantaneously adjust to their current counterparts, then it would hold $L = L^{D}, \quad S = L^{S}$. If. Moreover, also the wage instantaneously adjusts, as usually assumed in the standard

[5] This assumption follows Chichilnisky et al. (1995).

neoclassical framework, then also $L=S$, and the system can not show involuntary unemployment. We rather consider the possibility that neither the current demand nor the current supply for labour (L,S) immediately adjust to their optimal counterparts (L^D,L^S), but rather follow an adaptive adjustment rule. Both these assumptions seem to be quite natural. Indeed, if we consider, for instance, the case of the firm and assume that a wage increase occurs, then the firm reacts to this change by computing a new value of the optimal demand for labour. This new value is unlikely to be effective, however, until a certain time has elapsed (for instance as a consequence of negotiations with the trade-union or other legal procedures). On the other hand consider the decision of hiring new labour as a consequence of a wage decrease. Even in this case it may take time – due to the search for the skills corresponding to the specific tasks required by the firm - to adjust the actual demand to the desired one. As a second example consider a female worker with an additional new-born baby. Assume an increase of the wage occurs. As a consequence she will re-compute her optimal labour supply. If her labour supply curve is positively sloped (b>0 in this paper), she will offer more labour. This increase in the desired labour supply may become effective only if she can find extra day care for the baby. But to find extra day care will usually take time. Therefore, quite likely, the increase in her desired labour supply will not become immediately effective. Recent econometric investigations on actual and desired supply of work suggest significant differences in the wage elasticities of British male employees reporting that their work hours have been constrained vis-a-vis those reporting satisfaction with actual hours (Brown and Sessions, 2001).

Finally, the wage is assumed to continuously adjust to the current excess demand for labour, according to the following adaptive rule:

$$\dot{w} = lz(w) = l(L - S) \qquad (6)$$

Equation (6) is the bare bone of the Walrasian dynamic theory.

The final model takes the form:[6]

$$\begin{aligned} \dot{L} &= g\left(L^D - L\right) = g\left(\left(\frac{w}{aD}\right)^{1/(a-1)} - L\right) && g > 0 \\ \dot{S} &= d(L^S - S) = d\left(\frac{1}{1 + w^{b/(b-1)}} - S\right) && d > 0 \qquad (7) \\ \dot{w} &= l\left(L - S\right) && l > 0 \end{aligned}$$

which can be written as follows:

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[6] Due to the fact that the maximum of labour supply is set to one, it should also be assumed that the upper bound for L is necessarily one. We do not impose explicitly this bound in order to avoid the further non-linearity due to the existence of a "barrier" in the dynamics (in this event also locally unstable *linear systems* could behave chaotically (Simonovits, 1982)). Our main goal is in fact to investigate the dynamical behaviour of our economy under the "minimal" number of nonlinear ingredients, i.e. just those due to the neo-classical assumptions. Nevertheless we can assume that, temporarily, during the cycle a situation of overemployment, due to, for instance, overtime work and so on, could exist. Note that this problem arise also in Goodwin's model (1967), in which the rate of employment is not bounded, and for which similar temporary overemployment situations can be postulated, as in Flaschel-Groth (1995).

$$\begin{aligned} \dot{L} &= g\left(f_1(w) - L\right) \qquad & g > 0 \\ \dot{S} &= d\left(f_2(w) - S\right) \qquad & d > 0 \\ \dot{w} &= l\left(L - S\right) \qquad & l > 0 \end{aligned} \tag{7'}$$

where the functions $f_1(w)$, $f_2(w)$ respectively denote the desired labour demand and labour supply schedule.

The analysis of the dynamics of wage adjustments, such as (6), has been typically performed by considering an exogenous cyclical fluctuation either of demand (Hansen, 1970), or both of demand and supply (Bowden, 1980). We rather consider a fully endogenous mechanism relating demand, supply and wage adjustments.

It is worth noting that equation (6) not only is the back-bone of the Walrasian system, but is nothing else than the Phillips equation in the famous interpretation of Lipsey (1960). From this latter Lipsey has attempted to relegate the Phillips effect among the limbo of transient phenomena: it would be just a consequence of frictions in an otherwise equilibrated world. Instead this paper reaches totally opposite conclusions: the neoclassical model with frictions can well embody a long run Phillips curve.

3. Main theoretical results

The equilibria of the system are defined as the solutions of the equation:

$$k w^{-\frac{1}{1-a}} = \frac{1}{1 + w^{\frac{b}{b-1}}} \qquad where \quad k = \left(\frac{1}{aD}\right)^{\frac{1}{a-1}} > 0 \tag{8}$$

for $0<a<1, b<1$.

The following proposition summarises our steady state analysis:

Proposition 1: the model (7) always admits a unique equilibrium point $E_1=(L^*,S^*,w^*)$, which is always economically meaningful. This result holds when consumption and leisure are substitutes as well as when they are complements.

The previous proposition (proved in appendix b)) deals with a labour market in which the firms' demand for labour has the traditional decreasing shape, whereas the labour supply will have the traditional increasing shape for *b>0*, and it will bend backward for *b<0*. These two cases are depicted in fig. 1a,1b.

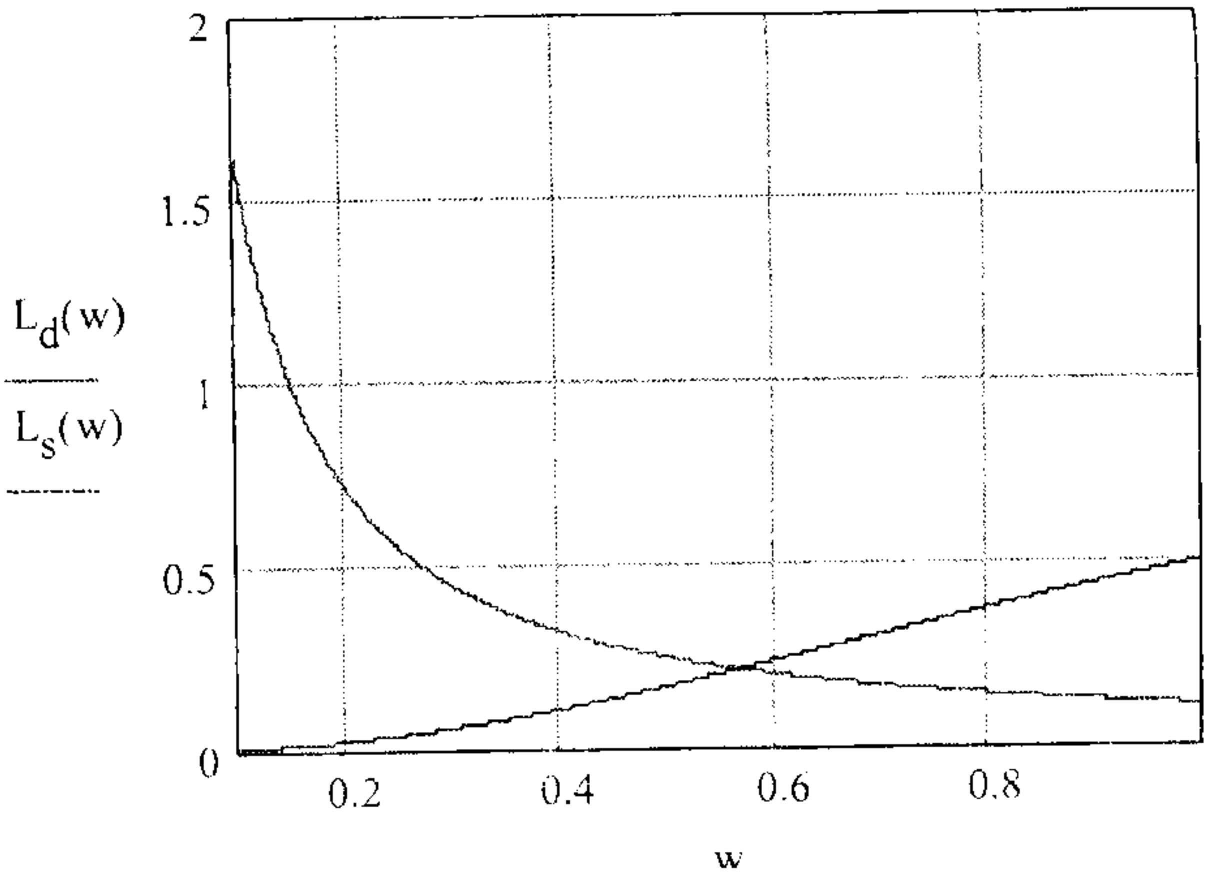

FIG 1a

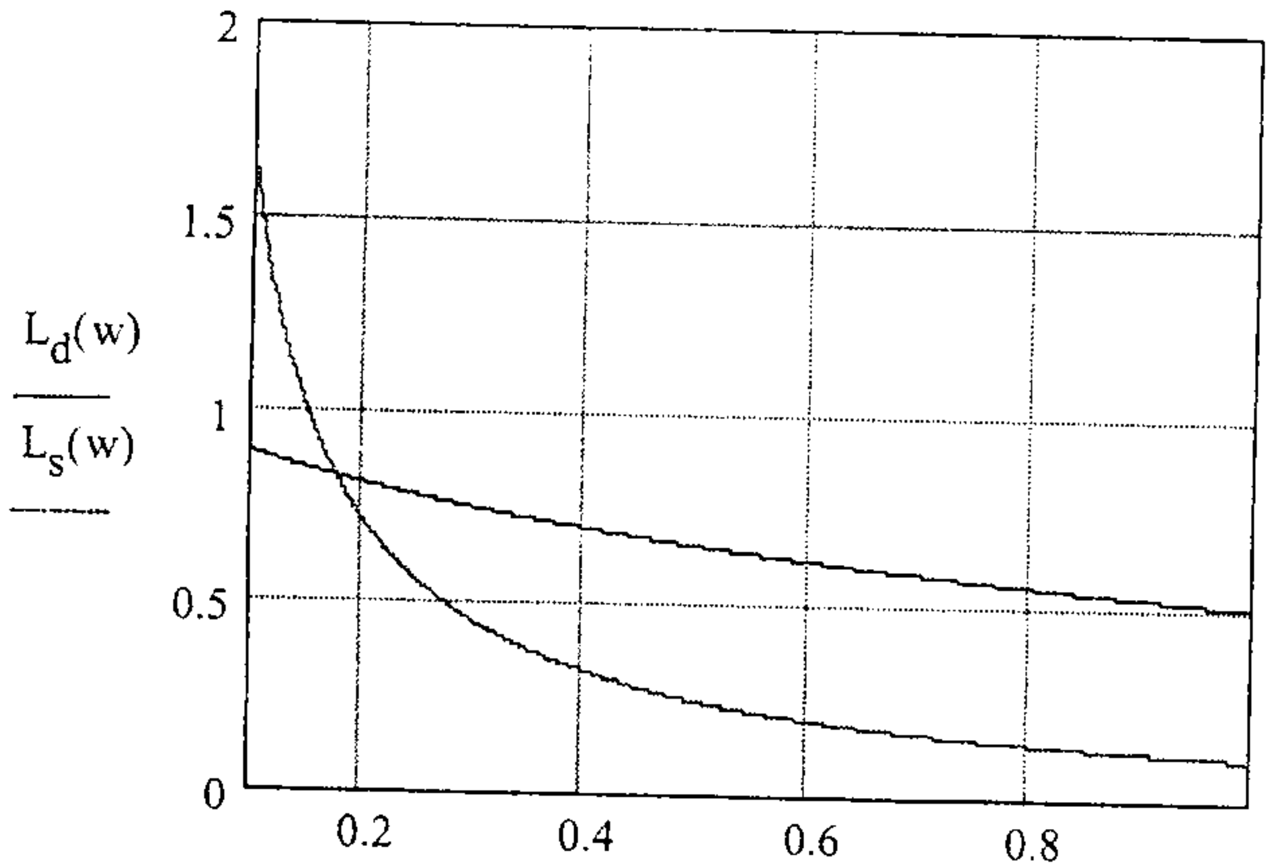

FIG 1b

Fig. 1. Demand and supply of labour in model (7): a) the labour supply is upward sloping; b) the labour supply is downward sloping

The local stability analysis of the unique Walrasian equilibrium of the system under an upward sloping labour supply (*b>0*) leads to the following results (see appendix b)):

Proposition 2: if consumption and leisure are “strong” substitutes ($0<b<1$) the equilibrium E_1 is always locally asymptotically stable (LAS). This also holds for $b=0$ (a rigid optimal labour supply).[7]

Let us now move to the case of a backward-bending labour supply ($b<0$) implying that consumption and leisure are weak substitutes. The importance of this case is argued in many recent studies. For instance Chichilnisky et al. (1995) argue that *b* can be a large negative number, so that consumption and leisure are used in approximately fixed proportions, as in the case of any recreational activity which requires consumer goods as inputs. Also Benhabib-Farmer (2000) argue that a representative agent with standard preferences over leisure and consumption will choose to consume more leisure at the same time that he consumes more consumption. In addition, as argued, for example, by Farmer-Guo (1995), the estimates of labour supply curves from first order conditions in aggregate data typically show a negatively sloped labour supply curve. Compared to the case $b>0$, the investigation of the case $b<0$ leads to more interesting results, summarised by the following:

Proposition 3: Since the demand curve at equilibrium is always steeper[8] than the supply curve at equilibrium, the Walrasian equilibrium is always locally unstable (a saddle-point); b) conversely, if the demand curve at equilibrium is steeper than the

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[7] For $b=0$ the equilibrium is explicit. The proof of its stability is trivial and is omitted.

[8]

supply curve at equilibrium[9], then: b1) the Walrasian equilibrium is always LAS when $g \geq d$ (i.e. when the speed of adjustment of the demand sector of the economy is larger of the corresponding quantity of the supply side of the economy); b2) instability may occur when $g<d$. This latter result occurs when:

$$(g+d)gd + l\left(Ag^2 - Bd^2\right) < 0 \qquad (9)$$

where the auxiliary variables A,B are defined as follows:

$$A = -f_1^{'}(w) \quad ; \quad B = f_2^{'}(w)$$

Finally, the locus:

$$(g+d)gd + l\left(Ag^2 - Bd^2\right) = 0 \qquad (10)$$

is a Hopf bifurcation locus for the Walrasian equilibrium.

Viewed in terms of the speeds of adjustment g,d, the bifurcation locus g_H (given as the only admissible solution for g of the quadratic (10)) lies always in the region $g<d$, and it is a strictly increasing and convex function of d. The straight line $g=d$ splits the plane into two regions. In the region above the line ($g>d$) the system is always LAS, and this happens on all the points of the line as well. Conversely, in the region below the line the system is stable in the region above the bifurcation locus.

9 This assumption is adopted for instance by Benhabib-Farmer (2000).

The previous findings on Hopf bifurcation, are summarised by the following (appendix b)):

Proposition 4: there exists a bifurcation value of the adjustment parameter of the current demand $g_H = g_H(a,D,b,d,l)$ such that for $0<g<g_H$, the unique equilibrium E_1 is locally unstable. Moreover, a suitable left neighbourhood of g_H exists in which a stable limit cycle (at least one) exists.

Inspection of the bifurcation curve $g_H(d)$ shows that the stability of the Walrasian equilibrium prevails for combination of values of the speeds of adjustment of the decisions of both the firm and the households which lie above a critical line, given by the bifurcation curve. The following remark holds

Remark: stickiness in the realisation of the employment decisions of the households as well as flexibility in those of the firms tend to favour the stability of the Walrasian equilibrium.

4. Simulative evidence

Simulations show that when g is further decreased, chaotic behaviour can emerge. This result is found by using the g parameter as a bifurcation (or "control") parameter and by analysing how the structure of the attractors evolve as the control parameter is varied while all the other parameters are kept fixed. The following parameter constellation has been considered: D=1, a=0.15, b= -10, d=4, l=4. The corresponding equilibrium values are $L^*=S^*$= 0.832

and w^*=0.182. All the reported simulations are performed with initial conditions very close to the equilibrium E_1 ($L°$= 0.825, $S°$= 0.835, $w°$= 0.18).

The main results are as follows[10]: E_1 is LAS for large g (a node for g>1.3, and then a stable focus). Then it undergoes a supercritical Hopf bifurcation when g falls below the critical threshold defined by (10) (given by g_H=1.068 in our parameter constellation): trajectories starting sufficiently close to E_1 initially diverge, and subsequently converge to a stable limit cycle, which seems to be unique[11]. The cycle exhibits small oscillations between 0.82 and 0.84 for both demand and supply. When g is further decreased complex behaviours arise[12]. The visual inspection reveals that the

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10 The sequence of windows of distinct dynamical behaviour reported below was observed for a very wide set of values of initial conditions and parameters.

11 The Hopf theorem does not predict the uniqueness of the involved periodic orbit.

12 The use of techniques for the global analysis of the system (8) in order to obtain an analytical or geometrical detection of "chaos" is beyond the scope of this paper (see Wiggins (1990)). The emergence of a chaotic attractor may be detected through several measures: 1) by "eye"; 2) through bifurcation diagrams; 3) through numerical and statistical tests. Among these, we remark the computation of 1) a Poincarè map by numerical-graphical techniques which in the case of a simple bi-dimensional surface of section, permits to identify different types of dynamic behaviour, such as limit cycle, subharmonic oscillations, quasiperiodic oscillations and the presence of a strange attractor; 2) the dominant first-order Lyapunov exponent for a reconstructed attractor which whether is positive gives a sign of

trajectories of the system wander erratically in a bounded region of the phase plan *L,S* (fig. 2). Furthermore the presence of SDIC (sensitive dependence on initial conditions), is neatly indicated by fig. 3 where the divergence between two trajectories starting from two arbitrarily close initial points ($w_1(0)$=0.180, $w_2(0)$=0.181, $L_1(0)$= $L_2(0)$= 0.825, $S_1(0)$= $S_2(0)$= 0.835) is reported. The bifurcation diagram (fig. 4) clearly indicates the onset of chaos: when still with reference to our initial parameter constellation, the *g* parameter is reduced below the threshold value g_C=0.62 the stable limit cycle, appeared via Hopf bifurcation, bifurcates in its turn, and the system exhibits an evident route to chaos of a quasi-periodic type (see fig. 4). Further reductions of *g* lead to a "catastrophic" crisis revealed by a sudden increases in the size of the chaotic attractor, until the final exploding crisis (for g_E<0.53), which leads the system to global instability.

The figure 2 also illustrates that chaotic fluctuations of the demand and supply of labor are on average below their equilibrium value whereas they are systematically exceeding the equilibrium value for the wage. In substantive terms, a wage increase tends to discourage both the optimal demand and the optimal supply though in different measures depending on their parameters. The balancing

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existence of SDIC (Wolf at al., 1985); 3) the correlation dimension of the (reconstructed) attractor which whether is a non integer number indicates a fractal structure of the attractor (Grassberger – Procaccia, 1983). Such computations (for sake of brevity not reported here) have confirmed the presence of deterministic chaos in the system (8).

effect of the reduction in both demand and supply occurs in a region which is far from the WE, and lies prevalently below the WE value:

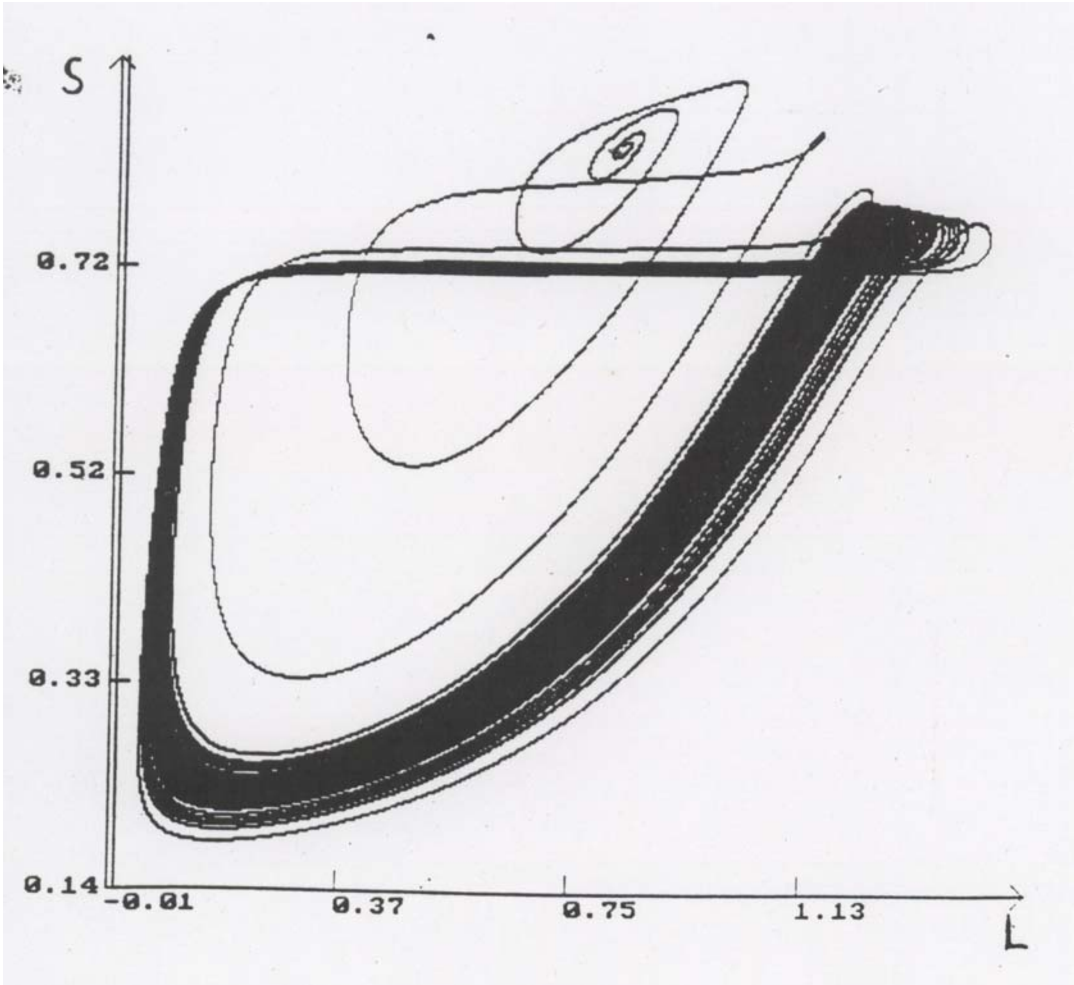

FIG. 2 – *Chaotic behaviour in the phase space S, L. (I.C.: L°=0.825, S°=0.835, w°=0.18)*

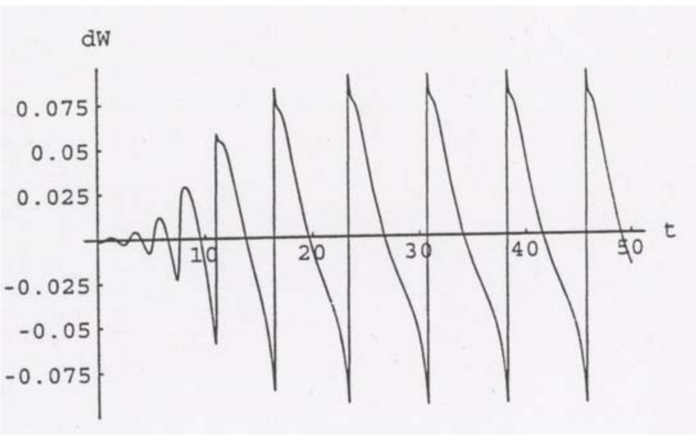

FIG. 3 – *Visualisation of SDIC: the time path of the difference between two trajectories of variable w arising from two very close initial conditions.*

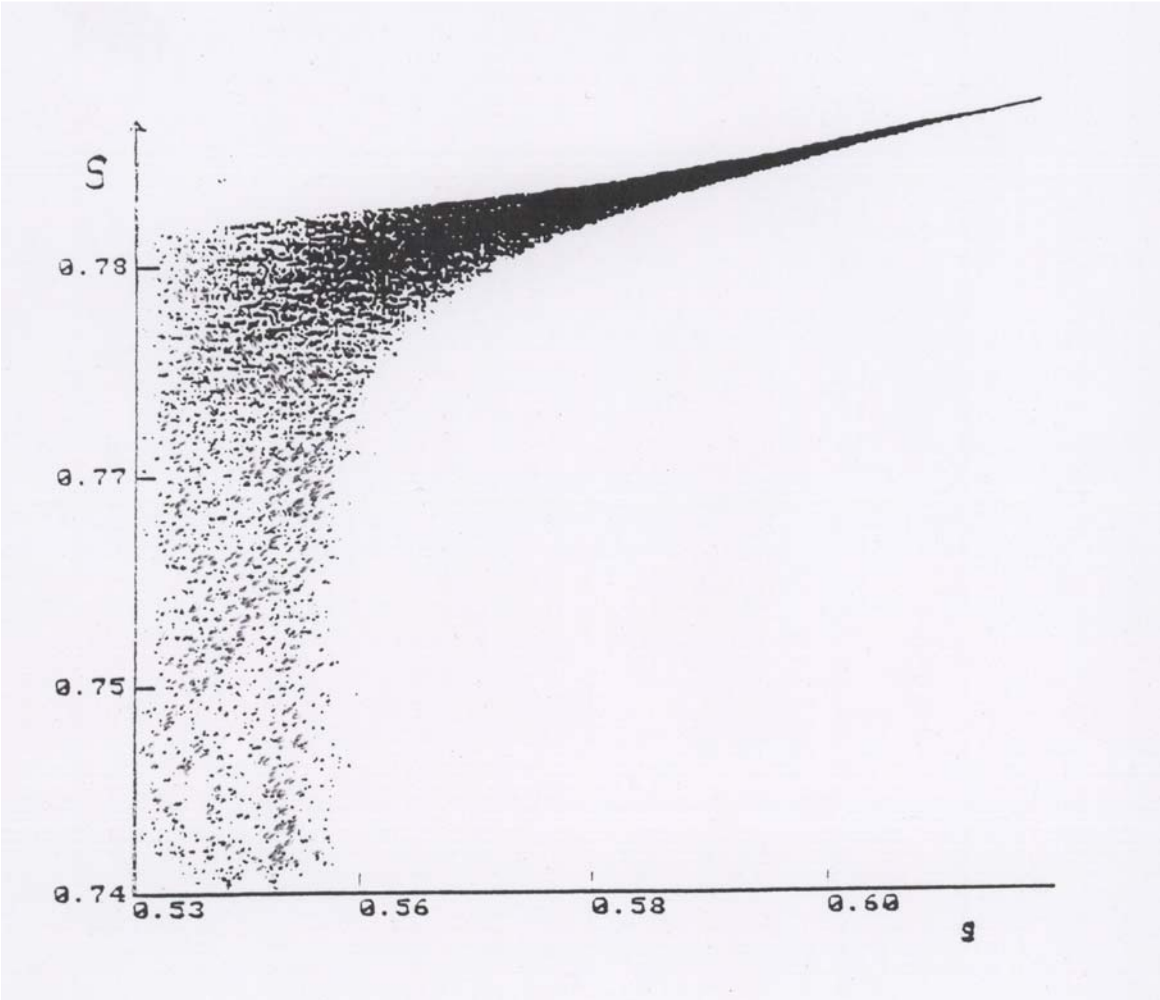

FIG. 4 – *Bifurcation diagram for the parameter g (between 0.535 and 0.62) and the labour supply S.*

Let us briefly summarise our results. As we have seen, in presence of a sufficiently limited flexibility of the employment decisions of the firm, i.e. for $g<g_H$, the stable Walrasian equilibrium is

destabilised: an initial excess demand for labour will not be reabsorbed. Two broad dynamical situations may then occur: i) provided that g is in a suitable intermediate window ($g_C<g<g_H$), an initial excess demand for labour gives rise to an increasing wage which eventually leads to a cyclical balancement between demand and supply; conversely, ii) when $g<g_C$, the increase of the wage does not end in a balanced (though cyclical) regime, but in chaotic fluctuations.

Thus, a first striking result arises: the system, once the WE is abandoned, behaves either regularly or chaotically, *fluctuating in a range of values of the employment and of the wage that are on average respectively lower and higher than the WE values*. This evidences the loss of any predictive ability by the notion of Walrasian equilibrium.

The previous dynamical results suggest a second, and to our mind more important, fact: despite Lipsey's interpretation, the Phillips curve is resurrected as a long term phenomenon by our model. This a direct consequence of the fact that oscillations (latu-sensu) and not stability are the rule for a "backward-bended" neoclassical labour market. In this case the revealed negative relationship between wage dynamics and unemployment which is persistent in the long period could not be interpreted by the policy-maker as a locus of equilibrium trade-offs between wage inflation and unemployment between which to choose. It is purely a dynamical consequences of the working of the standard "backward-bended" neoclassical labour market wage.

4.1 On the Re-emergence of the Phillips' curve

In this paper we used, as usual, the excess demand for labour as a proxy for unemployment. This allows to define the unemployment rate as a non-negative variable defined as the ratio between the excess demand for labour and the labour supply. Our model gives predictions on the shape of the relation between the rate of change of the wage and the rate of unemployment, which is usually known as the Phillips curve. We will call this relation the "true Phillips" relation. Indeed, our model also gives predictions on the shape of the relation between the change of the wage (rather than the rate of change) and the excess demand for labour (rather than the unemployment rate). This is what Chichilnisky & al (1995, this Journal) consider as a Phillips' relation. We term this relation, just to distinguish it from the previous one, a "pseudo-Phillips" relation. The emergence of the "pseudo-Phillips" in our model is illustrated in fig. 5 (which holds for both the regimes of long term stable as well as chaotic oscillations).[13] The nice linear shape is just a trivial consequence of the adopted definition of pseudo-Phillips. Fig. 5 is just Chichilnisky & al (1995) result which we may rephrase as follows: the relations underlying our model necessarily imply that every plot of wage changes against the associate levels of unemployment produce a negative (linear) association between

[13] Note that the Phillips' curves reported in fig. 5, and also in the subsequent fig. 6, predicts only wage deflation for strictly positive values of the unemployment.. This is an obvious consequence of the fact that in our model there is wage inflation only when the excess demand is strictly positive. More realistic forms of the Phillips' curve should obviously exhibit a positive natural rate of inflation, e,g, the natural level of wage inflation under full-employment.

wage changes (inflation) and excess demand for labour (unemployment).

Of course the relation collapses into a single point, e.g. the origin, when the model is simply asymptotically stable (and the transients have disappeared). In the latter case the Phillips curve does not persist in the long term, which is the famous Lipsey's result.

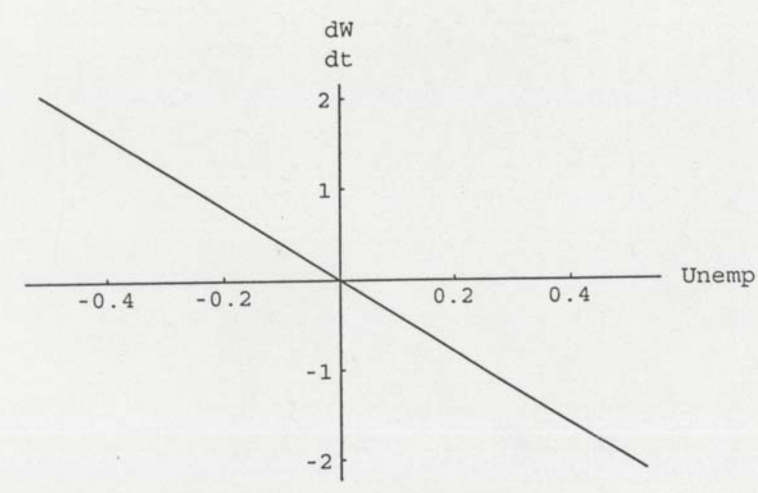

Figure 1: Phillips curve

Fig.5

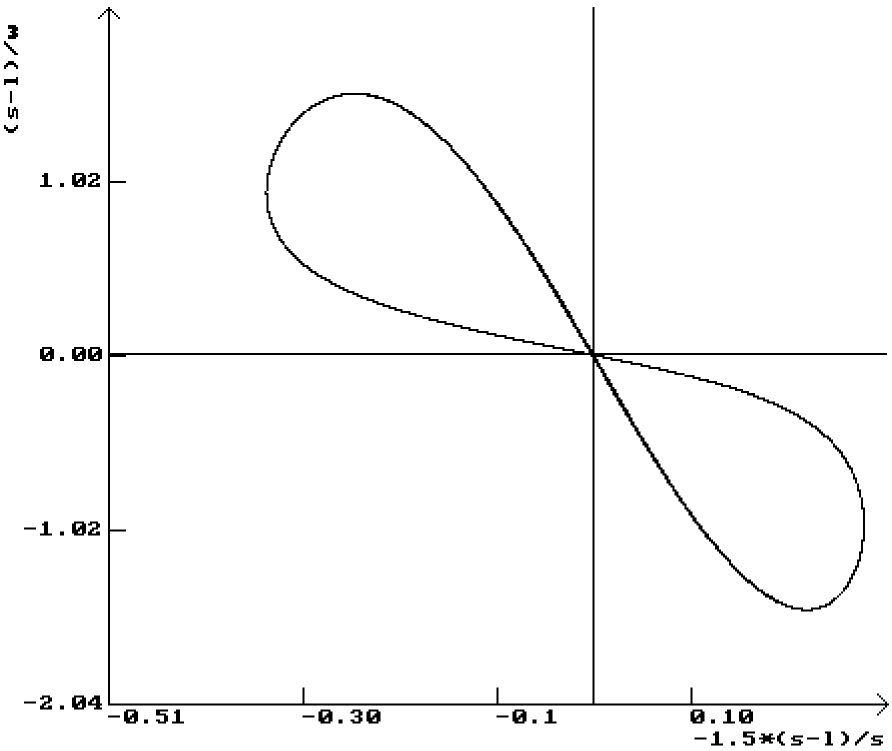

FIG 6

However when we look at the predictions of our Walrasian model on the "true-Phillips" curve, our results seem to go much beyond the finding of a linear association between the two involved variables. Indeed, for instance in the case of regular persistent cycles of the economy, the long term relation between the rate of change of the wage and the unemployment rate leads to the "statistical" plot reported in fig. 6. Once investigated statistically such relation would of course suggest a linear relation possibly with a high correlation between the two variables what and u. However, compared to Chichilnisky & al we have found two additional interesting features: a) a noise-generating mechanism which gives the relation a more scattered form, b) a simple explanation of the puzzling cyclical pattern which a lot of econometric work on

Phillips'data has put in evidence (by purely adding the time on the observed points).

Moreover, in the case of chaotic oscillations of the economy, which lead (when transients are discarded and we look at separate time windows) for the relation between what and u, to the strange plot of fig. 7, the wandering paths of what and u, seem to depict not just one but several Phillips' curves for different time windows, which seems to be consistent with the shifts of the Phillips' curves observed in many real economies (and which have been the base for the criticism of the curve as a policy tool in view of its "perceived" instability).

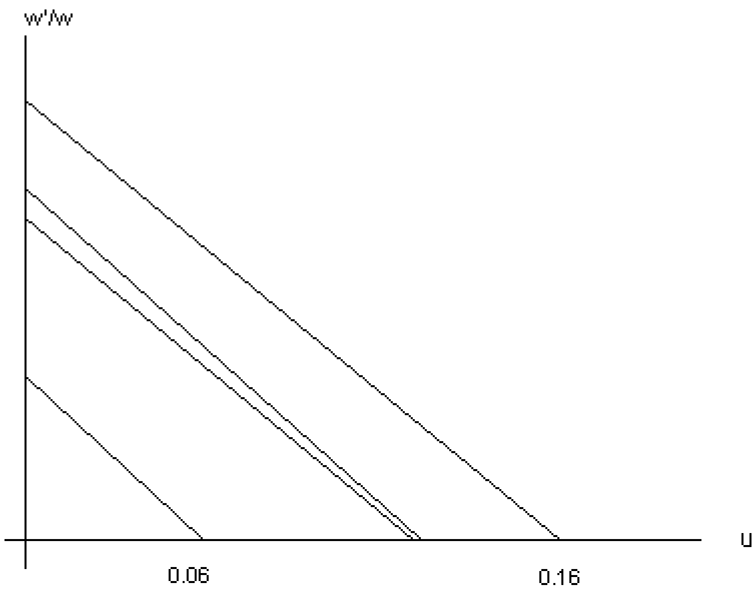

FIG. 7

To sum up we believe that our results in the chaotic case are capable of explaining by just one highly simplified model all the features typically described on empirical works on the Phillips' curve (negative slope between what and u, cyclical loops, and shifts in the whole curve). It is important to note at this stage, that some of our results – the Phillips curve as a true long-run phenomenon on the one hand, and the fact that the curve is a useless policy tool on the other hand, are already fully exhibited without the need for a chaotic attractor. Indeed these features hold also for in the simplified 2-dimensional model developed in appendix c), which is obtained by considering model (7) under the condition that the actual labour supply instantaneously adjusts to its desired counterpart (e.g. for $L^s=S$, corresponding to the d parameter going to infinity)[14]. However, this simplified model could not show the chaotic outcome exhibited by the full model (7), which in our opinion adds two very important features for the interpretation of the reminiscence of the Phillips' curve: i) the endogenous explanation of the possible shifts in the curve over time, ii) the fact that the onset of chaos is not neutral in the debate on the use of the Phillips' curve as a policy tool, given that the existence of chaos has far-reaching implications for policy: for instance (Goodwin-Pacini 1992, pp. 254-255) 1) inability to perfect forecasting, 2) impossibility of fine tuning, 3) persistence of temporary disturbances, 4) need for policy intervention of structural type (e.g. aimed to influence the structural parameters of the economy) rather than temporary, only affecting the state of the system.

It is important, as a final step, to illustrate the economic mechanisms underlying the resurrection of the Phillips curve as a long term phenomenon. It is to be pointed out that this Phillips'

[14] We thank a referee for having pointed out to us this important point.

curve happens to be "truly neoclassical", e.g. it results from the optimal agents behaviour, as opposite to the standard explanation (which states that when unemployment is high – and thus the availability of alternative jobs is reduced, workers would put less pressure to obtain wage increases).

First our model is purely neoclassical: both the sides of the labour market depend on the wage. Second, when the labour supply is backward bending wage changes determine changes in the labour supply and demand in the same direction: although the demand for labour is steeper than the labour supply, and thus the labour market is "basically stable"[15], the presence of stickiness at least on the demand side of the economy (as represented by the delay) may generate instability.

Indeed a decrease in the wage causes an increase in both the demand and supply for labour. In a perfectly flexible labour market this would never cause unemployment because the demand curve is steeper. In presence of stickiness unemployment may arise because the reaction of the labour demand and supply is not anymore temporally synchronised, and this causes unemployment. In addition this mechanism in an unstable one. Indeed in presence of a backward bending labour supply the excess labour supply leads to a decrease in the wage, which in turn stimulates a further increase in the labour supply, and so on.

Finally, the emergence of a "statistical" association having the features of the Phillips curve is a direct consequences of the presence of persistent instability in the model. In the simplified 2-dimensional model this instability takes the form of stable

[15] The model

oscillations, but as our analysis of the general model has shown, the phenomenon is more general.

We remark, finally, that the dynamical interaction between the various speeds of adjustment is not the only source of dynamical complexity in the model. As an illustration we report also some results regarding the dynamical role played by two crucial economic parameters, e.g. the b parameter tuning the shape of the consumption-leisure indifference map, and the parameter a tuning the shape of the firms' production function. Both b and a are potentially responsible of the full set of dynamic transitions observed in the model. For instance, under the same values of the other economic parameters as used in the above simulation, the values: $b=0.35$, $b=-0.23$, $b=-10$, represent thresholds identifying the following dynamic patterns: i) $b>0.35$ monotonic convergence to the equilibrium, ii) $-0.23<b<0.35$ oscillatory convergence, iii) $-10<b<-0.23$ stable oscillations, iv) b<-10 chaotic oscillations. The same dynamic windows are identified by the following values of the a parameter: $a=0.97$, $a=0.58$, $a=0.105$, under the same parametric set of the above simulation, by just varying the g parameter ($g=0.75$).

It can be of interest to figure out which are the individual (preferences) behaviour and the firm's (technological) behaviour underlying the previous dynamic patterns, as summarised by the individual consumption-leisure indifference curves and by the production function. Thus fig. 8a) depicts the shape of typical consumption-leisure indifference curves in correspondence of the three values of the b parameter reported above, whereas fig. 8b) depicts the shape of the production function corresponding to the three values of *a* parameter reported above.

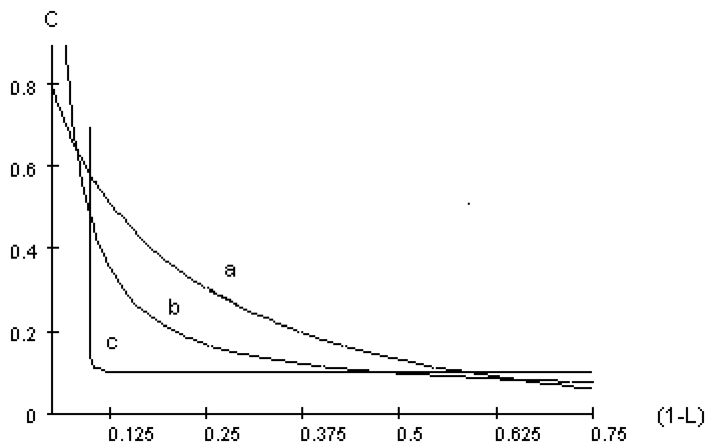

FIG. 8a

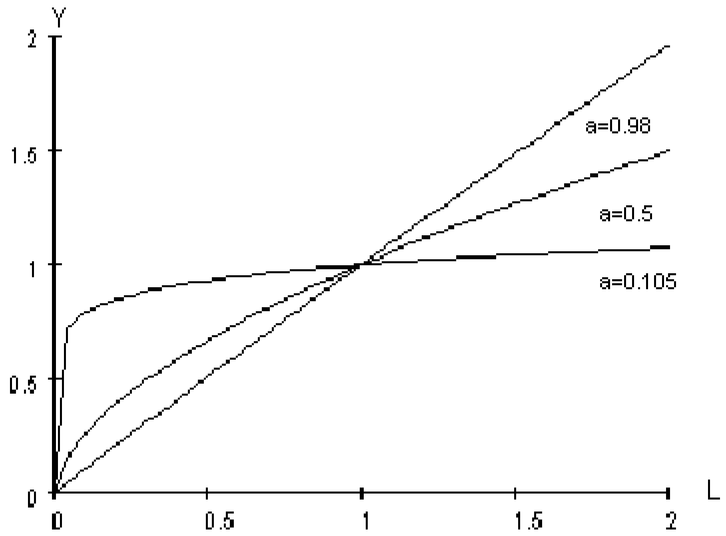

FIG.8b

Fig. 8.

a). Shape of typical consumption-leisure indifference curves in correspondence of three values of the b parameter implying switches between the dynamic regimes observed in the model. Curve a: b=0.35, curve b: b=-0.23, curve c: b=-10. The three indifference curves are drawn for different values of utility for scale reasons in drawing the picture.

b) Shape of the production function in correspondence of three values of the parameter a implying switches between the dynamic regimes observed in the model.

5. Conclusions

This paper has showed that regular fluctuations and chaotic behaviour of wages and employment may be a robust outcome of a market economy when consumption and leisure are low substitutes.

In particular when the unique Walrasian equilibrium of this economy is destabilised, then the economic variables evolve toward a stable attracting region within which their motion is chaotic. In this region the wage always moves in opposite direction to the unemployment: this is an evident "reminiscence" of the Phillips curve which appears as true a long run phenomenon and not only a transitory disequilibrium phenomenon for a wage adjustment process converging to the equilibrium as postulated by the neoclassical interpretation of the Phillips curve. We pointed out however that such a curve, is a totally useless policy tool.

Moreover, a further interesting point is that, although most Walrasian systems can be forced to show complex dynamics by suitable complications of the adjustment process, this work has shown that chaos is a feature of a neoclassical economy under the simplest (e.g. linear) adjustment rule.

Thus our result - which, interestingly, is based on a simple neoclassical framework - intends to be a further contribution to the literature on chaotic economic dynamics (Rosser, 1999).

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APPENDIX: Proofs of the main results.

a. Derivation of the optimal labour supply schedule

We sketch the decision process of the individuals leading to the labour supply function used in the main text (eq. 5).

The individuals, given the utility function (eq. 4), solve the maximisation problem:

Max $U = \left[C^b + (1-L)^b\right]^{\frac{1}{b}}$ (a.1)

subject to the budget constraint $C=wL$ (a.2)

By forming the usual Lagrangian function (LA)

Max $LA = \left[C^b + (1-L)^b\right]^{\frac{1}{b}} + \lambda(C - wL)$ (a.3)

the first order conditions (FOC) are

$$\frac{\partial U}{\partial(1-L)} = (1-L)^{b-1}\left[C^b + (1-L)^b\right]^{\frac{1.b}{b}} = \lambda w$$
$$\frac{\partial U}{\partial C} = C^{b-1}\left[C^b + (1-L)^b\right]^{\frac{1.b}{b}} = \lambda \qquad \text{(a.4)}$$

C=wL

By dividing the first two FOCs we have

$$\frac{(1-L)^{b-1}}{C^{b-1}} = w \qquad \text{(a.5)}$$

and substituting the third one we obtain

$$\frac{(1-L)^{b-1}}{L^{b-1}} = w^{-b} \qquad \text{(a.6)}$$

After some elementary manipulation of (a.6), we obtain the following labour supply function (eq. (5) in the main text)

$$L = \frac{1}{1 + w^{\frac{b}{1-b}}} \qquad \text{(a.7)}$$

b. Proof of the main dynamical results on the general model (7)

To investigate the stability of the Walrasian equilibrium E_1=(L*,S*,w*) let us consider our model in the form (7') used in the text

$$\begin{aligned} \dot{L} &= g(f_1(w) - L) \\ \dot{S} &= d(f_2(w) - S) \qquad g,d,l > 0 \qquad (b.1) \\ \dot{w} &= l(L - S) \end{aligned}$$

For simplicity let us suppress (*) and denote by (L,S,w) the equilibrium values of the state variables. The Jacobian evaluated at E_1 leads to a third order characteristic equation with coefficients a_i (i=0,1,2,3) defined as:

$$a_0 = 1; \quad a_1 = g + d; \quad a_2 = gd - l\left(gf_1^{'}(w) - df_2^{'}(w)\right); \quad a_3 = ldg\left(f_2^{'}(w) - f_1^{'}(w)\right)$$

where f'_i (i=1,2) denotes first derivative of f with respect to wage. We note that a_1 is always positive. Moreover a_3 is always positive when $b>0$ (Conversely when $b<0$, a_3 is not necessarily positive). In the case $b>0$ the Routh-Hurwicz test gives the following stability condition for E_1:

$$(g + d)gd + l\left(d^2 f_2^{'}(w) - g^2 f_1^{'}(w)\right) > 0 \qquad (b.2)$$

As $f_1'(w)<0$ the last inequality is always true, showing that when $b>0$ the E_1 equilibrium is always locally asymptotically stable. This proves Proposition 2 in the main text.

Let us now consider the case $b<0$. In this case both the derivatives of the optimal labour supply and demand are negative. Let us then introduce the quantities A,B used in the main text:

$A = -f_1^{'}(w)$, $B = -f_2^{'}(w)$.

As:

$$a_3 = ldg(f_2^{'}(w) - f_1^{'}(w)) = ldg(A - B)$$

for A-B<0 (A<B) (corresponding to the case of a supply curve steeper than the demand curve at equilibrium) the equilibrium is always locally unstable (a saddle point).[16] Conversely, for $A>B$ the

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16 The equality $A=B$ causes a saddle-node bifurcation. In this paper we have not considered the whole spectrum of possible bifurcation

coefficient a_3 remains positive and more interesting dynamical effects may appear. In this case the R-H test gives the condition:

$$g^2(d + Al) + d^2(g - Bl) > 0 \qquad (b.3)$$

For $d=g$ we get:

$$a_1 a_2 - a_3 = 2d^3 + ld^2(A - B) > 0$$

Hence, for $d=g$ the system is always locally stable in a neighborhood of the E_1 equilibrium. This shows that, a fortiori, the system remain LAS for all combinations (g,d) satisfying $g>d$. On the contrary, when g<d, instability arises when (a.3) does not holds. This proves Proposition 3.

We claim that a Hopf bifurcation arises when the equality:

$$(g + d)gd + l(Ag^2 - Bd^2) = 0 \qquad (b.4)$$

holds. By solving (a.4) as a quadratic with respect to g we get:

$$g_H = \frac{-d^2 \pm \sqrt{d^4 + 4lB(d + lA)d^2}}{2(d + lA)} \qquad (b.5)$$

---

patterns of the equilibrium as we are mainly interested in the mechanisms leading to complex behaviour (for sake of brevity we have omitted the full picture of the bifurcation patterns obtained via equilibria continuation methods (Kuznetzov 1995)).

The discriminant of the latter expression is always positive so that the solution (a.5) is always real. It is easy to check that only the greater among the solution (a.5) is positive and thus adequate to represent the desired bifurcation process. This shows that a bifurcation value always exists. It is of interest to study the shape of the bifurcation curve (a.5) as a relation between *g,d,* i.e. the speeds of adjustment of the current demand and supply curves. It is easy to see that the curve $g_H(d)$ lies always in the region $g<d$. Moreover a straightforward application of Dini's theorem to (a.4) shows that $g_H(d)$ is a strictly increasing function of d.

To complete the proof of the appearance of a Hopf bifurcation of the E_1 equilibrium, let us now show that the pair of bifurcating eigenvalues cross the imaginary axis with nonzero speed. This is equivalent to show that (Liu, 1994):

$$\left(\frac{d}{dg}(a_1a_2-a_3)\right)_{g=g_H} \neq 0$$

We quickly have:

$$\left(\frac{d}{dg}(a_1a_2-a_3)\right)_{g=g_H} = \left(2(d+lA)g+d^2\right)_{g=g_H} = \sqrt{d^4+4lB(d+lA)}$$

which is always positive, thereby completing the proof of Proposition 4.

c. The 2-dimensional model

This appendix sketches the main properties of a sub-model of our model (7), which is obtained from (7) under the condition that the actual labour supply instantaneously adjusts to its desired counterpart (e.g. for $L^s=S$, corresponding to the d parameter going to infinity). The suggested two-dimensional system is, according to the notation adopted in the paper, the following:

$$\dot{L} = g\left(\left(\frac{w}{aD}\right)^{1/(a-1)} - L\right) = g(f_1(w) - L)$$
$$\dot{w} = l\left(L - \frac{1}{1 + w^{b/(b-1)}}\right) = l(L - f_2(w))$$

The previous 2-dimensional system obviously leads to the same equilibria of the 3-dimensional system. Thus a unique economically meaninful equilibrium pair E_1=(L*,w*) always exists. The stability analysis around E_1 quickly gives: $Tr(J(E_1)) = (-1)\cdot(g + l \cdot f_2'(w_1))$ and $Det(J(E_1)) = g \cdot l \cdot (f_2'(w_1) - f_1'(w_1))$. It is thus a trivial matter to check that for a standard optimal labour supply curve ($b>0$) the trace is always negative and the determinant always positive, therefore implying that E_1 is always LAS. Conversely, in the case $b<0$, corresponding to the backward bending optimal labour supply curve, which is the one of major interest for the present paper, the determinant is always positive because (fig. 1b) the supply curve is less steep than the demand curve at equilibrium. In this event stability of E_1 requires $g + l \cdot f_2'(w_1) > 0$, which may be expanded as: $(g/l) > -f_2'(w_1)$. The latter condition may be interpreted in two-

ways: a) the ratio between the two involved speeds of adjustment g and l must exceed a prescribed threshold (e.g. the speed of adjustment of the actual demand for labour must be “large” compared to the speed of adjustment of the wage to the disequlibrium between labour demand and supply); b) the second interpretation may be given in terms of the slope of the optimal labour supply curve. When stability fails due to the violation of the previous requirements a Hopf bifurcation occurs. It is easy to show by a Poincaré-Bendixson argument that the limit cycle emerging when instability arises at E_1 is globally asymptotically stable.
Also in this case the Phillips’ curve re-emerges: for instance the discussion of fig. 6 in the main text straightforwardly holds also for this case.

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