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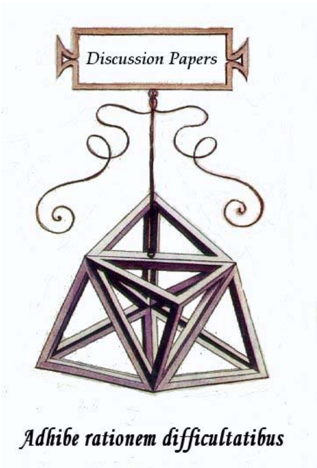

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Existence of Optimal Strategies in linear Multisector Models

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Abstract

In this paper we give a sufficient and almost necessary condition for the existence of optimal strategies in linear multisector models when time is continuous, consumption is limited to one commodity, the instantaneous utility is of the CES type, and available technology allows a positive growth rate.

Keywords: Endogenous growth, AK model, optimal control with mixed constraints, von Neumann growth model.
JEL classification: C61, C62, C41, C67, D91

Contents

I. Introduction

In this paper we give a sufficient and almost necessary condition for the existence of optimal strategies in linear multisector models when time is continuous, consumption is limited to one commodity, the instantaneous utility is of the CES type, largely used in the endogeneous growth literature, and available technology allows a positive growth rate.

Linear multisector models are largely studied in economic theory. Perhaps the first of these models was introduced by von Neumann in 1934. Von Neumann [16] proved existence of what could be said today a steady state equilibrium when the growth rate is maximazed so that consumption is nought. Later the model was generalized to introduce consumption and explicit labour growing at an exogeneous given rate by Kemeny, Morgenstern, and Thompson [12] and Morishima [14]. The model was generalized also in other directions. Instead of von Neumann' objective of a maximal growth rate, turnpike models adopted as objective the attainment of maximal capital stock of some specified composition at a given time in the future or the utility function which is defined with regard to only the stocks of commodities at a given (final) time as its argument (for a survey, see Takayama [19]). Finally the (multisector) optimal growth literature has introduced the possibility that the utility function is defined in every intermediate state as well as in the terminal state: for a survey, see again Takayama [19]).

In this paper we are interested to a formulation of a linear multisector model related to the recent literature on endogeneous growth. This is the reason why we do not consider explicit labour inputs and we avoid primary factors in general. The interpretation of the absence of explicit labour could be the same proposed in the original von Neumann paper: real wage rate(s) is (are) considered given and wage payments are included in the processes of production as inputs. Alternatively, the model can be interpreted as considering human capital inputs instead of labour inputs in the assumption that some of the sectors involved produce human capital of different qualities. More in general, all technologies which exibit constant returns in the reproducible factors can be treated, or approximated at whatever given degree, within the framework here provided. Since we avoid primary factors we can assume that a positive growth rate is technological feasible.

In this paper there are two main differences with the original von Neumann presentation and large part of the literature on it. One difference is connected to the fact that we consider time as continuous instead of discrete and will be clarified soon. In a multisector analysis capital stocks do not need to be in the proportions required and, unless a strong substitutability among factors is assumed, some commodities need to be disposed. Many linear multisector model, including that provided by von Neumann, adopt the rule of free disposal. In a discrete time setting, this means that a commodity non consumed and not used in production is destroyed in a period. In a continuous time setting a rule like this means instantaneous destruction, that is an infinite speed of disposal. This is not very realistic, since disposal requires time, and, more problematic from a theoretical point of view, it implies that stocks can vary in a discontinuous way. For this reason, in this paper we introduce a finite rate of depreciation for commodities not consumed and not used in production. Such a rate of depreciation may be higher than the rate of depreciation on the same commodities when they are used in production. We can also have a rate of depreciation for some commodities not consumed and not used in production lower than or equal to the rate of depreciation for commodities used in production when the aim is not the disposal, but the conservation of those commodities.

The other difference is connected to the fact that we are interested to the determination of an optimal path starting from an historically given capital stock. Unless we do not assume that all commodities are available at time 0, we have that even a commodity which is producible, in the sense that there are known processes that have it among its outputs, could be reduced to a primary factor since some of the commodities which are directly or indirectly necessary for its production are not available at any time. In this paper we introduce this possibility, but then put it on one side since it would impose that the economy cannot grow at a positive rate. Another paper will be devoted to this problem.

Whereas the model here presented has a production side close to the von Neumann model, in which commodities are produced out of each other, it has also Ramsey-like preferences in the sense that the optimal behavior of a representative agent determines the saving behavior of the system. This characteristic is shared with large part of the literature on optimal growth. In this paper we want to concentrate on the intertemporal choices instead than the intratemporal ones. This is the reason why we assume a single consumption good. Further, even if steady states are not an issue of this paper, we use the usual isoelastic utility function, which is the only one compatible with the existence of steady states. This will allow us to clarify the main differences with the single commodity analysis performed in large part of the literature. Therefore the preferences of the representative agent is characterized by two parameters: the rate of time discount ρ and the constant elasticity of substitution $\sigma > 0$.

In this paper we will prove that an important role for the existence is played by the upper bound of the uniform over time rates of reproduction of the consumption good Γ. In particular if

$$\Gamma > \frac{\Gamma - \rho}{\sigma},$$

then an optimal strategy exists, whereas if

$$\Gamma < \frac{\Gamma - \rho}{\sigma},$$

then no optimal strategy exists. This confirms in large part the analysis of the single commodity model studied in the received literature, where the upper bound of the uniform over time rates of reproduction of the consumption good is more appropriately interpreted as the rate of profit. In this analysis, of course, a uniform rate of profit does not need to exists. In another unpublished paper (see [9]) and in [8], however, we have investigated the steady states of this model and envisaged three possible steady states. In all these steady states a uniform rate of profit exists and in one of them it equals Γ.

Confirmation of existence results which are known to be valid in a single commodity model to a multisector setting is not the unique motivation of the paper. On the contrary, we will show that the multisector analysis is able to show difficulties which cannot be detected in a single commodity model. A simple example in which the economy can grow at a (non uniform over time) rate larger than Γ shows that if ρ and σ are such that

$$\Gamma = \frac{\Gamma - \rho}{\sigma},$$

then an optimal strategy may or may not exist, depending on ρ (and the corresponding σ). From the mathematical point of view the emergence of this complexity in the limiting cases is strictly related with the following two facts (depending on the structure of the input/output matrices $\mathbf{A}$ and $\mathbf{B}$): that the the upper bound of the uniform over time rates of reproduction of the consumption good Γ may be a maximum or not, and if it is a maximum, then there may be either polinomial terms (arising from non simple eigenvalues) that give a correction to the growth rate, or similar features in the dual space. These facts do not arise when the matrices have some prescribed structure (see Subsection III.A.). The study of the limiting cases became then quite complex and we leave it for future work. Here a couple of examples must suffice.

The plan of the paper is the following: first we describe the model in Section II., discussing also the main assumption on it. In Section III. we give the main results and a couple of examples to show the complexity of the limiting cases (two special cases are also mentioned in Subsection III.A.). The appendices are devoted to develop the technical part: Appendix A contains some preliminary estimates and properties of admissible strategies and trajectories; Appendix B is devoted to the proofs of main results; Appendix C deals with the limiting cases; Appendix D gives some controllability results that are used in some proofs.

II. The Model

There are $n \geq 1$ commodities, but only one of them is consumed, say commodity 1. Preferences with respect to consumption over time are such that they can be described by a single intertemporal utility function U_σ, which is the usual C.E.S. (Constant Elasticity of Substitution) function used in this kind of literature: for a given consumption path $c : [0, +\infty) \to \mathbb{R}$, ($c_t \geq 0$ a.e.), we set

$$U_\sigma (c(\cdot)) = \int_0^{+\infty} e^{-\rho t} u_\sigma (c(t))\, dt \tag{1}$$

where $\rho \in \mathbb{R}$ is the rate of time discount of the representative agent and the instantaneous utility function $u_\sigma : [0, +\infty) \to \mathbb{R} \cup \{-\infty\}$ depends on a single parameter $\sigma > 0$ (the elasticity of substitution) and is given by

$$u_\sigma (c) = \frac{c^{1-\sigma} - 1}{1 - \sigma} \qquad for \quad \sigma > 0,\ \sigma \neq 1$$

$$u_1 (c) = \log c \qquad for \quad \sigma = 1$$

(with the agreement that $u_\sigma(0) = -\infty$ for $\sigma \geq 1$). For the sake of simplicity we will drop the constant $-(1-\sigma)^{-1}$ in the following since this will not affect the optimal paths.

Technology is fully described by a pair of nonnegative matrices (the $m \times n$ material input matrix $\mathbf{A}$ and the $m \times n$ material output matrix $\mathbf{B}$, $m \geq 0$) and by a uniform rate of depreciation $\delta_{\mathbf{x}}$ of capital goods used for production. The rate of depreciation for goods not employed in production is $\delta_{\mathbf{z}}$. If $m = 0$, we say that matrices $\mathbf{A}$ and $\mathbf{B}$ are void. In this degenerate case production does not hold and the model reduces to the standard one-dimensional AK model with $A = -\delta_{\mathbf{z}} \leq 0$.

The amounts of commodities available as capital at time t are defined by vector $\mathbf{s}_t$. They may be either used for production (if $m > 0$) or disposed of. That is

$$\mathbf{s}_t^T = \mathbf{x}_t^T \mathbf{A} + \mathbf{z}_t^T,$$

where $\mathbf{x} \geq \mathbf{0}$ denotes the vector of the intensities of operation and $\mathbf{z} \geq \mathbf{0}$ the vector of the amounts of goods which are disposed of. Production consists in combining the productive services from the stocks to generate flows that add to the existing stocks. Decay and consumption, on the other hand, drain away the stocks:

$$\dot{\mathbf{s}}_t^T = \mathbf{x}_t^T [\mathbf{B} - \delta_{\mathbf{x}} \mathbf{A}] - \delta_{\mathbf{z}} \mathbf{z}_t^T - c_t \mathbf{e}_1^T; \qquad c_t \geq 0 \qquad \mathbf{s}_0 = \bar{\mathbf{s}}$$

By eliminating the variable $\mathbf{z}$ and setting $\delta = -\delta_{\mathbf{z}} + \delta_{\mathbf{x}}$, we obtain

$$\dot{\mathbf{s}}_t^T = \mathbf{x}_t^T [\mathbf{B} - \delta \mathbf{A}] - \delta_{\mathbf{z}} \mathbf{s}_t^T - c_t \mathbf{e}_1^T; \tag{2}$$

with the initial condition

$$\mathbf{s}_0 = \bar{\mathbf{s}} \geq \mathbf{0} \tag{3}$$

and the constraints

$$\mathbf{x}_t \geq \mathbf{0}, \qquad \mathbf{s}_t^T \geq \mathbf{x}_t^T \mathbf{A}, \qquad c_t \geq 0. \tag{4}$$

Our problem is then to maximize the intertemporal utility (1) over all production-consumption strategies $(\mathbf{x}, c)$ that satisfy the constraints (2), (3) and (4). This is an optimal control problem where $\mathbf{s}$ is the state variable and $\mathbf{x}$ and c are the control variables. We now describe this problem more formally.

A production-consumption strategy $(\mathbf{x}, c)$ is defined as a measurable and locally integrable function of $t : \mathbb{R}^+ \to \mathbb{R}^m \times \mathbb{R}$ (we will denote by $L^1_{\text{loc}}(0, +\infty; \mathbb{R}^{m+1})$ the set of such functions). Then the differential equation (2) has a unique solution $: \mathbb{R}^+ \mapsto \mathbb{R}^n$ which is absolutely continuous (we will denote by $W^{1,1}_{\text{loc}}(0, +\infty; \mathbb{R}^n)$ the set of such functions). Such a solution clearly depends on the initial datum $\bar{\mathbf{s}}$ and on the production-consumption strategy $(\mathbf{x}, c)$ so it will be denoted by the symbol $\mathbf{s}_{t;\bar{\mathbf{s}},(\mathbf{x},c)}$, omitting the subscript $\bar{\mathbf{s}}, (\mathbf{x}, c)$ when it is clear from the context.

Given an initial endowment $\bar{\mathbf{s}}$ we will say that a strategy $(\mathbf{x}, c)$ is admissible from $\bar{\mathbf{s}}$ if the triple $\left(\mathbf{x}, c, \mathbf{s}_{t;\bar{\mathbf{s}},(\mathbf{x},c)}\right)$ satisfies the constraints (4) and $U_1(c)$ is well defined[1]. The set of admissible control strategies starting at $\bar{\mathbf{s}}$ will be denoted by $\mathcal{A}(\bar{\mathbf{s}})$. We adopt the following definition of optimal strategies.

Definition II..1 *A strategy* $(\mathbf{x}^*, c^*) \in \mathcal{A}(\bar{\mathbf{s}})$ *will be called optimal if we have* $U_\sigma(c^*) > -\infty$ *and*

$$+\infty > U_\sigma(c^*) \geq U_\sigma(c)$$

for every admissible control pair $(\mathbf{x}, c) \in \mathcal{A}(\bar{\mathbf{s}})$.

We now provide some definitions useful to simplify the exposition. Then we introduce and comment on a set of assumptions that will be used throughout the paper.

If the j-th column of matrix $\mathbf{B}$ is semipositive we say that commodity j is *technologically reproducible*. If commodity j is not technologically producible ($\mathbf{B}\mathbf{e}_j = 0$) and it is available at time 0 ($\bar{\mathbf{s}}^T \mathbf{e}_j > 0$), we say that commodity j is a *primary factor*. If a primary factor is destructible ($\delta_{\mathbf{x}} + \delta_{\mathbf{z}} > 0$), we say that it is an *exhaustible resource*; if it is indestructible ($\delta_{\mathbf{x}} = \delta_{\mathbf{z}} = 0$), it is *Ricardian land*.

Assumption II..2 *Each row of matrix* $\mathbf{A}$ *is semipositive.*

This assumption means that no commodity can be produced without using some commodity as an input.

Assumption II..3 *Each row of matrix* $\mathbf{B}$ *is semipositive.*

[1]The condition on $U_1(c)$ is relevant only when $\sigma = 1$. Note that for $\sigma \in (0, 1)$ the function $t \to e^{-\rho t} u_\sigma(c_t)$ is always nonnegative so it is always semiintegrable (with the integral eventually $+\infty$). On the other hand for $\sigma > 1$ the function $t \to e^{-\rho t} u_\sigma(c_t)$ is always negative (and may be $-\infty$ when $c_t = 0$) and again it is always semiintegrable (with the integral eventually $-\infty$). This means that the intertemporal utility U_σ is always well defined for $\sigma \neq 1$. For $\sigma = 1$ the function $t \to e^{-\rho t} u_\sigma(c_t)$ may change sign so it may be not semiintegrable on $[0, +\infty)$. This is the reason why we need to require that $U_1(c)$ is well defined to define the admissibility of c.

This assumption means that each process produce something: i.e. that pure destruction processes are not dealt with as production processes.

Assumption II..4 *The initial datum* $\bar{\mathbf{s}} \geq \mathbf{0}$ *and the matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *are such that there is an admissible strategy* $(\mathbf{x}^*, c^*) \in \mathcal{A}(\bar{\mathbf{s}})$ *and a time* $t^* > 0$ *such that the first element of* $\mathbf{s}_{t^*;\bar{\mathbf{s}},(\mathbf{x}^*,c^*)}$ *is positive.*

If this assumption does not hold, then every admissible strategy must have $c = 0$ a.e. This case is not an interesting case to be investigated.

Assumption II..5 *For each* $j \neq 1$ *the initial datum* $\bar{\mathbf{s}} \geq \mathbf{0}$ *and the matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *are such that there is an admissible strategy* $(\mathbf{x}^*, c^*) \in \mathcal{A}(\bar{\mathbf{s}})$ *and a time* t_j^* *such that the* j*-th element of* $\mathbf{s}_{t_j^*;\bar{\mathbf{s}},(\mathbf{x}^*,c^*)}$ *is positive.*

Assumptions II..4 and II..5 imply that all commodities are available at any time $t > 0$: it can be proved that if both hold, then for any time $t_0 > 0$ there is an admissible strategy $(\mathbf{x}^*,c^*)$ such that $\mathbf{s}_{t_0;\bar{\mathbf{s}},(\mathbf{x}^*,c^*)} > 0$. See on this Appendix D below. Assumptions II..4 and II..5 could be stated in terms of the zero components of the initial datum $\bar{\mathbf{s}}$ and of the structure of the matrices $\mathbf{A}$ and $\mathbf{B}$, see on this Appendix D below.

Assumption II..5 is not really restrictive in the sense that when it does not hold, matrices $\mathbf{A}$ and $\mathbf{B}$ and vector $\bar{\mathbf{s}}$ can be redefined in order to obtain an equivalent model in which Assumption II..5 holds. Assume, in fact, that Assumption II..5 does not hold. Then there is a commodity j which is not available at any time $t \geq 0$ ($\mathbf{s}_t^T \mathbf{e}_j = 0$ for every $t \geq 0$). In this case any production process i in which commodity j is employed ($a_{ij} > 0$) cannot be used. The model is then equivalent to one in which matrices $\mathbf{B}$ and $\mathbf{A}$ and vector $\mathbf{s}$, in the state equation (2), are substituted with matrices $\mathbf{D}$ and $\mathbf{C}$ and vector $\mathbf{s}'$, respectively, where matrix $\mathbf{C}$ is obtained from $\mathbf{A}$ by deleting the j-th column and all rows which on the j-th column have a positive element, matrix $\mathbf{D}$ is obtained from matrix $\mathbf{B}$ by deleting the corresponding rows and the j-th column, and vector $\mathbf{s}'$ is obtained from vector $\mathbf{s}$ by deleting the j-th element. Note that if in the new equivalent model the Assumption II..5 does not hold and matrices $\mathbf{C}$ and $\mathbf{D}$ are not void, the argument can be iterated. If matrices $\mathbf{C}$ and $\mathbf{D}$ are void, then an equivalent model satisfying Assumption II..5 is obtained by deleting the nought elements of vector $\mathbf{s}'$. In any case the algorithm is able to detemine an equivalent model in which Assumption II..5 does hold. We will refer to the equivalent model found in this way as the *truncated model* and to the corresponding technology as the *truncated technology*, which then depends on $\bar{\mathbf{s}}$. It can easily be proved that if Assumptions II..2, II..3, II..4 hold in the original technology, then they hold in the truncated technology too (see Appendix D below). Except when it is not mentioned explicitly, all the following assumptions are referred to the truncated technology.

Let us define

$$\mathcal{G}_0 := \left\{\gamma | \exists \mathbf{x} \in \mathbb{R}^m : \mathbf{x} \geq \mathbf{0}, \mathbf{x} \neq \mathbf{0}, \mathbf{x}^T \left[\mathbf{B} - (\gamma + \delta_{\mathbf{x}}) \mathbf{A}\right] \geq \mathbf{0}\right\}, \qquad \Gamma_0 = \max \mathcal{G}_0$$

$$\mathcal{G}_1 := \left\{\gamma | \exists \mathbf{x} \in \mathbb{R}^m : \mathbf{x} \geq \mathbf{0}, \mathbf{x}^T \left[\mathbf{B} - (\gamma + \delta_{\mathbf{x}}) \mathbf{A}\right] \geq \mathbf{e}_1^T\right\}, \qquad \Gamma = \sup \mathcal{G}_1$$

Γ_0 is clearly the maximum among the uniform over time rates of growth feasible for this economy and corresponds to what von Neumann found both as growth rate and as rate of profit. Γ is the upper bound of the uniform over time rates of reproduction of the consumption good. Obviously $\Gamma \leq \Gamma_0$. It is easily proved that the Γ's relative to the truncated technology are not larger than the corresponding Γ's relative to the original one. If either $\mathbf{B}\mathbf{e}_1 = \mathbf{0}$ or matrices $\mathbf{A}$ and $\mathbf{B}$ are void, then $\Gamma = -\infty$. Moreover, if $\mathbf{B}\mathbf{e}_1 \neq \mathbf{0}$ and exhaustible resources are essential to the reproduction of the consumption good, then $\Gamma = -\delta_{\mathbf{x}}$.[2] Since this paper is devoted to the problem of endogenous growth we will eliminate all these cases by assuming that the upper bound of the uniform over time rates of reproduction of the consumption good is positive.

Assumption II..6

$$\Gamma > 0$$

As mentioned in the introduction this paper is mainly devoted to show the role that the following assumption plays for the existence of optimal strategies of the problem under analysis.

Assumption II..7

$$\Gamma > \frac{\Gamma - \rho}{\sigma}$$

The reader should have noticed that we have used the convoluted expression "the upper bound of the uniform over time rates of reproduction of the consumption good" instead of the most obvious "the upper bound of the rates of reproduction of the consumption good". This is so since for particular forms of matrices it could be possible to find growth rates of consumption which are higher, but not uniform over time. An example can clarify this point.

Example II..8 $\delta_{\mathbf{x}} = \delta_{\mathbf{z}} \in (0, 1)$ *and*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

---

[2] We say that commodity j is essential to the reproduction of consumption good when

$$\left(\mathbf{x} \geq 0, \varepsilon > 0, \mathbf{x}^T \left[\mathbf{B} - \varepsilon \mathbf{A}\right] \geq \mathbf{e}_1\right) \Rightarrow \mathbf{x}^T \mathbf{A} \mathbf{e}_j \neq 0.$$

It is immediately recognized that $\Gamma = 1-\delta_{\mathbf{x}} > 0$ *and* Γ *is a maximum. Nevertheless consumption can grow at the rate*

$$\frac{\dot{c}}{c} = \Gamma + \frac{\beta}{\alpha + \beta t} > \Gamma$$

where α *and* β *are given positive constants. It is enough that*

$$s_1 = 0$$

$$s_2 = c = x_1 = \alpha e^{\Gamma t} + \beta e^{\Gamma t} t$$

$$s_3 = x_2 = \beta e^{\Gamma t}$$

This is clearly a consequence of the fact that the matrix of the linear autonomous dynamical system arising from this choice of production-consumption strategy has a nonsimple eigenvalue $\lambda = 1$

III. The main results

The main goal of this paper is to show that in the general context outlined by Assumptions II..2, II..3, II..4, II..5, II..6, we have substantially an if and only if condition for the existence of optimal strategies. In fact in this paper we will prove the following results:

Theorem III..1 *If Assumptions II..2, II..3, II..4, II..5, II..6 and II..7 hold, then there is an optimal strategy* $(\mathbf{x}, c)$ *for problem* (P_σ) *starting at* $\bar{\mathbf{s}}$*. Moreover this strategy is unique in the sense that if* $(\hat{\mathbf{x}}, \hat{c})$ *is another optimal strategy, then* $\hat{c} = c$*.*

Theorem III..2 *Let Assumptions II..2, II..3, II..4, II..5, II..6 hold. If*

$$\Gamma < \frac{\Gamma - \rho}{\sigma}$$

then no optimal strategy exists for problem (P_σ) *starting at* $\bar{\mathbf{s}}$*.*

Theorem III..3 *Let Assumptions II..2, II..3, II..4, II..5, II..6 hold. Let*

$$\Gamma = \frac{\Gamma - \rho}{\sigma}.$$

Then we have the following:

1. *If* $\sigma = 1$ *then no optimal strategy exists for problem* (P_σ) *starting at* $\bar{\mathbf{s}}$*.*
2. *If* $\sigma \in (0, 1)$ *and* Γ *is a maximum then no optimal strategy exists for problem* (P_σ) *starting at* $\bar{\mathbf{s}}$*.*

3. *If $\sigma > 1$ and Γ is not a maximum, then no optimal strategy exists for problem (P_σ) starting at $\bar{\mathbf{s}}$.*

The limit cases

1. $\sigma \in (0,1)$ and Γ is not a maximum,

2. $\sigma > 1$ and Γ is a maximum,

are intrinsecally more complex than the others. Indeed in such cases we can have existence or nonexistence depending on the value of σ. We provide here below two examples of matrices A and B and scalars ρ, $\delta_{\mathbf{x}}$, $\delta_{\mathbf{z}}$ showing this fact.

In the first $\sigma > 1$ and we have existence when $\sigma > 2$ and nonexistence for $\sigma \in (1,2)$. In the second $\sigma \in (0,1)$ and we have existence when $\sigma < \frac{1}{2}$ and nonexistence for $\sigma > \frac{1}{2}$.

Example III..4 *The technology is that of the Example II..8. We now take the production strategy mentioned there and take account of the fact that $\rho = \Gamma(1-\sigma)$. It is easily checked that the functional (1) is in this case*

$$\frac{1}{1-\sigma}\int_0^{+\infty}\frac{1}{(\alpha+\beta t)^{\sigma-1}}dt$$

and, for $\sigma > 2$ this is finite whereas for $\sigma \in (1,2]$ it takes value $-\infty$. Further, it can be proved, with similar arguments, that if $\sigma < 2$, then all strategies take value $-\infty$. We omit this for brevity. See Appendix C for a discussion of this example. ∎

Example III..5 *Take $\delta_{\mathbf{x}} = \delta_{\mathbf{z}} \in (0,1)$ and*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}.$$

In this case it is easy to check that $\Gamma = 3 - \delta_{\mathbf{x}} > 0$ is not a maximum. Moreover take $\rho = \Gamma(1-\sigma)$. It is easily proved that if $\sigma \in \left(0, \frac{1}{2}\right)$ there exists an optimal strategy, whereas if $\sigma \in \left(\frac{1}{2}, 1\right)$ there is no optimal strategy (see Appendix C). ∎

III.A. Two special cases

We consider here two special cases correponding to the following two assumptions on technology:

Assumption III..6 *Each row of matrix* $\mathbf{B}$ *has one and only one positive element.*

Assumption III..6 avoids joint production of commodities. It is easily shown that if it holds in the original technology, then it holds in the truncated technology too. However, it could hold in the truncated technology without holding in the original one.

Assumption III..7 *The discount factor* ρ *is nonnegative and* $\sigma > 1$.

Proposition III..10 below proves that if Assumptions II..6 and III..6 hold, then Γ is not a maximum. As a consequence we have the following corollary.

Corollary III..8 *Let Assumptions II..2, II..3, II..4, II..5, II..6, and III..6 hold. Let* $\sigma > 1$. *Then there is an optimal strategy* $(\mathbf{x}, c)$ *for problem* (P_σ) *starting at* $\bar{\mathbf{s}}$ *if and only if Assumption II..7 holds. Moreover this strategy is unique in the sense that if* $(\hat{\mathbf{x}}, \hat{c})$ *is another optimal strategy, then* $\hat{c} = c$.

Moreover, if Assumptions II..6 and III..7 holds, then the equality $\Gamma = \frac{\Gamma - \rho}{\sigma}$ is not possible when $\sigma > 1$. In fact this would mean $\rho = \Gamma(1-\sigma) < 0$. So also the following corollary holds.

Corollary III..9 *If Assumptions II..2, II..3, II..4, II..5, II..6, and III..7 hold and* $\sigma > 1$, *then there is an optimal strategy* $(\mathbf{x}, c)$ *for problem* (P_σ) *starting at* $\bar{\mathbf{s}}$ *if and only if Assumption II..7 holds. Moreover this strategy is unique in the sense that if* $(\hat{\mathbf{x}}, \hat{c})$ *is another optimal strategy, then* $\hat{c} = c$.

We now state and prove Proposition III..10 announced above.

Proposition III..10 *If Assumptions II..6 and III..6 hold, then* Γ *is not a maximum.*

Proof. Assume that Γ is a maximum so that statement (8) of Lemma A.11 applies and let

$$\mathbf{y} \in \left\{ \mathbf{x} | \mathbf{x} \geq \mathbf{0}, \mathbf{x}^T \left[\mathbf{B} - (\Gamma + \delta_{\mathbf{x}}) \mathbf{A} \right] \geq \mathbf{e}_1^T \right\}$$

have as many zero elements as possible. Then, let $\overline{\mathbf{y}}$, $\overline{\mathbf{B}}$, $\overline{\mathbf{A}}$, $\overline{\mathbf{v}}_S$ be obtained by eliminating the zero elements of vector $\mathbf{y}$, the corresponding rows of matrices $\mathbf{B}$ and $\mathbf{A}$, the columns of the same matrices that otherwise would be nought in both matrices, and the corresponding elements of vector $\mathbf{v}_S$. Note that the number of the rows of matrices $\overline{\mathbf{B}}$ and $\overline{\mathbf{A}}$ is not lower than the number of the columns of the same matrices, since Assumptions II..6 and III..6 hold. With no loss of generality assume that the last columns of matrices $\overline{\mathbf{B}}$ and $\overline{\mathbf{A}}$ correspond to the positive elements of vector $\overline{\mathbf{v}}_S$ and the last rows of the same matrices are those whose positive element of matrix $\overline{\mathbf{B}}$ corresponds to a positive element of vector $\overline{\mathbf{v}}_S$. Then, by partitioning, obtain

$$\left[\overline{\mathbf{B}}_{11} - (\Gamma + \delta_{\mathbf{x}})\,\overline{\mathbf{A}}_{11}\mathbf{0} - (\Gamma + \delta_{\mathbf{x}})\,\overline{\mathbf{A}}_{21}\overline{\mathbf{B}}_{22} - (\Gamma + \delta_{\mathbf{x}})\,\overline{\mathbf{A}}_{22}\right]\left[\mathbf{0}\overline{\mathbf{v}}_{S2}\right] = \mathbf{0}$$

where $\overline{\mathbf{v}}_{S2} > \mathbf{0}$. By partitioning vector $\overline{\mathbf{y}}$, obtain

$$\left[\overline{\mathbf{y}}_1\overline{\mathbf{y}}_2\right]^T \left[\overline{\mathbf{B}}_{11} - (\Gamma + \delta_{\mathbf{x}})\,\overline{\mathbf{A}}_{11}\mathbf{0} - (\Gamma + \delta_{\mathbf{x}})\,\overline{\mathbf{A}}_{21}\overline{\mathbf{B}}_{22} - (\Gamma + \delta_{\mathbf{x}})\,\overline{\mathbf{A}}_{22}\right] \geq \mathbf{e}_1^T.$$

Hence

$$\overline{\mathbf{y}}_1^T \left[\overline{\mathbf{B}}_{11} - (\Gamma + \delta_{\mathbf{x}})\,\overline{\mathbf{A}}_{11}\right] \geq \mathbf{e}_1^T$$

which contradicts the assumption that $\mathbf{y}$ has as many zero elements as possible. ∎

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Appendix

A Properties of admissible strategies and trajectories

In this appendix we prove various useful properties of admissible triples $(\mathbf{s},(\mathbf{x},c))$.

AA. Some basic estimates

Here we prove some simple estimates for the admissible strategies. They do not involve the number Γ.

Proposition A.1 *Let Assumptions II..2 be verified. Let*

$$\delta_{\max} = \delta_{\mathbf{x}} \vee \delta_{\mathbf{z}}, \qquad \delta_{\min} = \delta_{\mathbf{x}} \wedge \delta_{\mathbf{z}}$$

We have the following estimates for every admissible strategy $(\mathbf{x},c)$*:*

$$\dot{\mathbf{s}}_t^T \geq \mathbf{x}_t^T \mathbf{B} - \delta_{\max} \mathbf{s}_t^T - c_t \mathbf{e}_1^T;$$
$$\dot{\mathbf{s}}_t^T \leq \mathbf{x}_t^T \mathbf{B} - \delta_{\min} \mathbf{s}_t^T - c_t \mathbf{e}_1^T;$$

so that

$$\mathbf{s}_t^T \geq e^{-\delta_{\max} t} \bar{\mathbf{s}}^T + \int_0^t e^{-\delta_{\max}(t-s)} \mathbf{x}_s^T \mathbf{B} ds - \int_0^t e^{-\delta_{\max}(t-s)} c_s \mathbf{e}_1^T ds$$
$$\mathbf{s}_t^T \leq e^{-\delta_{\min} t} \bar{\mathbf{s}}^T + \int_0^t e^{-\delta_{\min}(t-s)} \mathbf{x}_s^T \mathbf{B} ds - \int_0^t e^{-\delta_{\min}(t-s)} c_s \mathbf{e}_1^T ds$$

In particular for $j = 2, ..., n$,

$$\mathbf{s}_t^T \mathbf{e}_j \geq e^{-\delta_{\max} t} \bar{\mathbf{s}}^T \mathbf{e}_j$$

while, for j *exhaustible resource*

$$\mathbf{s}_t^T \mathbf{e}_j \leq e^{-\delta_{\min} t} \bar{\mathbf{s}}^T \mathbf{e}_j.$$

Morerover there exists a constant $\lambda > 0$ *such that*

$$\left|\left|\mathbf{x}_t^T \mathbf{A}\right|\right| \leq ||\mathbf{s}_t|| \leq e^{\lambda t} ||\bar{\mathbf{s}}||, \qquad ||\mathbf{x}_t|| \leq C e^{\lambda t} ||\bar{\mathbf{s}}|| \tag{5}$$

for suitable $C > 0$.

Proof. We use the constraint $\mathbf{s}_t^T \geq \mathbf{x}_t^T \mathbf{A}$ and put it into the state equation (2). Observe that, if $\delta = \delta_{\mathbf{x}} - \delta_{\mathbf{z}} < 0$ (so $\delta_{\max} = \delta_{\mathbf{z}}$) then

$$\mathbf{0} \leq -\delta \mathbf{x}_t^T \mathbf{A} \leq -\delta \mathbf{s}_t^T$$

so that, from the state equation

$$\dot{\mathbf{s}}_t^T = \mathbf{x}_t^T [\mathbf{B} - \delta \mathbf{A}] - \delta_{\mathbf{z}} \mathbf{s}_t^T - c_t \mathbf{e}_1^T$$

it follows

$$\begin{aligned} \dot{\mathbf{s}}_t^T &\geq \mathbf{x}_t^T \mathbf{B} - \delta_{\mathbf{z}} \mathbf{s}_t^T - c_t \mathbf{e}_1^T \\ \dot{\mathbf{s}}_t^T &\leq \mathbf{x}_t^T \mathbf{B} - \delta \mathbf{s}_t^T - \delta_{\mathbf{z}} \mathbf{s}_t^T - c_t \mathbf{e}_1^T = \mathbf{x}_t^T \mathbf{B} - \delta_{\mathbf{x}} \mathbf{s}_t^T - c_t \mathbf{e}_1^T. \end{aligned}$$

Similarly, if $\delta > 0$ (so $\delta_{\max} = \delta_{\mathbf{x}}$) then

$$\mathbf{0} \geq -\delta \mathbf{x}_t^T \mathbf{A} \geq -\delta \mathbf{s}_t^T$$

so that, from the state equation it follows

$$\begin{aligned} \dot{\mathbf{s}}_t^T &\geq \mathbf{x}_t^T \mathbf{B} - \delta_{\mathbf{x}} \mathbf{s}_t^T - c_t \mathbf{e}_1^T \\ \dot{\mathbf{s}}_t^T &\leq \mathbf{x}_t^T \mathbf{B} - \delta_{\mathbf{z}} \mathbf{s}_t^T - c_t \mathbf{e}_1^T. \end{aligned}$$

This prove the first two inequalities. The second two are consequences of the comparison theorem for ODE's (see e.g. [11]) and the third ones comes as special cases (using also that exhaustible resources cannot be produced). The inequality (5) comes as follows. By Assumption II..2 for every $i \in \{1, ..., m\}$ there exists $j = j(i) \in \{1, ..., n\}$ such that $a_{ij} > 0$ so that, if $\mathbf{x}^T \mathbf{A} \leq \mathbf{s}^T$ then

$$\mathbf{x}^T \mathbf{e}_i \leq a_{ij}^{-1} \mathbf{s}^T \mathbf{e}_j$$

and we can find a nonnegative matrix $\mathbf{C}$ $n \times m$ with exactly one nonzero element for every column such that $\mathbf{x}^T \leq \mathbf{s}^T \mathbf{C}$ ($\mathbf{C}$ is such that $\mathbf{AC} \geq \mathbf{I}$ and on the $i-$th column it has all 0 element except for $j(i)$ that is a_{ij}^{-1}). Consequently we have, for $\mathbf{x}^T \mathbf{A} \leq \mathbf{s}^T$,

$$\mathbf{x}^T \mathbf{B} \leq \mathbf{s}^T \mathbf{C} \mathbf{B}.$$

Now the matrix $\mathbf{D} = \mathbf{CB}$ is $n \times n$ and nonnegative. From the above equation it follows that for every admissible strategy we have

$$\dot{\mathbf{s}}_t^T \le \mathbf{s}_t^T \mathbf{D} - \delta_{\mathbf{z}} \mathbf{s}_t^T - c_t \mathbf{e}_1^T.$$

Since the control c is positive

$$\dot{\mathbf{s}}_t^T \le \mathbf{s}_t^T \left[\mathbf{D} - \delta_{\mathbf{z}} \mathbf{I}\right].$$

From the nonnegativity of the matrix $\mathbf{D}$ it then follows

$$\mathbf{s}_t^T \le \bar{\mathbf{s}}^T e^{t[\mathbf{D} - \delta_{\mathbf{z}} \mathbf{I}]}$$

so the claim follows taking any $\lambda > \max\{\mathrm{Re}\,\mu,\ \mu \text{ eigenvalue of } \mathbf{D}\} - \delta_{\mathbf{z}}$. ■

As a consequence of the proposition above we have the following corollary.

Corollary A.2 *Let Assumptions II..2 hold. Let* $\bar{\mathbf{s}} \ge 0$ *and let* $\mathbf{s}_t$ *an admissible trajectory starting at* $\bar{\mathbf{s}}$*. Then, for* $j = 2, \ldots, n$*,*

$$\bar{\mathbf{s}}^T \mathbf{e}_j > 0 \Rightarrow \mathbf{s}_t^T \mathbf{e}_j > 0 \quad \forall t > 0.$$

Proof. The implication is obvious from Proposition A.1 above.■

AB. Properties of the number Γ

We give here some preliminary results on Γ that will be useful in the following. Here we consider a more general case than what is done in Section II. allowing also negative values of Γ. For this reason we give a different definition of Γ that coincides with the one of the Section II. when $\Gamma > 0$.

Define first

$$\mathcal{G}_0 := \left\{\gamma | \exists \mathbf{x} \in \mathbb{R}^m : \mathbf{x} \ge \mathbf{0}, \mathbf{x} \ne \mathbf{0}, \mathbf{x}^T \left[\mathbf{B} - (\gamma + \delta_{\mathbf{x}})\,\mathbf{A}\right] \ge \mathbf{0}\right\}, \qquad \Gamma_0 = \sup \mathcal{G}_0$$

$$\mathcal{G}_1 := \left\{\gamma | \exists \mathbf{x} \in \mathbb{R}^m : \mathbf{x} \ge \mathbf{0}, \mathbf{x} \ne \mathbf{0}, \mathbf{x}^T \left[\mathbf{B} - (\gamma + \delta_{\mathbf{x}})\,\mathbf{A}\right] \ge \mathbf{e}_1^T\right\}, \qquad \Gamma_1 = \sup \mathcal{G}_1$$

and

$$\Gamma := \max\{-\delta_{\mathbf{z}}, \Gamma_1\} = -\delta_{\mathbf{z}} \vee \Gamma_1.$$

Then the following hold.

Proposition A.3 *Under Assumptions II..2, II..3, II..4, we have the following*

1. $\Gamma_1 \leq \Gamma_0 < +\infty$*; moreover* $\mathcal{G}_0 \supseteq \mathcal{G}_1$ *and both are half lines;*
2. $\mathcal{G}_0 \supseteq (-\infty, -\delta_{\mathbf{x}}]$*, so that* $\Gamma_0 \geq -\delta_{\mathbf{x}}$ *and it is always a maximum;*
3. $\Gamma_0 = -\delta_{\mathbf{x}}$ *if and only if*

   a) there exists $j \in \{1, ..., n\}$ *with*

$$\mathbf{B}\mathbf{e}_j = \mathbf{0}, \qquad \mathbf{A}\mathbf{e}_j \neq \mathbf{0}$$

   b) the truncated matrices $\mathbf{B}_1$ *and* $\mathbf{A}_1$ *obtained cutting all columns* j *with* $\mathbf{B}\mathbf{e}_j = \mathbf{0}$*, and the rows* i *where* $\mathbf{e}_i^T \mathbf{A}\mathbf{e}_j > 0$ *for some* j *such that* $\mathbf{B}\mathbf{e}_j = \mathbf{0}$ *are empty or satisfy again point a) and b), recursively.*

Proof of 1). The fact that both $\mathcal{G}_0$ and $\mathcal{G}_1$ are a negative half lines follows by definition since

$$\gamma_1 < \gamma_2 \Longrightarrow \mathbf{x}^T \left[\mathbf{B} - (\gamma_1 + \delta_{\mathbf{x}})\,\mathbf{A}\right] \geq \mathbf{x}^T \left[\mathbf{B} - (\gamma_2 + \delta_{\mathbf{x}})\,\mathbf{A}\right]$$

so, if $\gamma_2 \in \mathcal{G}_0$ (or $\mathcal{G}_1$), then also $\gamma_1 \in \mathcal{G}_0$ (or $\mathcal{G}_1$). The fact that $\mathcal{G}_0 \supseteq \mathcal{G}_1$ (and so $\Gamma_1 \leq \Gamma_0$) is obvious since

$$\mathbf{x}^T \left[\mathbf{B} - (\gamma + \delta_{\mathbf{x}})\,\mathbf{A}\right] \geq \mathbf{e}_1^T \Longrightarrow \mathbf{x}^T \left[\mathbf{B} - (\gamma + \delta_{\mathbf{x}})\,\mathbf{A}\right] \geq \mathbf{0}$$

so that

$$\gamma \in \mathcal{G}_1 \Longrightarrow \gamma \in \mathcal{G}_0.$$

It remain to prove that $\Gamma_0 < +\infty$. By the Farkas Lemma (see for instance Gale's theorem for linear inequalities; [10] or [13], pp. 33-34) we have that

$$\gamma > \Gamma_0 \Longleftrightarrow \exists \mathbf{v}_F^0 \geq \mathbf{0}, : \quad (\mathbf{B} - (\gamma + \delta_{\mathbf{x}})\,\mathbf{A})\,\mathbf{v}_F^0 < \mathbf{0}.$$

So if the right hand side is true for $\mathbf{v}_F^0 = (1, 1, ..., 1)$ then surely $\gamma > \Gamma_0$. But taking $\mathbf{v}_F^0 = (1, 1, ..., 1)$ would mean, for $i = 1, ..., m$

$$\mathbf{e}_i^T (\mathbf{B} - (\gamma + \delta_{\mathbf{x}})\,\mathbf{A})\,\mathbf{v}_F^0 = \sum_{j=1}^{n} (b_{ij} - (\gamma + \delta_{\mathbf{x}})\,a_{ij}) = \sum_{j=1}^{n} b_{ij} - (\gamma + \delta_{\mathbf{x}}) \sum_{j=1}^{n} a_{ij}$$

so, (recalling that Assumption II..2 guarantees that $\sum_{j=1}^{n} a_{ij} > 0$ for every $i = 1, ..., m$) if γ is such that

$$\sum_{j=1}^{n} b_{ij} - (\gamma + \delta_{\mathbf{x}}) \sum_{j=1}^{n} a_{ij} < 0, \qquad \forall i = 1, ..., m$$

i.e.

$$\gamma > -\delta_{\mathbf{x}} + \sup_{i=1,\dots,m} \frac{\sum_{j=1}^{n} b_{ij}}{\sum_{j=1}^{n} a_{ij}}$$

then also $\gamma > \Gamma_0$. This means that

$$\Gamma_0 \leq -\delta_{\mathbf{x}} + \sup_{i=1,\dots,m} \frac{\sum_{j=1}^{n} b_{ij}}{\sum_{j=1}^{n} a_{ij}} < +\infty$$

Proof of 2). It is clear that $\gamma = \delta_{\mathbf{x}}$ belongs to $\mathcal{G}_0$ since in this case

$$\mathbf{x}^T [\mathbf{B} - (\gamma + \delta_{\mathbf{x}}) \mathbf{A}] = \mathbf{x}^T \mathbf{B} \geq \mathbf{0} \qquad \forall \mathbf{x} \geq \mathbf{0}.$$

Moreover let $\gamma_k \to \Gamma_0^-$ for $k \to +\infty$. By definition there exists a sequence $\{\mathbf{x}_k\}_{k \in \mathbb{N}}$ such that $\mathbf{x}_k \geq \mathbf{0}$, $\mathbf{x}_k \neq \mathbf{0}$ and

$$\mathbf{x}_k^T [\mathbf{B} - (\gamma_k + \delta_{\mathbf{x}}) \mathbf{A}] \geq \mathbf{0}.$$

Then setting

$$\mathbf{y}_k = \frac{\mathbf{x}_k}{||\mathbf{x}_k||_{\mathbb{R}^m}}$$

it is clear that, for every $k \in \mathbb{N}$,

$$\mathbf{y}_k^T [\mathbf{B} - (\gamma_k + \delta_{\mathbf{x}}) \mathbf{A}] \geq \mathbf{0}, \qquad \textit{and} \qquad ||\mathbf{y}_k|| = 1.$$

This implies that there exists $\mathbf{y}_0 \in S_{\mathbb{R}^m}$ such that, on a subsequence (again denoted with $\mathbf{y}_k$ for simplicity), we have $\mathbf{y}_k \to \mathbf{y}_0$ and, passing to the limit in the above inequality

$$\mathbf{y}_0^T [\mathbf{B} - (\Gamma_0 + \delta_{\mathbf{x}}) \mathbf{A}] \geq \mathbf{0}$$

which means that Γ_0 is a maximum.

Proof of 3). Finally $\Gamma_0 = -\delta_{\mathbf{x}}$ implies that for every $\varepsilon > 0$ we have

$$\mathbf{x}^T [\mathbf{B} - \varepsilon \mathbf{A}] \not\geq \mathbf{0} \qquad \forall \mathbf{x} \geq \mathbf{0}, \mathbf{x} \neq \mathbf{0}$$

Now if for all $j \in \{1, ..., n\}$ we have either $\mathbf{Be}_j \neq \mathbf{0}$, or $\mathbf{Be}_j = \mathbf{0}$ and $\mathbf{Ae}_j = \mathbf{0}$, then choosing $\mathbf{x} = (1, 1, ..., 1)$ we would have

$$\mathbf{x}^T (\mathbf{B} - \varepsilon \mathbf{A}) \mathbf{e}_j = \sum_{i=1}^{m} b_{ij} - \varepsilon \sum_{i=1}^{m} a_{ij},\ j = 1, ..., n.$$

This implies that, choosing

$$0 < \varepsilon \leq \sup_{j : \mathbf{Be}_j \neq \mathbf{0}} \frac{\sum_{i=1}^{m} a_{ij}}{\sum_{i=1}^{m} b_{ij}}$$

we would have $\mathbf{x}^T (\mathbf{B} - \varepsilon \mathbf{A}) \geq \mathbf{0}$, a contradiction. So a) holds. let us now prove b) showing that if there exist $\varepsilon > 0$ and $\mathbf{x} \geq \mathbf{0}$ with $\mathbf{x}^T (\mathbf{B} - \varepsilon \mathbf{A}) \geq \mathbf{0}$ then $\mathbf{x} = \mathbf{0}$. Let J the set of all j such that $\mathbf{Be}_j = \mathbf{0}$, and $\mathbf{Ae}_j \neq \mathbf{0}$. Then, for $\mathbf{x} \geq \mathbf{0}$, and for such a j

$$\mathbf{x}^T (\mathbf{B} - \varepsilon \mathbf{A}) \mathbf{e}_j = -\varepsilon \mathbf{x}^T \mathbf{Ae}_j = -\varepsilon \sum_{i=1}^{m} x_i \mathbf{e}_i^T \mathbf{Ae}_j$$

so if $\mathbf{x}^T (\mathbf{B} - \varepsilon \mathbf{A}) \geq \mathbf{0}$ then $x_i = 0$ for each i such that there exists $j \in J$ with $\mathbf{e}_i^T \mathbf{Ae}_j > 0$. Then finding $\mathbf{x} \geq \mathbf{0}$ such that $\mathbf{x}^T (\mathbf{B} - \varepsilon \mathbf{A}) \geq \mathbf{0}$ it is equivalent to find $\mathbf{x}_1 \geq \mathbf{0}$ such that $\mathbf{x}_1^T (\mathbf{B}_1 - \varepsilon \mathbf{A}_1) \geq \mathbf{0}$ and the above argument recursively apply.

Viceversa if a) and b) hold it is clear that $\mathbf{x}^T (\mathbf{B} - \varepsilon \mathbf{A}) \geq \mathbf{0}$ for $\mathbf{x} \geq \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$. ∎

Proposition A.4 *Under Assumptions II..2, II..3, II..4, we have the following*

1. $\Gamma_1 \in \{-\infty\} \cup [-\delta_{\mathbf{x}}, +\infty)$; *moreover*
   - $\Gamma_1 = -\infty$ *and* $\mathcal{G}_1 = \emptyset$ *if and only if* $\mathbf{Be}_1 = \mathbf{0}$ *and* $\mathbf{Ae}_1 = \mathbf{0}$.
   - $\Gamma_1 = -\delta_{\mathbf{x}}$ *and* $\mathcal{G}_1 = (-\infty, -\delta_{\mathbf{x}})$ *if and only if* $\mathbf{Be}_1 = \mathbf{0}$ *and* $\mathbf{Ae}_1 \neq \mathbf{0}$.
   - $\Gamma_1 \geq -\delta_{\mathbf{x}}$ *and* $\mathcal{G}_1 \supseteq (-\infty, -\delta_{\mathbf{x}}]$ *if and only if* $\mathbf{Be}_1 \neq \mathbf{0}$.

2. *Let* $\mathbf{Be}_1 \neq \mathbf{0}$*. If* $\Gamma_0 = -\delta_{\mathbf{x}}$ *then also* $\Gamma_1 = -\delta_{\mathbf{x}}$*. If* $\Gamma_0 > -\delta_{\mathbf{x}}$ *then* $\Gamma_1 = -\delta_{\mathbf{x}}$ *(and* $\mathcal{G}_1 = (-\infty, -\delta_{\mathbf{x}}]$*) if and only if*
   - *(i) either for every* i *such that* $b_{i1} > 0$ *there exists* j *with* $\mathbf{Be}_j = \mathbf{0}$ *and* $\mathbf{e}_i^T \mathbf{Ae}_j > 0$
   - *(ii) or for the truncated matrices (cutting rows* i *such that the above holds) we have* $\Gamma_0 = -\delta_{\mathbf{x}}$*.*
3. *If* $\Gamma_1 > -\delta_{\mathbf{x}}$ *we can have or* $\Gamma_0 = \Gamma_1$ *or* $\Gamma_0 > \Gamma_1$

Proof of 1). Note first that, clearly

$$\mathbf{Be}_1 \neq \mathbf{0} \qquad \Longleftrightarrow \qquad \exists \mathbf{x} \geq \mathbf{0} : \mathbf{x}^T \mathbf{Be}_1 \geq 1 \qquad \Longleftrightarrow \qquad \delta_{\mathbf{x}} \in \mathcal{G}_1$$

Consider first the case when $\mathbf{Be}_1 = \mathbf{0}$. If we have $\mathbf{Ae}_1 \neq \mathbf{0}$ we easily find that for every $\gamma < \delta_{\mathbf{x}}$, $\gamma \in \mathcal{G}_1$. in fact, if $\mathbf{e}_i \mathbf{Ae}_1 > 0$ then for suitable $\alpha > 0$

$$\alpha \mathbf{e}_i \left[\mathbf{B} + \varepsilon \mathbf{A}\right] \geq \mathbf{e}_1.$$

This means that in this case $\mathcal{G}_1 = (-\infty, -\delta_{\mathbf{x}})$. On the other hand if $\mathbf{Be}_1 = \mathbf{0}$ and $\mathbf{Ae}_1 = \mathbf{0}$ then it is clear that $\mathcal{G}_1$ is empty as the first component of $\mathbf{x}^T \left[\mathbf{B} - (\gamma + \delta_{\mathbf{x}}) \mathbf{A}\right]$ is always zero.

For the case when $\mathbf{Be}_1 \neq \mathbf{0}$ there is nothing to prove.

Proof of 2). Since $\Gamma_0 \geq \Gamma_1$ it is clear from point 1) above that, being $\mathbf{Be}_1 \neq \mathbf{0}$, and $\Gamma_0 = -\delta_{\mathbf{x}}$ then it must be $\Gamma_1 = -\delta_{\mathbf{x}}$. Moreover let $\Gamma_0 > -\delta_{\mathbf{x}}$. We prove that $\Gamma_1 = -\delta_{\mathbf{x}}$ (and $\mathcal{G}_1 = (-\infty, -\delta_{\mathbf{x}}])$ if and only if (i) and (ii) hold. In fact assume that $\mathcal{G}_1 = (-\infty, -\delta_{\mathbf{x}}]$. In this case we have

$$\mathbf{x}^T \left[\mathbf{B} - \varepsilon \mathbf{A}\right] \ngeq \mathbf{e}_1^T \qquad \forall \mathbf{x} \geq \mathbf{0}, \mathbf{x} \neq \mathbf{0}.$$

This happens clearly when (i) holds. If (i) does not hold it means that for all i that satisfy (i) the component x_i must be 0. So cutting all such rows we should have again

$$\mathbf{x}_1^T \left[\mathbf{B}_1 - \varepsilon \mathbf{A}_1\right] \ngeq \mathbf{e}_1^T \qquad \forall \mathbf{x}_1 \geq \mathbf{0}, \mathbf{x}_1 \neq \mathbf{0}.$$

Proof of 3). It is enough to look at the examples below. ■

Example A.5 *First an example where* $\Gamma_1 = \Gamma_0 = -\delta_{\mathbf{x}}$ *and* Γ_1 *is a maximum*

$$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}; \qquad \mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix};$$

$$\Rightarrow$$

$$\mathbf{B}-\eta\mathbf{A} = \begin{pmatrix} 1-\eta & -2\eta \\ 2-3\eta & -4\eta \end{pmatrix},$$

$$\mathbf{x}^T [\mathbf{B}-\eta\mathbf{A}] = \begin{pmatrix} x_1 (1-\eta) + x_2 (2-3\eta) \\ -2\eta x_1 - 4\eta x_2 \end{pmatrix}$$

so

$$\mathbf{x}^T [\mathbf{B}-\eta\mathbf{A}] \geq \mathbf{0} \qquad \Longleftrightarrow \qquad \begin{cases} x_1 (1-\eta) + x_2 (2-3\eta) \geq 0 \\ -2\eta x_1 - 4\eta x_2 \geq 0 \end{cases}.$$

Clearly for $\eta > 0$ *the above system admits no solution* $\mathbf{x} \geq \mathbf{0}$, $\mathbf{x} \neq \mathbf{0}$. *Since* $\mathbf{Be}_1 \neq \mathbf{0}$ *then we have* $-\delta_{\mathbf{x}} \leq \Gamma_1 \leq \Gamma_0 = -\delta_{\mathbf{x}}$ *and so the claim.*

If above we modify $\mathbf{A}$ *taking*

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$$

then we see that $\Gamma_0 = \Gamma_1 = 1 + \delta_{\mathbf{x}}$ *and* Γ_1 *is not a maximum.* ■

Example A.6 *A case where one needs a first truncation to get the if and only if condition of Proposition A.3 and then* $\Gamma_0 = -\delta_{\mathbf{x}}$.

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \qquad \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here $\Gamma_0 = \Gamma_1 = \delta_{\mathbf{x}}$. ■

Example A.7 *Case when* $\Gamma_0 > \Gamma_1 = -\delta_{\mathbf{x}}$

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \qquad \mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\Gamma_0 = 1 - \delta_{\mathbf{x}}$ *or*

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \qquad \mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here $\Gamma_0 > \Gamma_1 = -\delta_{\mathbf{x}}$. ■

Example A.8 *Case when* $\Gamma_0 > \Gamma_1 = -\delta_{\mathbf{x}}$

$$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \qquad \mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix};$$

$$\mathbf{B}-\eta\mathbf{A} = \mathbf{B}-\eta\mathbf{A} = \begin{pmatrix} 1-2\eta & -\eta \\ 0 & 1-2\eta \end{pmatrix}$$

$$\mathbf{x}^T\left[\mathbf{B}-\eta\mathbf{A}\right] = \begin{pmatrix} x_1\left(1-2\eta\right) \\ -\eta x_1 + \left(1-\eta\right)x_2 \end{pmatrix}$$

Here $\Gamma_0 = 1$ *and* $\Gamma_1 = \frac{1}{2}$ *and it is not a maximum.* ■

Example A.9 *Case when* $\Gamma_0 = \Gamma_1 > -\delta_{\mathbf{x}}$

$$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \qquad \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix};$$

$$\mathbf{B}-\eta\mathbf{A} = \begin{pmatrix} 1-\eta & -\eta \\ 0 & 1-2\eta \end{pmatrix},$$

$$\mathbf{x}^T\left[\mathbf{B}-\eta\mathbf{A}\right] = \begin{pmatrix} x_1\left(1-\eta\right) \\ -\eta x_1 + \left(1-2\eta\right)x_2 \end{pmatrix}$$

Here $\Gamma_0 = 1$ *and* $\Gamma_1 = 1$ *and it is not a maximum.* ■

Remark A.10 *Note that, if* $\mathbf{B}\mathbf{e}_1 = \mathbf{0}$ *then we know that the first good (the only consumption good) is not producible. This implies that our optimal control problem reduces to a one dimensional problem with state equation (call* $s_{1t} = \mathbf{s}_t^T\mathbf{e}_1$*)*

$$\dot{s}_{1t} = -\delta_{\min}s_{1t} - c_t \qquad \textit{if } \mathbf{A}\mathbf{e}_1 \neq 0$$
$$\dot{s}_{1t} = -\delta_{\mathbf{z}}s_{1t} - c_t \qquad \textit{if } \mathbf{A}\mathbf{e}_1 = 0$$

and the same utility functional to maximize

$$\int_0^{+\infty} e^{-\rho t} u_\sigma (c_t)\, dt.$$

This comes from the fact that the role of the production strategy $\mathbf{x}$ *now is only the one of choosing at what rate to depreciate the available stock of the consumption good. Recall that in this case, for* $s_{10} > 0$*, we have existence if and only if* $\rho > -\delta_{\min} (1-\sigma)$ *if* $\mathbf{A}\mathbf{e}_1 \neq 0$ *and* $\rho > -\delta_{\mathbf{z}} (1-\sigma)$ *if* $\mathbf{A}\mathbf{e}_1 = 0$ *(*$s_{10} > 0$ *since Assumption II..4 holds). So in this case the if and only if conditions for existence and uniqueness are known and in this case our results are easily proved in a simple (and well known) one dimensional setting.* ■

In view of the above remark we consider from now on the case when $\mathbf{B}\mathbf{e}_1 \neq \mathbf{0}$. Since this means that $\Gamma_1 \geq -\delta_{\mathbf{x}}$ and $\mathcal{G}_1 \supseteq (-\infty, -\delta_{\mathbf{x}}]$, then $\Gamma \geq (-\delta_{\mathbf{x}}) \vee (-\delta_{\mathbf{z}}) = -\delta_{\min}$.

AC. Estimates involving Γ

In this section, we provide estimates that are useful to prove the main results stated in Section III..

Troughout all this section we will assume that Assumptions II..2, II..3, II..4, II..5, hold without explicitely mentioning them. We observe that we do not assume that Γ is strictly positive.

We start by the following Lemma that provides the basis for estimates of the state and control trajectories.

Lemma A.11 *If* Γ_1 *is not a maximum*

$$\gamma \geq \Gamma_1 \Longleftrightarrow \exists \mathbf{v}_F \geq \mathbf{0} : \quad (\mathbf{B} - (\gamma + \delta_{\mathbf{x}}) \mathbf{A}) \mathbf{v}_F \leq \mathbf{0}, \quad \mathbf{e}_1^T \mathbf{v}_F = 1. \quad (6)$$

If Γ *is a maximum*

$$\gamma > \Gamma_1 \Longleftrightarrow \exists \mathbf{v}_F \geq \mathbf{0} : \quad (\mathbf{B} - (\gamma + \delta_{\mathbf{x}}) \mathbf{A}) \mathbf{v}_F \leq \mathbf{0}, \quad \mathbf{e}_1^T \mathbf{v}_F = 1; \quad (7)$$

$$\exists \mathbf{v}_S \geq \mathbf{0} : \quad (\mathbf{B} - (\Gamma_1 + \delta_{\mathbf{x}}) \mathbf{A}) \mathbf{v}_S \leq \mathbf{0}, \mathbf{v}_S \neq \mathbf{0}. \quad (8)$$

Moreover,

$$\mathbf{e}_1^T \mathbf{v}_S = \mathbf{y}^T \left[\mathbf{B} - (\Gamma_1 + \delta_{\mathbf{x}}) \mathbf{A} \right] \mathbf{v}_S = 0 \tag{9}$$

where

$$\mathbf{y} \in \left\{ \mathbf{x} | \mathbf{x} \geq \mathbf{0}, \mathbf{x}^T \left[\mathbf{B} - (\Gamma_1 + \delta_{\mathbf{x}}) \mathbf{A} \right] \geq \mathbf{e}_1^T \right\}.$$

Proof. Statements (6) and (7) are obvious applications of the Farkas Lemma (see for instance Gale's theorem for linear inequalities; [10] or [13], pp. 33-34). Assume now that statement (8) does not hold and obtain, once again from the Farkas Lemma (see for instance Motzkin's theorem of the alternative; [15] or [13], pp. 28-29), that

$$\exists \mathbf{w} \geq \mathbf{0} : \mathbf{w}^T \left[\mathbf{B} - (\Gamma_1 + \delta_{\mathbf{x}}) \mathbf{A} \right] > \mathbf{0}^T.$$

Hence there is $\phi > 0$ so large and $\eta > 0$ so small that

$$\phi \mathbf{w}^T \left[\mathbf{B} - (\Gamma_1 + \delta_{\mathbf{x}}) \mathbf{A} \right] \geq \mathbf{e}_1^T + \eta \phi \mathbf{w}^T \mathbf{A}$$

Hence a contradiction since $\Gamma_1 = \sup \mathcal{G}_1$. By remarking that

$$0 \geq \mathbf{y}^T \left[\mathbf{B} - (\Gamma_1 + \delta_{\mathbf{x}}) \mathbf{A} \right] \mathbf{v}_S \geq \mathbf{e}_1^T \mathbf{v}_S \geq 0$$

the proof is completed. ∎

The next lemma and the subsequent corollary give various estimates for the state and control variables that will be the basis for the proof of existence and nonexistence (see [6, p.30] for analogous arguments in the one-dimensional case). Note that for the case $\sigma \in (0, 1)$ we are interested in an estimate from above of the integral $\int_0^t e^{-\rho s} c_s^{1-\sigma} ds$ giving finiteness of the value function for $\rho - (\Gamma + \varepsilon)(1 - \sigma) > 0$ (so we need terms that remain bounded when $t \to +\infty$), while for the case $\sigma \in (1, +\infty)$ we are interested in an estimate from below of the same integral to show that the value function equal to $-\infty$ when $\rho - (\Gamma + \varepsilon)(1 - \sigma) < 0$, (so we need terms that explode when $t \to +\infty$). These different targets require to use different estimates with different methods of proof. Of course, both methods can be applied to both cases yielding however estimates that are not useful for our target. In order to simplify notation we will set

$$a_\varepsilon = \rho - (\Gamma + \varepsilon)(1 - \sigma)$$

Recall that, by what is said at the end of the previous subsection we have $\Gamma \geq -\delta_{\min}$.

Lemma A.12 *Let $\sigma > 0$, $\bar{\mathbf{s}} \in \mathbb{R}^n$, $\bar{\mathbf{s}} \geq 0$. Fix $\varepsilon > 0$ ($\varepsilon = 0$ when $\Gamma = -\delta_{\mathbf{z}} > \Gamma_1$ or when Γ_1 is not a maximum). For every $0 \leq t < +\infty$, $\bar{\mathbf{s}} \in \mathbb{R}^n$, $\bar{\mathbf{s}} \geq 0$ we have, for every admissible control strategy $(\mathbf{x}, c) \in \mathcal{A}(\bar{\mathbf{s}})$,*

$$\mathbf{s}_t^T \mathbf{v}_F \leq e^{(\Gamma+\varepsilon)t} \bar{\mathbf{s}}^T \mathbf{v}_F, \tag{10}$$

and, for $\eta \in \mathbb{R}$

$$\int_0^t e^{-\eta s} \mathbf{s}_s^T \mathbf{v}_F ds \leq \bar{\mathbf{s}}^T \mathbf{v}_F \frac{e^{(\Gamma+\varepsilon-\eta)t} - 1}{\Gamma + \varepsilon - \eta}; \quad \eta \neq \Gamma + \varepsilon$$
$$\int_0^t e^{-\eta s} \mathbf{s}_s^T \mathbf{v}_F ds \leq \bar{\mathbf{s}}^T \mathbf{v}_F t; \quad \eta = \Gamma + \varepsilon \tag{11}$$

and, setting $I(t) := \int_0^t e^{-(\Gamma+\varepsilon)s} c_s ds$,

$$I(t) + e^{-(\Gamma+\varepsilon)t} \mathbf{x}_t^T \mathbf{A} \mathbf{v}_F \leq \bar{\mathbf{s}}^T \mathbf{v}_F, \tag{12}$$

and also

$$\mathbf{x}_\tau^T \mathbf{A} \mathbf{v}_F e^{-\eta\tau} + \int_t^\tau e^{-\eta s} c_s ds \leq e^{-\eta t} \bar{\mathbf{s}}_t^T \mathbf{v}_F e^{(\Gamma+\varepsilon-\eta)^+(\tau-t)} \tag{13}$$

Moreover, setting $a_\varepsilon = \rho - (\Gamma + \varepsilon)(1 - \sigma)$ we have for $\sigma \in (0, 1)$

$$\int_0^t e^{-\rho s} c_s^{1-\sigma} ds \tag{14}$$
$$\leq \left[\bar{\mathbf{s}}^T \mathbf{v}_F\right]^{1-\sigma} \left[t^\sigma e^{-a_\varepsilon t} + [a_\varepsilon]^+ \int_0^t s^\sigma e^{-a_\varepsilon s} ds \right]$$

while, for $\sigma \in (1, +\infty)$

$$\int_0^t e^{-\rho s} c_s^{1-\sigma} ds \geq t^\sigma \left(\bar{\mathbf{s}}^T \mathbf{v}_F \right)^{1-\sigma} e^{-(a_\varepsilon)^+ t} \tag{15}$$

and, for $\sigma = 1$ *and* $\rho \geq 0$

$$\int_0^t e^{-\rho s} \log c_s ds \leq e^{-\rho t} t \left[(\Gamma + \varepsilon)\, t + \log \frac{\bar{\mathbf{s}}^T \mathbf{v}_F}{t} \right] + \rho \int_0^t e^{-\rho s} s \left[(\Gamma + \varepsilon)\, s + \log \frac{\bar{\mathbf{s}}^T \mathbf{v}_F}{s} \right] ds \tag{16}$$

Proof. We prove the seven inequalities (10)–(16) in order of presentation.

(1) First we observe that, by multiplying the state equation (2) by $\mathbf{v}_F$ we obtain

$$\dot{\mathbf{s}}_t^T v_F = -\delta_{\mathbf{z}} \mathbf{s}_t^T \mathbf{v}_F + \mathbf{x}_t^T [\mathbf{B} - \delta \mathbf{A}]\, \mathbf{v}_F - c_t \mathbf{e}_1^T \mathbf{v}_F \quad t \in (0, +\infty),$$
$$\mathbf{s}_0^T \mathbf{v}_F = \bar{\mathbf{s}}^T \mathbf{v}_F \geq 0.$$

Now for every $\mathbf{x}$ and ε,

$$\mathbf{x}^T [\mathbf{B} - \delta \mathbf{A}] = \mathbf{x}^T [\mathbf{B} - (\Gamma_1 + \varepsilon + \delta_{\mathbf{x}})\, \mathbf{A}] + (\Gamma_1 + \varepsilon + \delta_{\mathbf{z}})\, \mathbf{x}^T \mathbf{A}$$

Moreover for $\mathbf{x} \geq 0$ we have by (7)-(6) $\mathbf{x}^T [\mathbf{B} - (\Gamma_1 + \varepsilon + \delta_{\mathbf{x}})\, \mathbf{A}]\, \mathbf{v}_F \leq 0$ with the agrement that $\varepsilon = 0$ when Γ_1 is not a maximum. Then

$$\begin{aligned} \dot{\mathbf{s}}_t^T \mathbf{v}_F &= -\delta_{\mathbf{z}} \mathbf{s}_t^T \mathbf{v}_F + \mathbf{x}_t^T [\mathbf{B} - (\Gamma_1 + \varepsilon + \delta_{\mathbf{x}})\, \mathbf{A}]\, \mathbf{v}_F + (\Gamma_1 + \varepsilon + \delta_{\mathbf{z}})\, \mathbf{x}^T \mathbf{A} \mathbf{v}_F - c_t \\ &\leq -\delta_{\mathbf{z}} \mathbf{s}_t^T \mathbf{v}_F + (\Gamma_1 + \varepsilon + \delta_{\mathbf{z}})\, \mathbf{x}^T \mathbf{A} \mathbf{v}_F - c_t \end{aligned}$$

Let now $\Gamma_1 \geq -\delta_{\mathbf{z}}$ (which means that $\Gamma_1 = \Gamma$). Then from the the constraint $\mathbf{s}_t^T \geq \mathbf{x}_t^T \mathbf{A}$ and from the nonnegativity of c, we get

$$\dot{\mathbf{s}}_t^T \mathbf{v}_F \leq (\Gamma_1 + \varepsilon)\, \mathbf{s}_t^T \mathbf{v}_F - c_t \leq (\Gamma_1 + \varepsilon)\, \mathbf{s}_t^T \mathbf{v}_F \qquad t \in (0, +\infty), \tag{17}$$

and so, by integrating on $[0, t]$ and using the Gronwall lemma (see e.g. [2, p. 218]) we get the first claim (10).

Take now $\Gamma_1 < -\delta_{\mathbf{z}}$ in this case $\Gamma_1 < \Gamma = -\delta_{\mathbf{z}}$ and for ε small we have $\Gamma_1 + \varepsilon + \delta_{\mathbf{z}} < 0$ so that $(\Gamma_1 + \varepsilon + \delta_{\mathbf{z}})\, \mathbf{x}^T \mathbf{A} \mathbf{v}_F \leq 0$ which gives

$$\dot{\mathbf{s}}_s^T \mathbf{v}_F \leq -\delta_{\mathbf{z}} \mathbf{s}_t^T \mathbf{v}_F - c_t \leq -\delta_{\mathbf{z}} \mathbf{s}_t^T \mathbf{v}_F, \qquad s \in (0, +\infty), \tag{18}$$

and so the claim (in this case we clearly can take $\varepsilon = 0$).

(2) To prove inequality (11) we multiply the inequality (10) by $e^{-\eta s}$ and integrate on $[0,t]$. We obtain for $\eta \neq \Gamma + \varepsilon$

$$\begin{aligned}\int_0^t e^{-\eta s}\mathbf{s}_s^T\mathbf{v}_F ds &\leq \int_0^t e^{-\eta s}e^{(\Gamma+\varepsilon)s}\overline{\mathbf{s}}^T\mathbf{v}_F ds \\ &= \overline{\mathbf{s}}^T\mathbf{v}_F e\int_0^t e^{(\Gamma+\varepsilon-\eta)s}ds \\ &= \overline{\mathbf{s}}^T\mathbf{v}_F\frac{e^{(\Gamma+\varepsilon-\eta)t}-1}{\Gamma+\varepsilon-\eta}\end{aligned}$$

whereas if $\eta = \Gamma + \varepsilon$ we have

$$\begin{aligned}\int_0^t e^{-\eta s}\mathbf{s}_s^T\mathbf{v}_F ds &\leq \int_0^t e^{-\eta s}e^{(\Gamma+\varepsilon)s}\overline{\mathbf{s}}^T\mathbf{v}_F ds \\ &= \overline{\mathbf{s}}^T\mathbf{v}_F t.\end{aligned}$$

(3) For the third claim (12) we observe that, from (17) and (18) (taking $\varepsilon = 0$ when allowed)

$$\dot{\mathbf{s}}_s^T\mathbf{v}_F \leq (\Gamma+\varepsilon)\,\mathbf{s}_s^T\mathbf{v}_F - c_s \qquad \forall s \in [0,t] \tag{19}$$

so that, by the comparison theorem for ODE's

$$\mathbf{s}_t^T\mathbf{v}_F \leq \overline{\mathbf{s}}^T\mathbf{v}_F e^{(\Gamma+\varepsilon)t} - \int_0^t e^{(\Gamma+\varepsilon)(t-s)}c_s ds$$

which implies

$$\int_0^t e^{-(\Gamma+\varepsilon)s}c_s ds + e^{-(\Gamma+\varepsilon)t}\mathbf{s}_t^T\mathbf{v}_F \leq \overline{\mathbf{s}}^T\mathbf{v}_F$$

From the inequality $\mathbf{x}_t^T\mathbf{A}\mathbf{v}_F \leq \mathbf{s}_t^T\mathbf{v}_F$ we get the claim.

(4) The fourth claim (13) easily follows by multiplying both sides of (19) by $e^{-\eta s}$ and then integrating. In fact (taking for simplicity the case $\tau = 0$) we have

$$0 \leq e^{-\eta s}c_s \leq e^{-\eta s}\left[(\Gamma+\varepsilon)\,\mathbf{s}_s^T\mathbf{v}_F - \dot{\mathbf{s}}_s^T\mathbf{v}_F\right] \qquad \forall s \in [t,\tau]$$

and integrating and using that $\mathbf{x}_t^T \mathbf{A} \mathbf{v}_F \leq \mathbf{s}_t^T \mathbf{v}_F$

$$\begin{aligned}\int_0^t e^{-\eta s} c_s ds &\leq \int_0^t e^{-\eta s} \left[(\Gamma + \varepsilon) \mathbf{s}_s^T \mathbf{v}_F - \dot{\mathbf{s}}_s^T \mathbf{v}_F \right] ds \\ &= \int_0^t e^{-\eta s} (\Gamma + \varepsilon) \mathbf{s}_s^T \mathbf{v}_F ds - e^{-\eta t} \mathbf{s}_t^T \mathbf{v}_F + \bar{\mathbf{s}}^T \mathbf{v}_F - \eta \int_0^t e^{-\eta s} \mathbf{s}_s^T \mathbf{v}_F d \\ &\leq \bar{\mathbf{s}}^T \mathbf{v}_F - e^{-\eta t} \mathbf{x}_t^T \mathbf{A} \mathbf{v}_F + (\Gamma + \varepsilon - \eta) \int_0^t e^{-\eta s} \mathbf{s}_s^T \mathbf{v}_F ds\end{aligned}$$

Now, if $\eta \geq \Gamma + \varepsilon$ the above inequality implies

$$\int_0^t e^{-\eta s} c_s ds + e^{-\eta t} \mathbf{x}_t^T \mathbf{A} \mathbf{v}_F \leq \bar{\mathbf{s}}^T \mathbf{v}_F$$

while, for $\eta < \Gamma + \varepsilon$ we get, by using (11),

$$\begin{aligned}\int_0^t e^{-\eta s} c_s ds + e^{-\eta t} \mathbf{x}_t^T \mathbf{A} \mathbf{v}_F &\leq \bar{\mathbf{s}}^T \mathbf{v}_F + (\Gamma + \varepsilon - \eta) \int_0^t e^{-\eta s} \mathbf{s}_s^T \mathbf{v}_F ds \\ &\leq \bar{\mathbf{s}}^T \mathbf{v}_F + \bar{\mathbf{s}}^T \mathbf{v}_F \left[e^{(\Gamma + \varepsilon - \eta) t} - 1 \right] \\ &= \bar{\mathbf{s}}^T \mathbf{v}_F e^{(\Gamma + \varepsilon - \eta) t}\end{aligned}$$

which gives the fourth claim (13)

(5) Concerning the fifth inequality (14) setting (see e.g. [6, p. 30])

$$h(s) = \int_0^s e^{-(\Gamma + \varepsilon)(1 - \sigma) r} c_r^{1 - \sigma} dr$$

we have, by Jensen's inequality, for $\sigma \in (0, 1)$

$$h(s) \leq s \left[\frac{1}{s} \int_0^s e^{-(\Gamma + \varepsilon) r} c_r dr \right]^{1 - \sigma} = s^\sigma I(s)^{1 - \sigma} \tag{20}$$

Now, integrating by parts we obtain (this holds in fact for $\sigma > 0$, $\sigma \neq 1$),

$$\begin{aligned}\int_0^t e^{-\rho r} c_r^{1 - \sigma} ds &= \int_0^t e^{-(\rho - (\Gamma + \varepsilon)(1 - \sigma)) r} e^{-(\Gamma + \varepsilon)(1 - \sigma) r} c_r^{1 - \sigma} ds \qquad (21) \\ &= \left[e^{-a_\varepsilon s} h(s) \right]_0^t + \int_0^t a_\varepsilon e^{-a_\varepsilon s} h(s) ds \\ &= e^{-a_\varepsilon t} h(t) + \int_0^t a_\varepsilon e^{-a_\varepsilon s} h(s) ds.\end{aligned}$$

If we apply the inequality (12) to (20) we obtain

$$h(s) \leq s^{\sigma}[\bar{\mathbf{s}}^T \mathbf{v}_F]^{1-\sigma}$$

which yields, together with (21),

$$\int_0^t e^{-\rho s} c_s^{1-\sigma} ds$$

$$\leq e^{-a_\varepsilon t} t^{\sigma} [\bar{\mathbf{s}}^T \mathbf{v}_F]^{1-\sigma} + [a_\varepsilon]^+ \int_0^t e^{-a_\varepsilon s} s^{\sigma} [\bar{\mathbf{s}}^T \mathbf{v}_F]^{1-\sigma} ds$$

which gives the claim.

(6) To prove inequality (15) dealing with the case when $\sigma \in (1, +\infty)$ we apply directly the Jensen inequality to the integral $\int_0^t e^{-\rho s} c_s^{1-\sigma} ds$. In fact

$$\begin{aligned}
\int_0^t e^{-\rho s} c_s^{1-\sigma} ds &= t \frac{1}{t} \int_0^t \left(e^{-\frac{\rho}{1-\sigma} s} c_s \right)^{1-\sigma} ds \\
&\geq t \left[\frac{1}{t} \int_0^t e^{-\frac{\rho}{1-\sigma} s} c_s ds \right]^{1-\sigma} \\
&= t^{\sigma} \left[\int_0^t e^{-\frac{\rho}{1-\sigma} s} c_s ds \right]^{1-\sigma}
\end{aligned}$$

so that, by inequality (13) with $\eta = \frac{\rho}{1-\sigma}$ we get , (recalling that $\Gamma + \varepsilon - \frac{\rho}{1-\sigma} = \frac{a_\varepsilon}{\sigma - 1}$)

$$\int_0^t e^{-\rho s} c_s^{1-\sigma} ds \geq t^{\sigma} \left[\int_0^t e^{-\frac{\rho}{1-\sigma} s} c_s ds \right]^{1-\sigma}$$

$$\geq t^{\sigma} \left[\bar{\mathbf{s}}^T \mathbf{v}_F e^{\left(\frac{a_\varepsilon}{\sigma-1}\right)^+ t} \right]^{1-\sigma}$$

$$= t^{\sigma} \left(\bar{\mathbf{s}}^T \mathbf{v}_F \right)^{1-\sigma} e^{(1-\sigma)\left(\frac{a_\varepsilon}{\sigma-1}\right)^+ t} = t^{\sigma} \left(\bar{\mathbf{s}}^T \mathbf{v}_F \right)^{1-\sigma} e^{-(a_\varepsilon)^+ t}$$

(7) Inequality (16) follows by similar arguments. In fact, calling

$$\begin{aligned}
h(s) &= \int_0^s \log c_r dr = \int_0^s \log e^{(\Gamma+\varepsilon) r} e^{-(\Gamma+\varepsilon) r} c_r dr \\
&= \int_0^s (\Gamma + \varepsilon) r dr + \int_0^s \log \left(e^{-(\Gamma+\varepsilon) r} c_r \right) dr
\end{aligned}$$

we have, because of Jensen's inequality and of (12)

$$\begin{aligned} h(s) &\le (\Gamma+\varepsilon)\frac{s^2}{2} + s\log\left[\frac{1}{s}\int_0^s e^{-(\Gamma+\varepsilon)r}c_r dr\right] \\ &\le (\Gamma+\varepsilon)\frac{s^2}{2} + s\log\left[\frac{1}{s}\bar{\mathbf{s}}^T\mathbf{v}_F\right]. \end{aligned} \tag{22}$$

Now, integrating by parts as in (21), we obtain

$$\int_0^t e^{-\rho s}\log c_s ds = e^{-\rho t}h(t) + \int_0^t \rho e^{-\rho s}h(s)ds.$$

which, together with (22) and (12), gives, for $\rho \ge 0$

$$\int_0^t e^{-\rho s}\log c_s ds$$

$$\begin{aligned} &\le e^{-\rho t}\left((\Gamma+\varepsilon)\frac{t^2}{2} + t\log\left[\frac{1}{t}\bar{\mathbf{s}}^T\mathbf{v}_F\right]\right) \\ &+ \rho\int_0^t e^{-\rho s}\left((\Gamma+\varepsilon)\frac{s^2}{2} + s\log\left[\frac{1}{s}\bar{\mathbf{s}}^T\mathbf{v}_F\right]\right)ds \\ &= te^{-\rho t}\left[\frac{(\Gamma+\varepsilon)}{2}t + \log\frac{\bar{\mathbf{s}}^T\mathbf{v}_F}{t}\right] \\ &+ \rho\int_0^t e^{-\rho s}s\left[\frac{(\Gamma+\varepsilon)}{2}s + \log\frac{\bar{\mathbf{s}}^T\mathbf{v}_F}{s}\right]ds \end{aligned}$$

which completes the proof. ∎

Remark A.13 We observe that, when the matrix A is square and indecomposable and $B = I$ then a possible choice of v_F is the Frobenius right eigenvector of A that we call $\mathbf{v}_{PF} > 0$. In this case, due to the strict positivity of $\mathbf{v}_{PF}$ (which comes from indecomposability of $\mathbf{A}$), the above estimates are in fact estimates on every component of the vectors $\mathbf{s}_t$, $\mathbf{x}_t$. In the general case they give estimates on the first component of the state variable, which is enough to derive conditions for the functional U_σ to be well defined and finite for every admissible strategy when $\sigma \in (0,1)$ and Assumption II..7 holds, i.e. $a_0 > 0$, and for U_σ to be $-\infty$ for every admissible strategy when $\sigma \in (1,+\infty)$ and $a_0 < 0$. The case $\sigma = 1$ is treated similarly. ∎

Remark A.14 *Observe that we cannot have $\bar{\mathbf{s}}^T \mathbf{v}_F = 0$ since this would mean that Assumption II..4 do not hold.* ∎

B Proof of the existence and inexistence theorem

In this section we prove the existence and non existence results stated above as Theorems III..1, III..2. The proof uses compactness arguments and to our knowledge, the results given in the literature do not apply to this case (see [5] and [18] for similar results). For this reason we give a complete proof. Troughout this subsection we will assume that Assumptions II..2, II..3, II..4 and II..5 hold true without mentioning them. We will clarify when other assumptions are used.

We note that in fact we do prove a more general result since we treat also the cases when Γ can be ≤ 0.

We start from an easy corollary of the above estimates that gives already some cases of nonexistence (recall that we denote by Γ_E is the Euler Gamma function).

Corollary B.1 *Let $a_\varepsilon = \rho - (\Gamma + \varepsilon)(1 - \sigma)$. Then, for any $\bar{\mathbf{s}} \geq \mathbf{0}$ the following hold.*

(i) Let $\sigma \in (0,1)$. Then for any $(\mathbf{x}, c) \in \mathcal{A}(\bar{\mathbf{s}})$ and $\varepsilon > 0$ ($\varepsilon = 0$ when allowed) such that $a_\varepsilon > 0$ we have,

$$0 \leq U_\sigma(c) \leq \frac{a_\varepsilon}{1-\sigma} \frac{\Gamma_E(1+\sigma)}{a_\varepsilon^{1+\sigma}} \left[\bar{\mathbf{s}}^T \mathbf{v}_F\right]^{1-\sigma} < +\infty \tag{23}$$

(ii) Let $\sigma = 1$ (in this case for every ε we have $a_\varepsilon = \rho$). If $\rho > 0$ then or any $(\mathbf{x}, c) \in \mathcal{A}(\bar{\mathbf{s}})$ we have

$$U_\sigma(c) \leq \rho \int_0^{+\infty} e^{-\rho s} s \left[(\Gamma + \varepsilon)\, s + \log \frac{\bar{\mathbf{s}}^T \mathbf{v}_F}{s}\right] ds < +\infty. \tag{24}$$

If $\rho \leq 0$ and $\Gamma < 0$, then $U_\sigma(c) = -\infty$ for every $(\mathbf{x}, c) \in \mathcal{A}(\bar{\mathbf{s}})$. The same if $\rho \leq 0$ and $\Gamma_1 = 0$ and Γ_1 is not a maximum.

(iii) If $\sigma > 1$, then

$$U_\sigma(c) \leq 0.$$

Moreover if $a_0 < 0$ then $U_\sigma(c) = -\infty$ for every $(\mathbf{x}, c) \in \mathcal{A}(\bar{\mathbf{s}})$. The same if $a_0 = 0$ and $\Gamma_1 < -\delta_{\mathbf{z}}$ or Γ_1 is not a maximum.

Proof.

(i) The estimate (23) comes from (14) letting $t \to +\infty$ and using that

$$\int_0^{+\infty} s^\sigma e^{-a_\varepsilon s} ds = \frac{\Gamma_E(1+\sigma)}{a_\varepsilon^{1+\sigma}}.$$

(ii) If $\sigma = 1$ and $\rho > 0$ we get (24) using (16) and letting $t \to +\infty$. If $\rho = 0$ then from (16) we get

$$\int_0^t \log c_s ds \leq t \left[(\Gamma + \varepsilon) t + \log \frac{\bar{\mathbf{s}}^T \mathbf{v}_F}{t} \right]$$

so for $\Gamma < 0$ (or for $\Gamma = 0$ and Γ is not a maximum) we get in the limit for $t \to +\infty$, that $U_1(c) = -\infty$. If $\rho < 0$ and $\Gamma < 0$ (or for $\Gamma = 0$ and Γ is not a maximum) then

$$\int_0^t e^{-\rho s} \log c_s ds = \int_0^t e^{-\rho s} [\log c_s]^+ ds + \int_0^t e^{-\rho s} [\log c_s]^- ds.$$

Now, thanks to the nonpositivity of the negative part and to the fact that $e^{-\rho s} \geq 1$,

$$\int_0^t e^{-\rho s} [\log c_s]^- ds \leq \int_0^t [\log c_s]^- ds.$$

Since the right hand side goes to $-\infty$ as $t \to +\infty$ (thanks to the case $\rho = 0$) we have $\int_0^{+\infty} e^{-\rho s} [\log c_s]^- ds = -\infty$. By admissibility this implies that the integral of the positive part is finite and so $U_1(c) = -\infty$.

(iii) When $\sigma > 1$ it is obvious that $U_\sigma(c) \leq 0$ by construction. Moreover using (15) we get

$$\frac{1}{1-\sigma}\int_0^t e^{-\rho s} c_s^{1-\sigma} ds \leq t^\sigma \left(\overline{\mathbf{s}}^T \mathbf{v}_F\right)^{1-\sigma} e^{-(a_\varepsilon)^+ t}.$$

Then, if $a_0 < 0$ we have $a_\varepsilon = a_0 - \varepsilon(1-\sigma) < 0$ for $\varepsilon > 0$ sufficiently small, so that the above becomes

$$\frac{1}{1-\sigma}\int_0^t e^{-\rho s} c_s^{1-\sigma} ds \leq t^\sigma \left(\overline{\mathbf{s}}^T \mathbf{v}_F\right)^{1-\sigma}$$

and letting $t \to +\infty$ we get $U_\sigma(c) = -\infty$ for every admissible strategy. The same happens if $a_0 = 0$ and $\Gamma = -\delta_{\mathbf{z}} > \Gamma_1$ or Γ_1 is not a maximum. ∎

Remark B.2 *This result shows in particular that, when $a_0 > 0$ and $\sigma \in (0,1)$, the intertemporal utility functional $U_\sigma(c)$ is finite and uniformly bounded for every admissible production-consumption strategy (while for $\sigma \geq 1$ it is only bounded from above). In the cases when*

1. $\sigma = 1$, $\rho \leq 0$, $\Gamma < 0$;
2. $\sigma = 1$, $\rho \leq 0$ *and* $\Gamma = 0$ *and* Γ *is not a maximum;*
3. $\sigma > 1$, $a_0 < 0$;
4. $\sigma > 1$, $a_0 = 0$ *and* $\Gamma = -\delta_{\mathbf{z}} > \Gamma_1$ *or when* Γ_1 *is not a maximum;*

Corollary B.1 shows that there are no optimal strategies in the sense of Definition II..1 since all stategies hav utility $-\infty$. ∎

We now have the following result.

Proposition B.3 *Let either $\sigma \in (0,1)$ and $a_0 < 0$ or $\sigma = 1$, $a_0 \leq 0$ and $\Gamma > 0$. Then, given any $\overline{\mathbf{s}} \geq \mathbf{0}$ satisfying Assumption II..5 for each $j = 2, ..., n$, we can find an admissible strategy $(\mathbf{x}, c) \in \mathcal{A}(\overline{\mathbf{s}})$ such that $U_\sigma(c) = +\infty$. If on the other hand $\sigma \geq 1$ and $a_0 > 0$ and then there exists an admissible strategy with $U_\sigma(c) > -\infty$.*

Proof. Consider first the case when $\sigma \in (0,1)$ and $a_0 < 0$. Then change variable in our maximization problem setting

$$\omega_t = e^{-\frac{\rho}{1-\sigma}t}c_t; \qquad \mathbf{w}_t = e^{-\frac{\rho}{1-\sigma}t}\mathbf{x}_t; \qquad \mathbf{y}_t = e^{-\frac{\rho}{1-\sigma}t}\mathbf{s}_t \tag{25}$$

transforming it into

$$\max \int_0^{+\infty} \frac{\omega_t^{1-\sigma}}{1-\sigma} dt$$

$$\begin{aligned}\dot{\mathbf{y}}_t^T &= -\frac{\rho}{1-\sigma}e^{-\frac{\rho}{1-\sigma}t}\mathbf{s}_t + e^{-\frac{\rho}{1-\sigma}t}\dot{\mathbf{s}}_t \\ &= -\frac{\rho}{1-\sigma}e^{-\frac{\rho}{1-\sigma}t}\mathbf{s}_t + e^{-\frac{\rho}{1-\sigma}t}\mathbf{x}_t^T\left(\mathbf{B}-\delta\mathbf{A}\right) - \delta_{\mathbf{z}}e^{-\frac{\rho}{1-\sigma}t}\mathbf{s}_t^T - e^{-\frac{\rho}{1-\sigma}t}c_t\mathbf{e}_1^T \\ &= -\left(\delta_{\mathbf{z}} + \frac{\rho}{1-\sigma}\right)\mathbf{y}_t + \mathbf{w}_t^T\left(\mathbf{B}-\delta\mathbf{A}\right) - \omega_t\mathbf{e}_1^T.\end{aligned}$$

Since $a_0 < 0$ then $\frac{\rho}{1-\sigma} = \Gamma - \varepsilon_0$ for suitable $\varepsilon_0 > 0$, so

$$\dot{\mathbf{y}}_t^T = -\left(\delta_{\mathbf{z}} + \Gamma - \varepsilon_0\right)\mathbf{y}_t + \mathbf{w}_t^T\left(\mathbf{B}-\delta\mathbf{A}\right) - \omega_t\mathbf{e}_1^T.$$

$$\mathbf{y}^T \geq \mathbf{w}^T\mathbf{A}, \quad \mathbf{w} \geq \mathbf{0}, \quad \omega \geq 0.$$

Then first let evolve the system to reach a state $\mathbf{y}_0 > \mathbf{0}$ (this is possible since Assumption II..5 holds for every component). This means that we can take from the beginning $\bar{\mathbf{s}} > \mathbf{0}$. At this point for any $\varepsilon > 0$ we can find $\mathbf{w}_\varepsilon \geq \mathbf{0}$ such that

$$\mathbf{w}_\varepsilon^T\left(\mathbf{B}-\left(\Gamma_1 - \varepsilon + \delta_{\mathbf{x}}\right)\mathbf{A}\right) \geq \mathbf{e}_1^T \qquad \Rightarrow \qquad \mathbf{w}_\varepsilon^T\left(\mathbf{B}-\delta\mathbf{A}\right) \geq \mathbf{e}_1^T + \left(\Gamma_1 - \varepsilon + \delta_{\mathbf{z}}\right)\mathbf{w}_\varepsilon^T$$

This means that taking $\varepsilon = \varepsilon_0$ we find $\mathbf{w}_0 \geq \mathbf{0}$, $\mathbf{w}_0 \neq \mathbf{0}$ such that

$$\mathbf{w}_0^T\left(\mathbf{B}-\left(\Gamma_1 - \varepsilon_0 + \delta_{\mathbf{x}}\right)\mathbf{A}\right) \geq \mathbf{0} \qquad \Rightarrow \qquad \mathbf{w}_0^T\left(\mathbf{B}-\delta\mathbf{A}\right) \geq \mathbf{e}_1^T + \left(\Gamma_1 - \varepsilon_0 + \delta_{\mathbf{z}}\right)\mathbf{w}_0^T$$

Take now $\mathbf{w}_t = \alpha\mathbf{w}_0$ and $\omega_t = \beta$ for suitable $\alpha, \beta > 0$. To prove the claim it is enough to show that this couple is admissible. The associated solution of the state equation (2) is given by:

$$\mathbf{y}_t^T = e^{-(\delta_{\mathbf{z}}+\Gamma-\varepsilon_0)t}\bar{\mathbf{s}}^T + \int_0^t e^{-(\delta_{\mathbf{z}}+\Gamma-\varepsilon_0)(t-s)}\mathbf{w}_s^T\left[\mathbf{B}-\delta\mathbf{A}\right]ds - \int_0^t e^{-(\delta_{\mathbf{z}}+\Gamma-\varepsilon_0)(t-s)}\omega_s\mathbf{e}_1^T d$$

$$= e^{-(\delta_{\mathbf{z}}+\Gamma-\varepsilon_0)t}\left[\bar{\mathbf{s}}^T+\alpha\mathbf{w}_0^T\left(\mathbf{B}-\delta\mathbf{A}\right)\int_0^t e^{(\delta_{\mathbf{z}}+\Gamma-\varepsilon_0)s}ds-\beta\mathbf{e}_1^T\int_0^t e^{(\delta_{\mathbf{z}}+\Gamma-\varepsilon_0)s}ds\right]$$

$$= e^{-(\delta_{\mathbf{z}}+\Gamma-\varepsilon_0)t}\left[\bar{\mathbf{s}}^T+\left[\alpha\mathbf{w}_0^T\left(\mathbf{B}-\delta\mathbf{A}\right)-\beta\mathbf{e}_1^T\right]\int_0^t e^{(\delta_{\mathbf{z}}+\Gamma-\varepsilon_0)s}ds\right].$$

Now, since α is positive we have for $\delta_{\mathbf{z}}+\Gamma-\varepsilon_0\neq 0$

$$\mathbf{y}_t^T\geq e^{-(\delta_{\mathbf{z}}+\Gamma-\varepsilon_0)t}\left\{\bar{\mathbf{s}}^T+\left[\left(\alpha-\beta\right)\mathbf{e}_1^T+\alpha\left(\Gamma_1-\varepsilon_0+\delta_{\mathbf{z}}\right)\mathbf{w}_0^T\mathbf{A}\right]\left[\frac{e^{(\delta_{\mathbf{z}}+\Gamma-\varepsilon_0)t}-1}{\delta_{\mathbf{z}}+\Gamma-\varepsilon_0}\right]\right\}$$

and for $\delta_{\mathbf{z}}+\Gamma-\varepsilon_0=0$

$$\mathbf{y}_t^T\geq\bar{\mathbf{s}}^T+t\left[\left(\alpha-\beta\right)\mathbf{e}_1^T+\alpha\left(\Gamma_1-\varepsilon_0+\delta_{\mathbf{z}}\right)\mathbf{w}_0^T\mathbf{A}\right].$$

Now if $\Gamma=\Gamma_1$, we get, for $\delta_{\mathbf{z}}+\Gamma-\varepsilon_0\neq 0$ and $\alpha=\beta$

$$\mathbf{y}_t^T\geq e^{-(\delta_{\mathbf{z}}+\Gamma-\varepsilon_0)t}\left\{\bar{\mathbf{s}}^T+\alpha\mathbf{w}_0^T\mathbf{A}\left[e^{(\delta_{\mathbf{z}}+\Gamma-\varepsilon_0)t}-1\right]\right\}=e^{-(\delta_{\mathbf{z}}+\Gamma-\varepsilon_0)t}\left[\bar{\mathbf{s}}^T-\alpha\mathbf{w}_0^T\mathbf{A}\right]+\mathbf{w}_t^T\mathbf{A}$$

In this case it is clear that the constraints $\mathbf{y}_t^T\geq\mathbf{w}_t^T\mathbf{A}$ are satisfied if for every $t\geq 0$

$$\bar{\mathbf{s}}^T-\alpha\mathbf{w}_0^T\mathbf{A}\geq 0.$$

To have this we need to set α_0 sufficiently small so that

$$\bar{\mathbf{s}}^T-\alpha\mathbf{w}_0^T\mathbf{A}\geq 0,$$

which is always possible since $\bar{\mathbf{s}}>0$.

Moreover, if $\delta_{\mathbf{z}}+\Gamma-\varepsilon_0=0$ and $\Gamma=\Gamma_1$ we have $\mathbf{y}_t^T\geq\bar{\mathbf{s}}^T\geq\alpha_0\mathbf{w}_0^T\mathbf{A}=\mathbf{w}_t^T\mathbf{A}$ for α_0 sufficiently small.

If $\Gamma>\Gamma_1$ then necessarily $\Gamma=-\delta_{\mathbf{z}}$ and $\delta_{\mathbf{z}}+\Gamma-\varepsilon_0=-\varepsilon_0\neq 0$. In this case we have, for $\alpha=\beta$

$$\mathbf{y}_t^T\geq e^{\varepsilon_0 t}\left\{\bar{\mathbf{s}}^T+\alpha\left(\Gamma_1-\Gamma-\varepsilon_0\right)\mathbf{w}_0^T\mathbf{A}\left[\frac{e^{-\varepsilon_0 t}-1}{-\varepsilon_0}\right]\right\}$$
$$=e^{\varepsilon_0 t}\left[\bar{\mathbf{s}}^T-\alpha\frac{\Gamma-\Gamma_1+\varepsilon_0}{\varepsilon_0}\mathbf{w}_0^T\mathbf{A}\right]+\alpha\frac{\Gamma-\Gamma_1+\varepsilon_0}{\varepsilon_0}\mathbf{w}_0^T\mathbf{A}$$

so that

$$\mathbf{y}_t^T-\mathbf{w}_t^T\mathbf{A}\geq e^{\varepsilon_0 t}\left[\bar{\mathbf{s}}^T-\alpha\frac{\Gamma-\Gamma_1+\varepsilon_0}{\varepsilon_0}\mathbf{w}_0^T\mathbf{A}\right]+\alpha\frac{\Gamma-\Gamma_1}{\varepsilon_0}\mathbf{w}_0^T\mathbf{A}$$

and again for α sufficiently small the claim holds.

Take now the case when $\sigma = 1$ and $a_0 \leq 0$ and $\Gamma > 0$. Let ε such that $\Gamma > \varepsilon$. Then we can change variable setting

$$\omega_t = e^{-(\Gamma-\varepsilon)t} c_t; \qquad \mathbf{w}_t = e^{-(\Gamma-\varepsilon)t} \mathbf{x}_t; \qquad \mathbf{y}_t = e^{-(\Gamma-\varepsilon)t} \mathbf{s}_t$$

so that

$$U_1(c) = \int_0^{+\infty} e^{-\rho s} \log c_s ds = \int_0^{+\infty} e^{-\rho s} \log \left(\omega_s e^{(\Gamma-\varepsilon)s} \right) ds.$$

Arguing as above we find a constant strategy $\omega_s = \alpha > 0$. Clearly the function $e^{-\rho s} \log \left(\omega_s e^{(\Gamma-\varepsilon)s} \right)$ is locally bounded, definitely positive, and goes to $+\infty$ for $s \to +\infty$. Then for this strategy we have $U_1(c) = +\infty$.

Let $a_0 > 0$ and $\sigma \in [1, +\infty)$. We observe that the we can do the same change of variable as above (the second) taking $\varepsilon > 0$ so that $a_\varepsilon > 0$. Then again we find that the strategy $\mathbf{w}_t = \alpha \mathbf{w}_0$ and $\omega_t = \alpha$ with $\bar{\mathbf{s}}^T - \alpha \mathbf{w}_0^T \mathbf{A} \geq 0$, is still admissible (since admissibility does not depend on the value of σ). We then have, for $\sigma \in (1, +\infty)$

$$\begin{aligned} U_\sigma(c) &= \frac{1}{1-\sigma} \int_0^{+\infty} e^{-\rho s} c_s^{1-\sigma} ds = \frac{\alpha^{1-\sigma}}{1-\sigma} \int_0^{+\infty} e^{-\rho s} e^{(\Gamma-\varepsilon)(1-\sigma)s} ds \\ &= \frac{\alpha^{1-\sigma}}{1-\sigma} \int_0^{+\infty} e^{-a_\varepsilon s} ds = \frac{\alpha^{1-\sigma}}{1-\sigma} \frac{1}{a_\varepsilon} > -\infty \end{aligned}$$

For $\sigma = 1$ we have

$$U_\sigma(c) = \int_0^{+\infty} e^{-\rho s} \log c_s ds = \int_0^{+\infty} e^{-\rho s} \log \left(\omega_s e^{(\Gamma-\varepsilon)s} \right) ds$$

Since the function $\log \left(\omega_s e^{(\Gamma-\varepsilon)s} \right)$ is less than polinomially growing and $\rho = a_0 > 0$ then the integral above is finite, so $U_\sigma(c) > -\infty$. ∎

Remark B.4 *The above result shows in particular that, when $a_0 > 0$ and $\sigma \in [1, +\infty)$, the intertemporal utility functional $U_\sigma(c)$ is not always $-\infty$ so it is bounded from below (recall that from Corollary B.1 we already know that in these case $U_\sigma(c)$ is bounded from above). Moreover in the cases when*

1. $\sigma \in (0,1)$ *and* $a_0 < 0$ *or* $\sigma = 1$, $a_0 \leq 0$ *and* $\Gamma > 0$
2. $\sigma = 1$, $a_0 \leq 0$ *and* $\Gamma > 0$

Proposition B.3 shows that there are no optimal strategies in the sense of Definition II..1 since the supremum of the utility is $+\infty$. ∎

Summing up the informations taken from Corollary B.1 and Proposition B.3 we can say the following.

- In the cases when $a_0 > 0$ we know that the functional is uniformly bounded (case $\sigma \in (0,1)$) or bounded from from above and not identically $-\infty$ (case $\sigma \geq 1$);
- In the cases when $a_0 \leq 0$ we have nonexistence when
  1. $\sigma \in (0,1)$ and $a_0 < 0$;
  2. $\sigma = 1$, $a_0 \leq 0$, $\Gamma \neq 0$ or $\Gamma = 0$ and Γ is not a maximum;
  3. $\sigma > 1$ and $a_0 < 0$ or $a_0 = 0$ and $\Gamma = -\delta_{\mathbf{z}} > \Gamma_1$ or when Γ_1 is not a maximum;

We observe that, to end the treatment of nonexistence result one should deal with with the following limiting cases:

- 1. $\sigma \in (0,1)$ and $a_0 = 0$;
  2. $\sigma = 1$, $a_0 \leq 0$, $\Gamma = 0$ and Γ is a maximum;
  3. $\sigma > 1$ and $a_0 = 0$ and $\Gamma = \Gamma_1$ and Γ_1 is a maximum;

Proof of Theorem III..2. It follows directly from Corollary B.1, Proposition B.3 and the remarks above. ∎

Now we come to prove existence when $a_0 > 0$ using compactness arguments. We first observe that, thanks to estimates of Lemma A.12 the set $\mathcal{A}(\bar{\mathbf{s}})$ of admissible control strategies starting at $\bar{\mathbf{s}}$ is a closed subset of the space $L_{g_1}^{\infty}(0,+\infty;\mathbb{R}^m) \times L_{g_2}^{1}(0,+\infty;\mathbb{R})$ (recall that given a measurable function $g_1 : \mathbb{R}^+ \to \mathbb{R}$ we denote by $L_{g_1}^{\infty}(0,+\infty;\mathbb{R}^m)$ the set of measurable functions $f : \mathbb{R}^+ \to \mathbb{R}^m$ such that the product $f \cdot g_1$ is bounded on $\mathbb{R}^+$. Moreover given a measurable function $g_2 : \mathbb{R}^+ \to \mathbb{R}$ we denote by $L_{g_2}^{1}(0,+\infty;\mathbb{R})$ the set of measurable functions $f : \mathbb{R}^+ \to \mathbb{R}$ such that the product $f \cdot g_2$ is integrable in $\mathbb{R}^+$).where $g_1(t) = e^{\lambda t}$ (λ is given by (5) of Proposition A.1)and $g_2(t) = e^{(\Gamma+\varepsilon)t}$.

The next proposition sets up some basic properties of the set $\mathcal{A}(\bar{\mathbf{s}})$ needed to prove existence.

Proposition B.5 *Let Assumptions II..2 and II..3 be verified. Let also $\sigma \in (0,1) \cup (1,+\infty)$. Given any $\bar{\mathbf{s}} \geq \mathbf{0}$ the set $\mathcal{A}(\bar{\mathbf{s}})$ of admissible control strategies starting at $\bar{\mathbf{s}}$ is a closed convex subset of the space $L_{g_1}^{\infty}(0,+\infty;\mathbb{R}^m) \times L_{g_2}^{1}(0,+\infty;\mathbb{R}^n)$. Finally*

$$(\mathbf{x},c) \in \mathcal{A}(\bar{\mathbf{s}}), \quad \lambda \in [0,1] \Rightarrow (\lambda\mathbf{x},\lambda c) \in \mathcal{A}(\bar{\mathbf{s}}) \tag{26}$$

and the functional U_σ is strictly concave with respect to the argument c. The same holds when $\sigma = 1$ and $\rho > 0$.

Proof. Convexity. Let $i = 1,2$ and let $(\mathbf{x}_i,c_i) \in \mathcal{A}(\bar{\mathbf{s}})$, and $\lambda \in [0,1]$. Calling

$$(\mathbf{x}_\lambda,c_\lambda) = \lambda(\mathbf{x}_1,c_1) + (1-\lambda)(\mathbf{x}_2,c_2)$$

we have, due to the linearity of the state equation (2) that

$$\mathbf{s}_{t,\bar{\mathbf{s}},(\mathbf{x}_\lambda,c_\lambda)} = \lambda\mathbf{s}_{t,\bar{\mathbf{s}},(\mathbf{x}_1,c_1)} + (1-\lambda)\,\mathbf{s}_{t,\bar{\mathbf{s}},(\mathbf{x}_2,c_2)}.$$

Since all constraints on $(\mathbf{s},(\mathbf{x},c))$ (i.e. $\mathbf{x} \geq \mathbf{0}$, $c \geq 0$, $\mathbf{x}^T\mathbf{A} \leq \mathbf{s}^T$) are linear it follows that, since $(\mathbf{x}_i,c_i)$ $(i = 1,2)$ satisfy them, also $(\mathbf{x}_\lambda,c_\lambda)$ does so. This yields $(\mathbf{x}_\lambda,c_\lambda) \in \mathcal{A}(\bar{\mathbf{s}})$ when $\sigma \in (0,1) \cup (1,+\infty)$. If $\sigma = 1$ we also have to prove that $(\mathbf{x}_\lambda,c_\lambda)$ is semiintegrable. This follows from point (ii) of Corollary B.1. In fact if $\rho > 0$, thanks to estimate (24) we know all admissible strategies are upper semiintegrable, so also their convex combinations are upper semiintegrable (we observe that from the proof of Corollary B.1 (ii) it follows that also for $\rho \leq 0$ and $\Gamma < 0$ or for $\Gamma = 0$ and when $\varepsilon = 0$ is allowed again all admissible strategies are upper semiintegrable and again their convex combinations are upper semiintegrable but this is not of our interest here).

Closedness follows from the fact that all constraints are linear so all of them preserves in the limit in the topology of $L^\infty_{g_1}(0,+\infty;\mathbb{R}^m)\times L^1_{g_2}(0,+\infty;\mathbb{R}^n)$. For $\sigma = 1$ we need to know that the limit of semiintegrable sequences is again semiintegrable. For $\rho > 0$ this follows from the estimate (24).

Homogeneity (26) follows from convexity and from the fact that the strategy $(\mathbf{0},0)$ is always admissible.

Strict concavity of the functional U is a standard result (see e.g. [7]) and we omit the proof. ∎

Next Proposition prove the existence (and uniqueness) result for optimal strategies when $a_0 > 0$. The proof uses standard compactness for weak topologies (see for a reference on this e.g. [3] or [20]) combined with the special setting and estimates of our problem.

Proposition B.6 *Assume that $a_0 > 0$. Then there exists an optimal production-consumption strategy $(\mathbf{x}, c)$ maximizing U_σ. This strategy is unique in the sense that, if $(\hat{\mathbf{x}}, \hat{c})$ is another optimal strategy, then $\hat{c} = c$ a.e..*

Proof. The uniqueness property follows from the strict concavity of U_σ proved in Proposition B.5. The existence result follows applying a suitable modification of Theorems 21 and 22 in [18, p. 406] (see also [17]). We divide it in three cases depending on the value of σ. For the case $\sigma = \sigma(0,1)$ we give a self contained proof. For the other cases we limit ourself to verify exact conditions of theorems present in the literature.

Case $\sigma \in (0,1)$.

In the case $\sigma \in (0,1)$ as the other cases are completely analogous. Let $a_0 > 0$ and $\bar{\mathbf{s}} \geq \mathbf{0}$ be the initial datum. Take a sequence $(\mathbf{x}_n, c_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}(\bar{\mathbf{s}})$ of production-consumption strategies such that

$U_\sigma(c_n) \nearrow \sup_{(\mathbf{x},c)\in\mathcal{A}(\bar{\mathbf{s}})} U(c)$ as $n \to +\infty$. Then it is clear that, for every $n \in \mathbb{N}$ and $t \geq 0$, we have

$$\mathbf{s}_{nt}^T = e^{-\delta_{\mathbf{z}}t}\bar{\mathbf{s}}^T + \int_0^t e^{-\delta_{\mathbf{z}}(t-s)}\mathbf{x}_{ns}^T\left[\mathbf{B} - \delta\mathbf{A}\right]ds - \int_0^t e^{-\delta_{\mathbf{z}}(t-s)}c_{ns}\mathbf{e}_1^T ds \qquad \mathbf{s}_{nt}^T \geq \mathbf{x}_{nt}^T\mathbf{A} \tag{27}$$

By estimate (13) we know that the functions $t \to f_n(t) = c_{nt}^{1-\sigma}$ belong to the space $L^{1/(1-\sigma)}_{\Gamma+\varepsilon}(0,+\infty;\mathbb{R})$ i.e. the space of functions: $(0,+\infty) \mapsto \mathbb{R}$ that, elevated to $1/(1-\sigma)$ and multiplied by the weight function $e^{-(\Gamma+\varepsilon)t}$ are integrable. Moreover, denoting by $||\cdot||_{1/(1-\sigma),\Gamma+\varepsilon}$ the norm in this space we have that $||f_n||_{1/(1-\sigma),\Gamma+\varepsilon} \leq K\bar{\mathbf{s}}^T\mathbf{v}_F$ for a suitable $K > 0$ independent of n. It follows that (by weak compactness theorems, see e.g. [3, Ch. 4]), on a subsequence (that we still denote by f_n for simplicity of notation) we have $f_n \to f_0$ weakly in $L^{1/(1-\sigma)}_{\Gamma+\varepsilon}(0,+\infty;\mathbb{R})$, for a suitable $f_0 \in L^{1/(1-\sigma)}_{\Gamma+\varepsilon}(0,+\infty;\mathbb{R})$. Let us call $c_0 = f_0^{1/(1-\sigma)}$. Clearly $c_0 \in L^1_{\Gamma+\varepsilon}(0,+\infty;\mathbb{R})$. Similarly by estimate (12) we know that the functions $\mathbf{x}_n \in L^\infty_{\Gamma+\varepsilon}(0,+\infty;\mathbb{R}^n)$ and that $||\mathbf{x}_n||_{\infty,\Gamma+\varepsilon} \leq K\bar{\mathbf{s}}^T\mathbf{v}_F$ for a suitable $K > 0$ independent of n. So, as before, there exists $\mathbf{x}_0 \in L^\infty_{\Gamma+\varepsilon}(0,+\infty;\mathbb{R}^n)$ such that $\mathbf{x}_n \to \mathbf{x}_0$ (on a subsequence) weakly star in $L^\infty_{\Gamma+\varepsilon}(0,+\infty;\mathbb{R}^n)$. We prove that the strategy $(\mathbf{x}_0, c_0)$ is admissible and optimal.

First it is clear that $\mathbf{x}_0 \geq \mathbf{0}$, and $c_0 \geq 0$, since the above convergencies preserve the sign constraints on the limit (see e.g. [3, Ch. 4]).

Second, consider the associated state trajectory $\mathbf{s}_0$. It is clear that

$$\mathbf{s}_{0t}^T = e^{-\delta_{\mathbf{z}}t}\bar{\mathbf{s}}^T + \int_0^t e^{-\delta_{\mathbf{z}}(t-s)}\mathbf{x}_{0s}^T\left[\mathbf{B} - \delta\mathbf{A}\right]ds - \int_0^t e^{-\delta_{\mathbf{z}}(t-s)}c_{0s}\mathbf{e}_1^T ds$$

Moreover, by definition of weak star convergence in $L^\infty_{\Gamma+\varepsilon}(0,+\infty;\mathbb{R}^n)$ we have that

$$\int_0^t e^{-\delta_{\mathbf{z}}(t-s)}\mathbf{x}_{0s}^T\left[\mathbf{B} - \delta\mathbf{A}\right]ds = \lim_{n\to+\infty}\int_0^t e^{-\delta_{\mathbf{z}}(t-s)}\mathbf{x}_{ns}^T\left[\mathbf{B} - \delta\mathbf{A}\right]ds \qquad \forall t \geq 0$$

and, by the lower semicontinuity of convex functions with respect to the the weak convergence in $L^{1/(1-\sigma)}_{\Gamma+\varepsilon}(0,+\infty)$,

$$\int_0^t e^{-\delta_{\mathbf{z}}(t-s)} c_{0s} ds \leq \liminf_{n\to+\infty} \int_0^t e^{-\delta_{\mathbf{z}}(t-s)} c_{ns} ds$$

so that, by (27) we have, for almost every $t \geq 0$

$$\mathbf{s}_{0t}^T \geq \limsup_{n\to+\infty} \mathbf{s}_{nt}^T \geq \limsup_{n\to+\infty} \mathbf{x}_{nt}^T \mathbf{A} \geq \mathbf{x}_{0t}^T \mathbf{A}$$

where in the last inequality we have still used the properties of the weak star convergence in $L^\infty_{\Gamma+\varepsilon}(0,+\infty;\mathbb{R}^n)$. This gives admissibility of $(\mathbf{x}_0, c_0)$. The optimality easily follows by the concavity of U_σ which implies the weak upper semicontinuity, so that

$$\sup_{c\in\mathcal{A}(\bar{\mathbf{s}})} U_\sigma(c) = \limsup_{n\to+\infty} U_\sigma(c_n) \leq U_\sigma(c_0)$$

Case $\sigma > 1$.

Here we can apply directly Theorem 22 and note 26 of [18, p. 406] plus [18, note 20, p.137]. In fact this theorem asks the following:

1. the set U where the controls take values is closed (in our case U is $\mathbb{R}^n_+ \times \mathbb{R}_+$ i.e. the positive orthant of $\mathbb{R}^{n+1}$);
2. the functions defining the running utility $((\mathbf{s},(\mathbf{x},c),t) \to e^{-\rho t} u_\sigma(c))$, the dynamics of the state equation $((\mathbf{s},(\mathbf{x},c),t) \to -\delta_{\mathbf{z}}\mathbf{s}^T + \mathbf{x}^T(\mathbf{B}-\delta\mathbf{A}) - c\mathbf{e}_1^T)$ and the constraints $((\mathbf{s},(\mathbf{x},c),t) \to \mathbf{s}^T - \mathbf{x}^T\mathbf{A})$ are defined on the set
$$S = \left\{(\mathbf{s},(\mathbf{x},c),t) \in \mathbb{R}^n_+ \times [\mathbb{R}^n_+ \times \mathbb{R}_+] \times \mathbb{R}_+ : \mathbf{s}^T - \mathbf{x}^T\mathbf{A} \geq \mathbf{0}\right\}$$
are linear (or sum of linear and nondecreasing) in the variable $\mathbf{s}$ and continuous on the set
$$S' = \left\{(\mathbf{s},(\mathbf{x},c),t) \in \mathbb{R}^n_+ \times [\mathbb{R}^n_+ \times (0,+\infty)] \times \mathbb{R}_+ : \mathbf{s}^T - \mathbf{x}^T\mathbf{A} \geq \mathbf{0}\right\};$$
3. for each $t \geq 0$ the set
$$S'(t) = \left\{(\mathbf{s},(\mathbf{x},c)) \in \mathbb{R}^n_+ \times [\mathbb{R}^n_+ \times (0,+\infty)] : \mathbf{s}^T - \mathbf{x}^T\mathbf{A} \geq \mathbf{0}\right\}$$
is contained in the closure $\overline{S^0(t)}$ of the set
$$S^0(t) = \left\{(\mathbf{s},(\mathbf{x},c)) \in \mathbb{R}^n_+ \times U : \mathbf{s}^T - \mathbf{x}^T\mathbf{A} > \mathbf{0}\right\};$$

4. for each $n \in \mathbb{N}$ and $t \geq 0$ the set
$$\Gamma_t^n = \left\{ (\mathbf{s}, (\mathbf{x}, c)) : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} \geq \mathbf{0}, (\mathbf{x}, c) \in U, \right.$$
$$\left. \left|\left| \left(e^{-\rho t} u_\sigma (c), -\delta_{\mathbf{z}} \mathbf{s}^T + \mathbf{x}^T (\mathbf{B} - \delta \mathbf{A}) - c \mathbf{e}_1^T \right) \right|\right| \leq n, \right\}$$
is closed and is contained in $S'(t)$. The same for the set
$$\Gamma^n = \{ \mathbf{s} : (\mathbf{s}, (\mathbf{x}, c)) \in \Gamma_t^n \};$$

5. there exists an admissible strategy with finite value;

6. the set
$$N(\mathbf{s}, U, t) = \left\{ \left(e^{-\rho t} u_\sigma (c) + \gamma, -\delta_{\mathbf{z}} \mathbf{s} + \mathbf{x}^T (\mathbf{B} - \delta \mathbf{A}) - c \mathbf{e}_1^T + \gamma \right) : \right.$$
$$\left. (\gamma, \gamma) \leq 0, \mathbf{s} - \mathbf{x}^T \mathbf{A} \geq \mathbf{0}, (\mathbf{x}, c) \in U \right\}$$
is convex for all $(\mathbf{s}, t) \in \mathbb{R}^n \times [0, +\infty)$;

7. the set $N(\mathbf{s}, U, t)$ has closed graph for each t as a function of $\mathbf{s} \in \Gamma^n$. Closed graph means that
$$\mathbf{s}_n \in \Gamma^n, \mathbf{v}_n \in N(\mathbf{s}, U, t), \mathbf{s}_n \to \mathbf{s}, \mathbf{v}_n \to \mathbf{v} \Rightarrow \mathbf{s} \in \Gamma^n.$$

8. there exists $\mathbf{q}' \in \mathbb{R}^{n+1}$, $\mathbf{q}' \geq 0$ such that for every $\mathbf{q} \geq \mathbf{q}'$ $(\mathbf{q} = (q_0, q_1, ..., q_n) = (q_0, \mathbf{q}_1))$ there exists locally integrable functions ϕ_q and ψ_q defined for $t \in [0, +\infty)$ such that
$$e^{-\rho t} u_\sigma (c) q_0 + \left(-\delta_{\mathbf{z}} \mathbf{s}^T + \mathbf{x}^T (\mathbf{B} - \delta \mathbf{A}) - c \mathbf{e}_1^T \right) \mathbf{q}_1 \leq \phi_q (t) + \psi_q (t) \cdot \max \left[0, \mathbf{s}^T \mathbf{e}_1, \right.$$
for every $(\mathbf{s}, (\mathbf{x}, c), t) \in S$.

9. for every $i = 1, ..., n$ and every admissible state trajectory $\mathbf{s}_t$, we $\mathbf{s}_t^T \mathbf{e}_i \geq 0$ for every $t \geq 0$. eMorover for every $q_0 \in \mathbb{R}$, $q_0 \geq 0$, there exists an integrable function ν_{q_0} defined for $t \in [0, +\infty)$ such that, ,
$$e^{-\rho t} u_\sigma (c_t) (1 + q_0) \leq \nu_{q_0} (t),$$
for every admissible strategy $(\mathbf{x}, c) \in \mathcal{A}(\bar{\mathbf{s}})$.

All points 1-4 and 6-7 are easily checked in our case thanks to the linearity of the state equation and of the constraints. We omit the verification of them for brevity. Point 5 is known from previous results (Corollary B.1 and Proposition B.3). Point 9 comes simply recalling that for $\sigma > 1$ the utility is negative and so one can choose $\nu_{q_0}(t) = 0$ for every $t \geq 0$. Point 8 is more delicate. Setting

$$g(c) = e^{-\rho t} u_\sigma(c)\, q_0 - c\mathbf{e}_1^T \mathbf{q}_1$$

we have

$$g(c) \leq \frac{\sigma}{1-\sigma} \mathbf{e}_1^T \mathbf{q}_1 \cdot \left(\frac{\mathbf{e}_1^T \mathbf{q}_1}{e^{-\rho t} q_0} \right)^{-\frac{1}{\sigma}} = \frac{\sigma}{1-\sigma} \left(\mathbf{e}_1^T \mathbf{q}_1 \right)^{1-\frac{1}{\sigma}} q_0^{\frac{1}{\sigma}} e^{-\frac{\rho}{\sigma} t}.$$

Moreover

$$-\delta_{\mathbf{z}} \mathbf{s}^T + \mathbf{x}^T (\mathbf{B} - \delta \mathbf{A}) = -\delta_{\mathbf{z}} \left(\mathbf{s}^T - \mathbf{x}^T \mathbf{A} \right) + \mathbf{x}^T (\mathbf{B} - \delta_{\mathbf{x}} \mathbf{A}) \leq \mathbf{x}^T \mathbf{B}.$$

Now, recalling the proof of (5) we have that

$$\mathbf{x}^T \mathbf{B} \leq \mathbf{s}^T \mathbf{D} \leq M \max \left[0, \mathbf{s}^T \mathbf{e}_1, ..., \mathbf{s}^T \mathbf{e}_n \right]$$

where M depends only on the coefficient of $\mathbf{D}$. Setting, for $t \geq 0$, $\phi_q(t) = \frac{\sigma}{1-\sigma} \left(\mathbf{e}_1^T \mathbf{q}_1 \right)^{1-\frac{1}{\sigma}} q_0^{\frac{1}{\sigma}} e^{-\frac{\rho}{\sigma} t}$ and $\psi_q(t) = M$ we see that ϕ_q and ψ_q are locally integrable functions and satisfy point 8.

Case $\sigma = 1$.

Also this case goes applying Theorem 22 and note 26 of [18, p. 406] (see also [17] or [18, Exercise 6.8.3, p.410]). In fact this theorem asks the following:

1. the set U where the controls take values is closed (in our case U is $\mathbb{R}_+^n \times \mathbb{R}_+$ i.e. the positive orthant of $\mathbb{R}^{n+1}$);

2. the functions defining the running utility ($(\mathbf{s}, (\mathbf{x}, c), t) \to e^{-\rho t} u_\sigma(c)$), the dynamics of the state equation ($(\mathbf{s}, (\mathbf{x}, c), t) \to -\delta_{\mathbf{z}} \mathbf{s}^T + \mathbf{x}^T (\mathbf{B} - \delta \mathbf{A}) - c\mathbf{e}_1^T$) and the constraints ($(\mathbf{s}, (\mathbf{x}, c), t) \to \mathbf{s}^T - \mathbf{x}^T \mathbf{A}$) are defined on the set

$$S = \left\{ (\mathbf{s}, (\mathbf{x}, c), t) \in \mathbb{R}_+^n \times [\mathbb{R}_+^n \times \mathbb{R}_+] \times \mathbb{R}_+ : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} \geq \mathbf{0} \right\}$$

are linear (or sum of linear and nondecreasing) in the variable $\mathbf{s}$ and continuous on the set

$$S' = \left\{ (\mathbf{s}, (\mathbf{x},c), t) \in \mathbb{R}^n_+ \times [\mathbb{R}^n_+ \times (0,+\infty)] \times \mathbb{R}_+ : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} \geq \mathbf{0} \right\};$$

3. for each $t \geq 0$ the set

$$S'(t) = \left\{ (\mathbf{s}, (\mathbf{x},c)) \in \mathbb{R}^n_+ \times [\mathbb{R}^n_+ \times (0,+\infty)] : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} \geq \mathbf{0} \right\}$$

is contained in the closure $\overline{S^0(t)}$ of the set

$$S^0(t) = \left\{ (\mathbf{s}, (\mathbf{x},c)) \in \mathbb{R}^n_+ \times U : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} > \mathbf{0} \right\};$$

4. for each $n \in \mathbb{N}$ and $t \geq 0$ the set

$$\Gamma^n_t = \left\{ (\mathbf{s}, (\mathbf{x},c)) : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} \geq \mathbf{0}, (\mathbf{x},c) \in U, \right.$$
$$\left. \left|\left| \left(e^{-\rho t} u_\sigma(c), -\delta_{\mathbf{z}} \mathbf{s}^T + \mathbf{x}^T (\mathbf{B} - \delta \mathbf{A}) - c \mathbf{e}_1^T \right) \right|\right| \leq n, \right\}$$

is closed and is contained in $S'(t)$. The same for the set

$$\Gamma^n = \{ \mathbf{s} : (\mathbf{s}, (\mathbf{x},c)) \in \Gamma^n_t \};$$

5. there exists an admissible strategy with finite value;

6. the set

$$N(\mathbf{s},U,t) = \left\{ \left(e^{-\rho t} u_\sigma(c) + \gamma, -\delta_{\mathbf{z}} \mathbf{s} + \mathbf{x}^T (\mathbf{B} - \delta \mathbf{A}) - c \mathbf{e}_1^T + \gamma \right) : \right.$$
$$\left. (\gamma, \gamma) \leq 0, \mathbf{s} - \mathbf{x}^T \mathbf{A} \geq \mathbf{0}, (\mathbf{x},c) \in U \right\}$$

is convex for all $(\mathbf{s},t) \in \mathbb{R}^n \times [0,+\infty)$;

7. the set $N(\mathbf{s},U,t)$ has closed graph for each t as a function of $\mathbf{s} \in \Gamma^n$. Closed graph means that

$$\mathbf{s}_n \in \Gamma^n, \mathbf{v}_n \in N(\mathbf{s},U,t), \mathbf{s}_n \to \mathbf{s}, \mathbf{v}_n \to \mathbf{v} \Rightarrow \mathbf{s} \in \Gamma^n.$$

8. there exists $\mathbf{q}' \in \mathbb{R}^{n+1}$, $\mathbf{q}' \geq 0$ such that for every $\mathbf{q} \geq \mathbf{q}'$ $(\mathbf{q} = (q_0, q_1, ..., q_n) = (q_0, \mathbf{q}_1))$ there exists locally integrable functions ϕ_q and ψ_q defined for $t \in [0,+\infty)$ such that

$$e^{-\rho t} u_\sigma(c) q_0 + \left(-\delta_{\mathbf{z}} \mathbf{s}^T + \mathbf{x}^T (\mathbf{B} - \delta \mathbf{A}) - c \mathbf{e}_1^T \right) \mathbf{q}_1 \leq \phi_q(t) + \psi_q(t) \cdot \max \left[0, \mathbf{s}^T \mathbf{e}_1, \right.$$

for every $(\mathbf{s}, (\mathbf{x},c), t) \in S$.

9. for every $i=1,...,n$ and every admissible state trajectory $\mathbf{s}_t$, we have$\mathbf{s}_t^T\mathbf{e}_i \geq 0$ for every $t \geq 0$. Moreover for every $q_0 \in \mathbb{R}$, $q_0 \geq 0$, there exists an integrable function ν_{q_0}, continuous functions χ_q^i and $\theta_{q_0}^i$ $(i=1,...,n)$ defined for $t \in [0,+\infty)$ such that, for every admissible strategy $(\mathbf{x},c) \in \mathcal{A}(\bar{\mathbf{s}})$,

$$e^{-\rho t}\ln c_t\left(1+q_0\right)+\sum_{i=1}^{n}\chi_{q_0}^i\left(t\right)\left(-\delta_{\mathbf{z}}\mathbf{s}_t^T+\mathbf{x}_t^T\left(\mathbf{B}-\delta\mathbf{A}\right)-c_t\mathbf{e}_1^T\right)\mathbf{e}_i \leq \nu_{q_0}\left(t\right),$$

and

$$-\int_s^{+\infty}\chi_{q_0}^i\left(t\right)\left(-\delta_{\mathbf{z}}\mathbf{s}_t^T+\mathbf{x}_t^T\left(\mathbf{B}-\delta\mathbf{A}\right)-c_t\mathbf{e}_1^T\right)\mathbf{e}_i dt \leq \theta_{q_0}^i\left(s\right)$$

where $\lim_{s\to+\infty}\theta_{q_0}^i\left(s\right)=0$.

All points 1-4 and 6-7 are easily checked in our case thanks to the linearity of the state equation and of the constraints. We omit the verification of them for brevity. Point 5 is known from previous results (Corollary B.1 and Proposition B.3). Point 8 follows arguing exactly as in the case $\sigma > 1$. Point 9 is more delicate. Set $\chi_{q_0}^i\left(t\right) = e^{-dt}\mathbf{e}_i^T\mathbf{v}_F$ for suitable d to choose later and consider the term containing c_t first. They are

$$e^{-\rho t}\ln c_t\left(1+q_0\right)-e^{-dt}c_t.$$

Then, setting

$$g\left(c\right)=e^{-\rho t}\ln c\left(1+q_0\right)-e^{-dt}c$$

we have for every $c \geq 0$,

$$g\left(c\right) \leq e^{-\rho t}\left(1+q_0\right)\cdot\left[\ln\left(\frac{e^{-\rho t}\left(1+q_0\right)}{e^{-dt}}\right)-1\right]=e^{-\rho t}\left(1+q_0\right)\cdot\left[\left(-\rho+d\right)t+\ln\left(1+q_0\right)-\right.$$

The right hand side is integrable for $\rho > 0$. Moreover

$$-\delta_{\mathbf{z}}\mathbf{s}_t^T+\mathbf{x}_t^T\left(\mathbf{B}-\delta\mathbf{A}\right)=-\delta_{\mathbf{z}}\left(\mathbf{s}_t^T-\mathbf{x}_t^T\mathbf{A}\right)+\mathbf{x}_t^T\left(\mathbf{B}-\delta_{\mathbf{x}}\mathbf{A}\right) \leq \mathbf{x}_t^T\left(\mathbf{B}-\delta_{\mathbf{x}}\mathbf{A}\right).$$

Then, for every $\varepsilon > 0$,

$$\begin{aligned}\left[-\delta_{\mathbf{z}}\mathbf{s}_t^T+\mathbf{x}_t^T\left(\mathbf{B}-\delta\mathbf{A}\right)\right]\mathbf{v}_F e^{-dt} &\leq \mathbf{x}_t^T\left(\mathbf{B}-\delta_{\mathbf{x}}\mathbf{A}\right)\mathbf{v}_F e^{-dt}\\ &\leq \mathbf{x}_t^T\mathbf{A}\mathbf{v}_F\left(\Gamma+\varepsilon\right)e^{-dt} \leq e^{(\Gamma+\varepsilon)t}\bar{\mathbf{s}}^T\mathbf{v}_F\left(\Gamma+\varepsilon\right)e^{-dt}.\end{aligned}$$

So if $d > \Gamma$ the first part of point 9 is true. Now observe that

$$\begin{aligned}
&-\int_s^{+\infty} \chi_{q_0}^i(t)\left(-\delta_{\mathbf{z}}\mathbf{s}_t^T + \mathbf{x}_t^T(\mathbf{B}-\delta\mathbf{A}) - c_t\mathbf{e}_1^T\right)\mathbf{e}_i dt \\
&= -\int_s^{+\infty} e^{-dt}\mathbf{e}_i^T\mathbf{v}_F \cdot \left(-\delta_{\mathbf{z}}\mathbf{s}_t^T + \mathbf{x}_t^T(\mathbf{B}-\delta\mathbf{A}) - c_t\mathbf{e}_1^T\right)\mathbf{e}_i dt \\
&\leq \int_s^{+\infty} e^{-dt}\mathbf{e}_i^T\mathbf{v}_F \cdot \left(\delta_{\mathbf{z}}\mathbf{s}_t^T + \delta_{\mathbf{x}}\mathbf{x}_t^T\mathbf{A} + c_t\mathbf{e}_1^T\right)\mathbf{e}_i dt.
\end{aligned}$$

Now from the estimates 10.12

$$\mathbf{e}_i^T\mathbf{v}_F \cdot \left(\delta_{\mathbf{z}}\mathbf{s}_t^T + \delta_{\mathbf{x}}\mathbf{x}_t^T\mathbf{A}\right)\mathbf{e}_i \leq Me^{(\Gamma+\varepsilon)t}$$

so that

$$\int_s^{+\infty} e^{-dt}\mathbf{e}_i^T\mathbf{v}_F \cdot \left(\delta_{\mathbf{z}}\mathbf{s}_t^T + \delta_{\mathbf{x}}\mathbf{x}_t^T\mathbf{A}\right)\mathbf{e}_i dt \leq M_1 e^{-(d-\Gamma-\varepsilon)s}.$$

Moreover, thanks to stima 13-10 we get, for $d > \Gamma + \varepsilon$

$$\int_s^{\tau} e^{-dr}c_r dr \leq e^{-(d-\Gamma-\varepsilon)s}\overline{\mathbf{s}}^T\mathbf{v}_F$$

so that, sending $\tau \to +\infty$,

$$\int_s^{+\infty} e^{-dt}c_t dt \leq M_2 e^{-(d-\Gamma-\varepsilon)s}.$$

and this completes the proof. ∎

C Limit Cases

In this appendix we study the limit cases where Theorem III..3 do not hold, i.e. when

$$\rho = \Gamma(1-\sigma)$$

and, *either* $\sigma \in (0,1)$ and Γ is not a maximum, *or* $\sigma > 1$ and Γ is a maximum.

First we have the following result.

Proposition C.1 *Let the same assumption of Theorem III..3 hold true. Assume that there exist* $\mathbf{w}_0 \geq \mathbf{0}$, $\mathbf{w}_0 \neq \mathbf{0}$ *such that* $\mathbf{w}_0^T \left(\mathbf{B} - \left(\Gamma_1 + \delta_{\mathbf{x}}\right)\mathbf{A}\right) \geq \mathbf{0}$ *and* $\mathbf{w}_0^T \mathbf{A}\mathbf{e}_1 > 0$. *In this case for* $\sigma \in (0,1)$ *there exist an admissible strategy with utility* $+\infty$.

Proof. Let first apply the change of variable (25). Then let the system evolve to reach a state $\mathbf{y}_0 > \mathbf{0}$ (this is possible since Assumption II..5 holds for every component). This means that we can take from the beginning $\bar{\mathbf{s}} > \mathbf{0}$. Take now $\mathbf{w}_0 \geq \mathbf{0}$, $\mathbf{w}_0 \neq \mathbf{0}$ such that

$$\mathbf{w}_0^T \left(\mathbf{B} - \left(\Gamma_1 + \delta_{\mathbf{x}}\right)\mathbf{A}\right) \geq \mathbf{0} \qquad \Rightarrow \qquad \mathbf{w}_0^T \left(\mathbf{B} - \delta\mathbf{A}\right) \geq \left(\Gamma_1 + \delta_{\mathbf{z}}\right)\mathbf{w}_0^T \mathbf{A}$$

and

$$\mathbf{w}_0^T \mathbf{A}\mathbf{e}_1 > 0$$

Take also a positive differentiable decreasing function $\alpha : [0,+\infty) \longmapsto [0,+\infty)$ defined as $\alpha(t) = \alpha_0 \left(1+t\right)^{-b}$ for suitable $\alpha_0 > 0$ and $b > 0$ (to be fixed later on). Set, for suitable $\beta > 0$,

$$\mathbf{w}_t = \alpha(t)\mathbf{w}_0, \qquad \omega_t = -\beta\alpha'(t) \tag{28}$$

Then the associated solution of the state equation (2) is given by:

$$\mathbf{y}_t^T = e^{-(\delta_{\mathbf{z}}+\Gamma)t}\bar{\mathbf{s}}^T + \int_0^t e^{-(\delta_{\mathbf{z}}+\Gamma)(t-s)}\mathbf{w}_s^T \left[\mathbf{B} - \delta\mathbf{A}\right] ds - \int_0^t e^{-(\delta_{\mathbf{z}}+\Gamma)(t-s)}\omega_s \mathbf{e}_1^T ds$$

$$= e^{-(\delta_{\mathbf{z}}+\Gamma)t}\left[\bar{\mathbf{s}}^T + \mathbf{w}_0^T \left(\mathbf{B} - \delta\mathbf{A}\right)\int_0^t \alpha(s)e^{(\delta_{\mathbf{z}}+\Gamma)s}ds + \beta\mathbf{e}_1^T \int_0^t \alpha'(s)e^{(\delta_{\mathbf{z}}+\Gamma)s}ds\right].$$

Now, since α is positive we have by

$$\mathbf{y}_t^T \geq e^{-(\delta_{\mathbf{z}}+\Gamma)t}\left[\bar{\mathbf{s}}^T + \left(\Gamma_1 + \delta_{\mathbf{z}}\right)\mathbf{w}_0^T \mathbf{A}\int_0^t \alpha(s)e^{(\delta_{\mathbf{z}}+\Gamma)s}ds + \beta\mathbf{e}_1^T \int_0^t \alpha'(s)e^{(\delta_{\mathbf{z}}+\Gamma)s}ds\right]$$

Integrating by parts the first integral becomes:

$$\int_0^t \alpha(s)e^{(\delta_{\mathbf{z}}+\Gamma)s}ds = \frac{1}{\delta_{\mathbf{z}}+\Gamma}\left[\alpha(t)e^{(\delta_{\mathbf{z}}+\Gamma)t} - \alpha_0 - \int_0^t \alpha'(s)e^{(\delta_{\mathbf{z}}+\Gamma)s}ds\right], \qquad \text{for } \delta_{\mathbf{z}}+\Gamma > 0$$

$$\int_0^t \alpha(s)ds = \left[t\alpha(t) - \int_0^t s\alpha'(s)ds\right], \qquad \text{for } \delta_{\mathbf{z}}+\Gamma = 0$$

so that, for $\delta_{\mathbf{z}}+\Gamma>0$ (the argument for $\delta_{\mathbf{z}}+\Gamma=0$ is similar and we omit it for brevity),

$$\mathbf{y}_t^T \geq e^{-(\delta_{\mathbf{z}}+\Gamma)t}\cdot$$

$$\cdot\left\{\bar{\mathbf{s}}^T+\frac{\Gamma_1+\delta_{\mathbf{z}}}{\delta_{\mathbf{z}}+\Gamma}\mathbf{w}_0^T\mathbf{A}\left[\alpha(t)e^{(\delta_{\mathbf{z}}+\Gamma)t}-\alpha_0-\int_0^t\alpha'(s)e^{(\delta_{\mathbf{z}}+\Gamma)s}ds\right]+\beta\mathbf{e}_1^T\int_0^t\alpha'(s)e^{(\delta_{\mathbf{z}}+\right.$$

and, since in this case $\Gamma=\Gamma_1$,

$$= e^{-(\delta_{\mathbf{z}}+\Gamma)t}\left[\bar{\mathbf{s}}^T-\alpha_0\mathbf{w}_0^T\mathbf{A}\right]+\alpha\left(t\right)\mathbf{w}_0^T\mathbf{A}$$

$$+\left[-\mathbf{w}_0^T\mathbf{A}+\beta\mathbf{e}_1^T\right]\int_0^t\alpha'\left(s\right)e^{-(\delta_{\mathbf{z}}+\Gamma)(t-s)}ds$$

$$=\mathbf{w}_t^T\mathbf{A}+e^{-(\delta_{\mathbf{z}}+\Gamma)t}\left(\left[\bar{\mathbf{s}}^T-\alpha_0\mathbf{w}_0^T\mathbf{A}\right]+\left[-\mathbf{w}_0^T\mathbf{A}+\beta\mathbf{e}_1^T\right]\int_0^t\alpha'(s)e^{(\delta_{\mathbf{z}}+\Gamma)s}ds\right)$$

It is clear that the constraints $\mathbf{y}_t^T\geq\mathbf{w}_t^T\mathbf{A}$ are satisfied if for every $t\geq 0$

$$\left[\bar{\mathbf{s}}^T-\alpha_0\mathbf{w}_0^T\mathbf{A}\right]+\left[-\mathbf{w}_0^T\mathbf{A}+\beta\mathbf{e}_1^T\right]\int_0^t\alpha'(s)e^{(\delta_{\mathbf{z}}+\Gamma)s}ds\geq 0.$$

To have this we need to set α_0 sufficiently small so that

$$\bar{\mathbf{s}}^T-\alpha_0\mathbf{w}_0^T\mathbf{A}\geq 0,$$

which is always possible since $\bar{\mathbf{s}}>0$. Moreover, since $\mathbf{w}_0^T\mathbf{A}\mathbf{e}_1>0$ then we choose $\beta=\mathbf{w}_0^T\mathbf{A}\mathbf{e}_1$ and use that $\alpha'\leq 0$ to find that the above strategy (28) is admissible but we have, for $\sigma\in(0,1)$,

$$U_\sigma(\omega)=\frac{1}{1-\sigma}\int_0^{+\infty}\omega_s^{1-\sigma}ds=\frac{1}{1-\sigma}\int_0^{+\infty}(-\beta\alpha'(s))^{1-\sigma}ds$$

$$=\frac{(\beta\alpha_0 b)^{1-\sigma}}{1-\sigma}\int_0^{+\infty}(1+s)^{(-1-b)(1-\sigma)}ds$$

and the last integral is infinite if $(-1-b)(1-\sigma)>-1$ i.e. if $b<\frac{\sigma}{1-\sigma}$).

∎

CA. Two Examples for $\sigma \neq 1$

Consider the case when $\sigma \neq 1$ and $a_0 = 0$ (i.e. $\rho = \Gamma(1-\sigma)$). Then change variable in our maximization problem setting

$$\omega_t = e^{-\frac{\rho}{1-\sigma}t}c_t; \qquad \mathbf{w}_t = e^{-\frac{\rho}{1-\sigma}t}\mathbf{x}_t; \qquad \mathbf{y}_t = e^{-\frac{\rho}{1-\sigma}t}\mathbf{s}_t.$$

The functional to maximize is now transformed into

$$\max \int_0^{+\infty} \frac{\omega_t^{1-\sigma}}{1-\sigma} dt.$$

The state equation becomes

$$\begin{aligned}
\dot{\mathbf{y}}_t^T &= -\frac{\rho}{1-\sigma} e^{-\frac{\rho}{1-\sigma}t}\mathbf{s}_t + e^{-\frac{\rho}{1-\sigma}t}\dot{\mathbf{s}}_t \\
&= -\frac{\rho}{1-\sigma} e^{-\frac{\rho}{1-\sigma}t}\mathbf{s}_t + e^{-\frac{\rho}{1-\sigma}t}\mathbf{x}_t^T\left(\mathbf{B}-\delta\mathbf{A}\right) - \delta_{\mathbf{z}} e^{-\frac{\rho}{1-\sigma}t}\mathbf{s}_t^T - e^{-\frac{\rho}{1-\sigma}t}c_t\mathbf{e}_1^T \\
&= -\left(\delta_{\mathbf{z}} + \frac{\rho}{1-\sigma}\right)\mathbf{y}_t + \mathbf{w}_t^T\left(\mathbf{B}-\delta\mathbf{A}\right) - \omega_t\mathbf{e}_1^T.
\end{aligned}$$

Since $a_0 = 0$ then $\frac{\rho}{1-\sigma} = \Gamma$ so

$$\dot{\mathbf{y}}_t^T = -\left(\delta_{\mathbf{z}} + \Gamma\right)\mathbf{y}_t + \mathbf{w}_t^T\left(\mathbf{B}-\delta\mathbf{A}\right) - \omega_t\mathbf{e}_1^T$$

with the constraints

$$\mathbf{y}^T \geq \mathbf{w}^T\mathbf{A}, \quad \mathbf{w} \geq \mathbf{0}, \quad \omega \geq 0.$$

If we take the case when $\delta_{\mathbf{x}} = \delta_{\mathbf{z}}$ (so $\delta = 0$) then we have

$$\dot{\mathbf{y}}_t^T = -\left(\delta_{\mathbf{z}} + \Gamma\right)\mathbf{y}_t + \mathbf{w}_t^T\mathbf{B} - \omega_t\mathbf{e}_1^T.$$

So we are reduced to study the following problem.

Maximize

$$\int_0^{+\infty} \frac{\omega_t^{1-\sigma}}{1-\sigma} dt$$

with state equation

$$\dot{\mathbf{y}}_t^T = -\left(\delta_{\mathbf{z}} + \Gamma\right)\mathbf{y}_t + \mathbf{w}_t^T\mathbf{B} - \omega_t\mathbf{e}_1^T$$

and pointwise constraints

$$\mathbf{y}^T \geq \mathbf{w}^T\mathbf{A}, \quad \mathbf{w} \geq \mathbf{0}, \quad \omega \geq 0.$$

CA.i. Example 1: $\sigma \in (0,1)$

Take $\sigma \in (0,1)$, $\delta_{\mathbf{x}} = \delta_{\mathbf{z}} \in (0,1)$ and

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}.$$

We check that $\Gamma = 3 - \delta_{\mathbf{x}} > 0$ and Γ is not a maximum. In fact

$$\mathbf{B} - \eta \mathbf{A} = \begin{bmatrix} 0 & 3-\eta \\ 3-\eta & -\eta \end{bmatrix}$$

$$\mathbf{w}^T [\mathbf{B} - \eta \mathbf{A}] = \begin{matrix} w_2 (3-\eta) \\ w_1 (3-\eta) - w_2 \eta \end{matrix}$$

It is clear that

$$\mathbf{w}^T [\mathbf{B} - \eta \mathbf{A}] \geq \mathbf{0}$$

means

$$\begin{aligned} w_2 (3-\eta) &\geq 0 \\ w_1 (3-\eta) - w_2 \eta &\geq 0 \end{aligned}$$

If $\eta > 3$ then it is clear that the above system implies $w = 0$. If $\eta = 3$ then $w = (1,0)$ is a solution so $\Gamma_0 = 3 - \delta_{\mathbf{x}}$. Similarly, looking at the inequality

$$\mathbf{w}^T [\mathbf{B} - \eta \mathbf{A}] \geq \mathbf{e}_1^T$$

i.e.

$$\begin{aligned} w_2 (3-\eta) &\geq 1 \\ w_1 (3-\eta) - w_2 \eta &\geq 0 \end{aligned}$$

we see that for $\eta = 3$ there is no solution since the left hand side of the first inequality is zero. For $\eta < 3$ we have

$$\begin{aligned} w_2 &\geq \frac{1}{3-\eta} \\ w_1 &\geq w_2 \frac{\eta}{3-\eta} \end{aligned}$$

so a solution is

$$\mathbf{w}^{\eta} = \left(\frac{\eta}{(3-\eta)^2}, \frac{1}{3-\eta} \right).$$

Note that, when $(3-\eta) \to 0$ then $||\mathbf{w}|| \to +\infty$ with order $O\left(\frac{1}{(3-\eta)^2}\right)$, this fact will be a key point to establish the nonexistence result. In particular we have proved that $\Gamma = \Gamma_0 = 3 - \delta_{\mathbf{x}}$ and that Γ is not a maximum.

Now observe that the state equation is

$$\dot{\mathbf{y}}_t^T = -3\mathbf{y}_t + \mathbf{w}_t^T \mathbf{B} - \omega_t \mathbf{e}_1^T = \begin{cases} -3y_{1t} + 3w_{2t} - \omega_t \\ -3y_{2t} + 3w_{1t} \end{cases} \tag{29}$$

and the pointwise constraints are

$$\mathbf{w} \geq \mathbf{0}, \quad \omega \geq 0$$

$$\mathbf{w}^T \mathbf{A} \leq \mathbf{y} \Longleftrightarrow \begin{cases} w_2 \leq y_1 \\ w_1 + w_2 \leq y_2 \end{cases} . \tag{30}$$

It is possible to prove that all suboptimal strategies ω satisfy the following key estimate holds for a suitable positive constant M depending only on initial data:

$$\int_0^t (1+s)\, \omega_s ds \leq M, \qquad \forall t \geq 0. \tag{31}$$

From the above we get, using the Schwarz inequality

$$\begin{aligned} \int_0^t \omega_s^{1-\sigma} ds &\leq \int_0^t ((1+s)\, \omega_s)^{1-\sigma} \frac{1}{(1+s)^{1-\sigma}} ds \\ &\leq \left(\int_0^t (1+s)\, \omega_s ds \right)^{1-\sigma} \left(\int_0^t \left(\frac{1}{(1+s)^{1-\sigma}} \right)^{\frac{1}{\sigma}} ds \right)^{\sigma} \\ &\leq M^{1-\sigma} \left(\int_0^t \frac{1}{(1+s)^{\frac{1}{\sigma}-1}} ds \right)^{\sigma} . \end{aligned}$$

Letting $t \to +\infty$ we get that, for $\frac{1}{\sigma} - 1 > 1$, i.e $\sigma < \frac{1}{2}$, the value of the problem is finite. Once this estimate is proven, the proof of existence result can be done along the same lines as the proof of Theorem III..1. Moreover, in this case one can see that the sufficient condition given in [7] holds. The system of sufficient conditions has a solution in this case and so we can get from it the existence of an optimal strategy.

In the case when $\sigma > \frac{1}{2}$, we have nonexistence. It is enough to take the following admissible strategy, for suitable $\alpha > 0$:

$$\begin{aligned}\mathbf{w}_t &= \alpha (3-\eta)^2 \mathbf{w}^{\eta} e^{-(3-\eta)t} \\ \omega_t &= \alpha (3-\eta)^2 e^{-(3-\eta)t}.\end{aligned}$$

Arguing as in the proof of Proposition C.1 we see that for α sufficiently small (i.e. such that $\alpha (3-\eta)^2 w^{\eta} \leq y_0$) the above strategy is admissible and the functional is

$$\begin{aligned}\int_0^{+\infty} \omega_t^{1-\sigma} dt &= \alpha^{1-\sigma} (3-\eta)^{2(1-\sigma)} \frac{1}{1-\sigma} \int_0^{+\infty} e^{-(3-\eta)(1-\sigma)t} dt \\ &= \alpha^{1-\sigma} (3-\eta)^{2(1-\sigma)} \frac{1}{1-\sigma} \frac{1}{(3-\eta)(1-\sigma)} \\ &= \alpha^{1-\sigma} \frac{1}{(1-\sigma)^2} (3-\eta)^{1-2\sigma}.\end{aligned}$$

If $1 - 2\sigma < 0$ i.e. $\sigma > \frac{1}{2}$ for $\eta \to 3$ we get that the value of the problem goes to $+\infty$ so the supremum is infinite. ∎

To prove that (31) we need first the following lemma.

Lemma C.2 *The above problem is equivalent to the following problem* ***(P reduced)****:*

Maximize

$$\int_0^{+\infty} \frac{\omega_t^{1-\sigma}}{1-\sigma} dt$$

with state equation state equation

$$\begin{aligned}y'_{1t} &= -3y_{1t} + 3w_{2t} - \omega_t \\ y'_{2t} &= -3w_{2t}\end{aligned} \tag{32}$$

and constraints

$$\begin{aligned}\omega &\geq 0,\\ w_2 &\geq 0,\\ w_2 &\leq y_1,\\ w_2 &\leq y_2.\end{aligned}$$

Using the state equation (32) the last two constraints can be expressed equivalently in integral form as follows

$$w_{2t} \leq e^{-3t} y_{10} + \int_0^t e^{-3(t-s)} \left(3w_{2s} - \omega_s\right) ds \tag{33}$$

$$w_{2t} \leq y_{20} - \int_0^t 3w_{2s} ds.$$

Proof. We observe that, substituting the second constraint of (30)

$$w_1 + w_2 \leq y_2 \Longleftrightarrow w_1 \leq y_2 - w_2 \tag{34}$$

in the second line of the state equation (29) we get

$$y'_{2t} = -3y_{2t} + 3w_{1t} \leq -3y_{2t} + 3\left(y_{2t} - w_{2t}\right) = -3w_{2t}. \tag{35}$$

Choose now a new production-consumption strategy $(\bar{\mathbf{w}},\bar{\omega})$ as follows:

$$\begin{aligned}\bar{\omega} &= \omega\\ \bar{w}_2 &= w_2\end{aligned}$$

and, setting $\bar{\mathbf{y}}$ the associated state trajectory:

$$\bar{w}_1 = \bar{y}_2 - \bar{w}_2 = \bar{y}_2 - w_2. \tag{36}$$

We show that this new strategy is admissible. Indeed with this choice we have

$$\bar{y}'_{2t} = -3\bar{y}_{2t} + 3\bar{w}_{1t} = -3\bar{w}_{2t} = -3w_{2t}. \tag{37}$$

Comparing (35) with (37) we get that,

$$\bar{y}_{2t} \geq y_{2t}, \qquad \forall t \geq 0,$$

so that, from (36) and (34) we get, for a.e $t \geq 0$,

$$\bar{w}_{1t} = \bar{y}_{2t} - w_{2t} \geq y_{2t} - w_{2t} \geq w_{1t} \geq 0.$$

This means that $(\bar{\mathbf{w}}, \bar{\omega})$ is admissible at $\mathbf{y}_0$ as the other constraints are trivially satisfied. The statement follows. ∎

Now observe that if $w_{2t} = y_{1t}$ at almost every $t \geq 0$ then the state equation must be

$$y'_{1t} = -\omega_t$$
$$y'_{2t} = -3y_{1t}$$

with $y_{1t} \geq y_{2t}$ a.e. In this case the estimate

$$\int_0^t (1+s)\,\omega_s ds \leq M, \qquad \forall t \geq 0$$

easily follows from the presence of nonsimple eigenvalues in the linear part of the system. Indeed the solution is

$$y_{1t} = y_{10} - \int_0^t \omega_s ds$$

$$\begin{aligned} y_{2t} &= y_{20} - \int_0^t 3y_{1s} ds \\ &= y_{20} - \int_0^t 3\left[y_{10} - \int_0^s \omega_r dr\right] ds \\ &= y_{20} - 3ty_{10} + 3\int_0^t \int_0^s \omega_r dr ds \\ &= y_{20} - 3ty_{10} + 3t\int_0^t \omega_s ds - 3\int_0^t s\omega_s ds \\ &= y_{20} - 3ty_{1t} - \int_0^t s\omega_s ds \end{aligned}$$

From the first equation it follows

$$\int_0^t \omega_s ds \leq y_{10}$$

and from the second

$$3\int_0^t s\omega_s ds \le y_{20}$$

from which the claim follows. It is now enough to show that we can reduce our problem to this case.

To reduce to the above case we approximate w_2 with regular functions. We omit the details for brevity. ∎

CA.ii. Example 2: $\sigma \in (1, +\infty)$

Take $\delta_{\mathbf{x}} = \delta_{\mathbf{z}} \in (0,1)$ and

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

In this case it is easy to check that $\Gamma = 1 - \delta_{\mathbf{x}} > 0$ and Γ is a maximum. To see this observe that $\mathbf{B}\mathbf{e}_1 \neq \mathbf{0}$ so $\mathcal{G}_1 \supseteq (-\infty, -\delta_{\mathbf{x}}]$. Moreover

$$\mathbf{B}-\eta\mathbf{A} = \begin{bmatrix} 1 & 1-\eta & 0 \\ 0 & 1 & 1-\eta \end{bmatrix}$$

$$\mathbf{w}^T\left[\mathbf{B}-\eta\mathbf{A}\right] = \begin{pmatrix} w_1 \\ w_1(1-\eta) + w_2 \\ w_2(1-\eta) \end{pmatrix}.$$

It is clear that

$$\mathbf{w}^T\left[\mathbf{B}-\eta\mathbf{A}\right] \ge \mathbf{0}$$

for $\eta > 1$ gives $\mathbf{w} = \mathbf{0}$. For $\eta = 1$ it is true fo every $\mathbf{w} \ge \mathbf{0}$. Similarly, looking at the inequality

$$\mathbf{w}^T\left[\mathbf{B}-\eta\mathbf{A}\right] \ge \mathbf{e}_1^T$$

we see that $\Gamma_1 = 1 - \delta_{\mathbf{x}}$ as it is satisfied for $\eta = 1$ and $\mathbf{w} = (1,0)$. Now take $\sigma > 1$ and $\rho = \Gamma(1-\sigma)$. In this case we have existence when $\sigma > 2$ and nonexistence for $\sigma < 2$. The state equation is

$$\dot{\mathbf{y}}_t^T = -\mathbf{y}_t + \mathbf{w}_t^T\mathbf{B} - \omega_t\mathbf{e}_1^T.$$

with $\mathbf{w} \geq \mathbf{0}$, $\omega \geq 0$, and

$$\mathbf{w}^T \mathbf{A} \leq \mathbf{y}^T \Longleftrightarrow \begin{pmatrix} 0 \\ w_1 \\ w_2 \end{pmatrix} \leq \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

To prove existence it is enough to exhibit a strategy fit utility $> -\infty$. Setting

$$\mathbf{w}_t = \begin{pmatrix} y_3(0)\,t + y_2(0) - y_3(0) \\ y_3(0) \end{pmatrix}; \quad \omega_t = \alpha + \beta t$$

we get

$$\mathbf{y}_t = \begin{pmatrix} y_1(0) + [y_2(0) - \alpha]\,t + [y_3(0) - \beta]\,\frac{t^2}{2} \\ y_2(0) + y_3(0)\,t \\ y_3(0) \end{pmatrix}.$$

If we choose the initial datum so that $y_2(0) - y_3(0) \geq 0$ and $\alpha \leq y_2(0)$, $\beta \leq y_3(0)$ then all constraints are satisfied. Moreover we have,

$$\int_0^{+\infty} \frac{\omega_t^{1-\sigma}}{1-\sigma} dt = \frac{1}{1-\sigma} \int_0^{+\infty} \frac{1}{(\alpha + \beta t)^{\sigma - 1}} dt$$

and, for $\sigma > 2$ this is finite.

The proof of nonexistence follows the lines of the previous example. We omit it for brevity. ■

D Controllability results

Proposition D.1 *Let Assumptions II..2 and II..3 be verified.*

1. *Let II..4 be verified. Then for every* $t_0 > 0$ *there exists an admissible strategy* $(\mathbf{x},c)$ *such that* $\mathbf{s}^T_{t_0,\bar{\mathbf{s}},(\mathbf{x},c)}\mathbf{e}_1 > 0$.
2. *Let* $j = 2, ..., n$, *and let Assumption II..5 be verified for such* j. *Then for every* $t_0 > 0$ *there exists an admissible strategy* $(\mathbf{x},c)$ *such that* $\mathbf{s}^T_{t_0,\bar{\mathbf{s}},(\mathbf{x},c)}\mathbf{e}_j > 0$.

3. *Let II..4 and II..5 be verified. Then for every $t_0 > 0$ there exists an admissible strategy $(\mathbf{x},c)$ such that $\mathbf{s}^T_{t_0,\bar{\mathbf{s}},(\mathbf{x},c)} > 0$.*

Proof of 1). Let us start from the first statement. Of course it is verified when $\bar{\mathbf{s}}^T\mathbf{e}_1 > 0$. Consider the case when $\bar{\mathbf{s}}^T\mathbf{e}_1 = 0$. Given an admissible strategy $(\mathbf{x},c)$ and the associate state trajectory $\mathbf{s}$ we define (with the agreement that $\inf\emptyset = +\infty$)

$$T_1(\mathbf{x},c) = \inf\left\{t > 0 : \quad \mathbf{s}^T_{t,\bar{\mathbf{s}},(\mathbf{x},c)}\mathbf{e}_1 > 0\right\}.$$

By the above definition and the continuity of the trajectory $\mathbf{s}$ it is clear that $\mathbf{s}^T_{T_1(\mathbf{x},c),\bar{\mathbf{s}},(\mathbf{x},c)}\mathbf{e}_1 = 0$. It is clear also that $T_1(\mathbf{x},c) \geq T_1(\mathbf{x},0)$. From Assumption II..4 it follows that there exists an admissible strategy $(\mathbf{x},c) \in \mathcal{A}(\bar{\mathbf{s}})$ with $T_1(\mathbf{x},c) < +\infty$. In particular this implies that also $T_1(\mathbf{x},0) < +\infty$. We need to prove that, defining

$$T_{\text{inf}} = \inf\left\{T_1(\mathbf{x},0) : \quad \mathbf{x} \text{ s.t. } (\mathbf{x},0) \in \mathcal{A}(\bar{\mathbf{s}})\right\}$$

we have $T_{\text{inf}} = 0$.

To prove this we prove that given $\mathbf{x}$ with $(\mathbf{x},0) \in \mathcal{A}(\bar{\mathbf{s}})$ there exist $\tilde{\mathbf{x}}$ such that $(\tilde{\mathbf{x}},0) \in \mathcal{A}(\bar{\mathbf{s}})$ and $T_1(\tilde{\mathbf{x}},0) \leq T_1(\mathbf{x},0)/2$.

The strategy we use is based on the following intuitive fact: to pass from a time t_1 where $\mathbf{s}^T_{t_1}\mathbf{e}_1 = 0$ to a subsequent one t_2 where $\mathbf{s}^T_{t_2}\mathbf{e}_1 > 0$ one needs to produce the commodity 1. This may be done immediately if all goods that enters in the production of the first good are available. If not one needs also to produce the goods that enters in the production of the first one and so on. So the key point is activating new production processes which make positive some zero cohordinates of $\mathbf{s}$. Substantially we want to show that if this activation can be done at a certain times t_1, t_2,... then it can be done from the beginning.

For $t \geq 0$ we have

$$\mathbf{s}^T_t = e^{-\delta_{\mathbf{z}}t}\bar{\mathbf{s}}^T + \int_0^t e^{-\delta_{\mathbf{z}}(t-s)}\mathbf{x}^T_s\left[\mathbf{B} - \delta\mathbf{A}\right]ds$$

so, being $\bar{\mathbf{s}}^T\mathbf{e}_1 = 0$

$$\mathbf{s}^T_t\mathbf{e}_1 = \int_0^t e^{-\delta_{\mathbf{z}}(t-s)}\mathbf{x}^T_s\left[\mathbf{B} - \delta\mathbf{A}\right]\mathbf{e}_1 ds.$$

Now we have, by definition of $T_1(\mathbf{x},0)$,

$$\mathbf{s}_t^T \mathbf{e}_1 = 0, \quad \forall t \in [0, T_1(\mathbf{x},0)] .$$

This implies that

$$\int_0^t e^{-\delta_{\mathbf{z}}(t-s)} \mathbf{x}_s^T [\mathbf{B} - \delta \mathbf{A}] \mathbf{e}_1 ds = 0, \quad \forall t \in [0, T_1(\mathbf{x},0)] \Longrightarrow \mathbf{x}_t^T [\mathbf{B} - \delta \mathbf{A}] \mathbf{e}_1 = 0, \quad \forall$$

(this comes form the density of step functions in $L^2(0, T_1(\mathbf{x},0); \mathbb{R}^m)$). Since $\mathbf{s}_t^T \geq \mathbf{x}_t^T \mathbf{A}$ we also have, for $t \in [0, T_1(\mathbf{x},0)]$

$$\mathbf{x}_t^T \mathbf{A} \mathbf{e}_1 = 0, \quad \forall t \in [0, T_1(\mathbf{x},0)]$$

so from the above formula also

$$\mathbf{x}_t^T \mathbf{B} \mathbf{e}_1 = 0, \quad \forall t \in [0, T_1(\mathbf{x},0)] .$$

Moreover for $t > T_1(\mathbf{x},0)$ we have

$$\mathbf{s}_t^T \mathbf{e}_1 = \int_{T_1(\mathbf{x},0)}^t e^{-\delta_{\mathbf{z}}(t-s)} \mathbf{x}_s^T [\mathbf{B} - \delta \mathbf{A}] \mathbf{e}_1 ds$$

Assume for the moment that the trajectory $\mathbf{s}$ have the same number of zero components till time $T_1(\mathbf{x},0)$. Then there exists $\lambda > 0$ such that

$$\bar{\mathbf{s}} \geq \lambda \mathbf{s}_{T_1(\mathbf{x},0)}.$$

So fix such λ and consider the control $(\tilde{\mathbf{x}},0)$ with

$$\tilde{\mathbf{x}}_s = \lambda \mathbf{x}_{s+T_1(\mathbf{x},0)}, \quad s \geq 0.$$

We show that this control is admissible brings the first component to be strictly positive immediately. In fact call $\tilde{\mathbf{s}}$ the associated state trajectory and set for simplicity $T_1 = T_1(\mathbf{x},0)$. We have

$$\begin{aligned}\tilde{\mathbf{s}}_t &= e^{-\delta_{\mathbf{z}} t} \bar{\mathbf{s}} + \int_0^t e^{-\delta_{\mathbf{z}}(t-s)} \tilde{\mathbf{x}}_s^T [\mathbf{B} - \delta \mathbf{A}] ds \\ &= e^{-\delta_{\mathbf{z}} t} \bar{\mathbf{s}} + \lambda \int_0^t e^{-\delta_{\mathbf{z}}(t-s)} \mathbf{x}_{s+T_1}^T [\mathbf{B} - \delta \mathbf{A}] ds\end{aligned}$$

and, in particular, since by the construction $\bar{\mathbf{s}}^T\mathbf{e}_1 = 0$

$$\begin{aligned}\tilde{\mathbf{s}}_t^T\mathbf{e}_1 &= \lambda\int_0^t e^{-\delta_{\mathbf{z}}(t-s)}\mathbf{x}_{s+T_1(\mathbf{x},0)}^T\left[\mathbf{B}-\delta\mathbf{A}\right]\mathbf{e}_1 ds \\ &= \lambda\int_{T_1}^{t+T_1} e^{-\delta_{\mathbf{z}}(t+T_1-s_1)}\mathbf{x}_{s_1}^T\left[\mathbf{B}-\delta\mathbf{A}\right]\mathbf{e}_1 ds_1 \\ &= \lambda\mathbf{s}_{t+T_1}^T\mathbf{e}_1\end{aligned}$$

which means that $\tilde{\mathbf{s}}_t^T\mathbf{e}_1 > 0$ when $\mathbf{s}_{t+T_1}^T\mathbf{e}_1 > 0$ so that $T_1(\tilde{\mathbf{x}},0) = 0$. Concerning admissibility it is clear that $\tilde{\mathbf{x}}_s \geq \mathbf{0}$ by its definition. Let us show that, for the required value of the parameter λ, we have $\tilde{\mathbf{s}}_t \geq \tilde{\mathbf{x}}_t^T\mathbf{A}$ for $t > 0$. In fact we have for $t > 0$

$$\begin{aligned}\tilde{\mathbf{s}}_t - \tilde{\mathbf{x}}_t^T\mathbf{A} &= e^{-\delta_{\mathbf{z}}t}\bar{\mathbf{s}} + \lambda\int_0^t e^{-\delta_{\mathbf{z}}(t-s)}\mathbf{x}_{s+T_1}^T\left[\mathbf{B}-\delta\mathbf{A}\right]ds - \lambda\mathbf{x}_{t+T_1}^T\mathbf{A} \\ &= e^{-\delta_{\mathbf{z}}t}\bar{\mathbf{s}} + \lambda\int_{T_1}^{t+T_1} e^{-\delta_{\mathbf{z}}(t+T_1-s_1)}\mathbf{x}_{s_1}^T\left[\mathbf{B}-\delta\mathbf{A}\right]ds_1 - \lambda\mathbf{x}_{t+T_1}^T\mathbf{A}\end{aligned}$$

Now recall that, for $t_1 > T_1$

$$\mathbf{s}_{t_1} = e^{-\delta_{\mathbf{z}}(t_1-T_1)}\mathbf{s}_{T_1} + \int_{T_1}^{t_1} e^{-\delta_{\mathbf{z}}(t_1-s_1)}\mathbf{x}_{s_1}^T\left[\mathbf{B}-\delta\mathbf{A}\right]ds_1$$

so that for $t > 0$ (setting $t_1 = t + T_1$ above)

$$\begin{aligned}\tilde{\mathbf{s}}_t - \tilde{\mathbf{x}}_t^T\mathbf{A} &= e^{-\delta_{\mathbf{z}}t}\bar{\mathbf{s}} + \lambda\left[\mathbf{s}_{t+T_1} - e^{-\delta_{\mathbf{z}}t}\mathbf{s}_{T_1}\right] - \lambda\mathbf{x}_{t+T_1}^T\mathbf{A} \\ &= e^{-\delta_{\mathbf{z}}t}\left[\bar{\mathbf{s}} - \lambda\mathbf{s}_{T_1}\right] + \lambda\left[\mathbf{s}_{t+T_1} - \mathbf{x}_{t+T_1}^T\mathbf{A}\right] \\ &\geq e^{-\delta_{\mathbf{z}}t}\left[\bar{\mathbf{s}} - \lambda\mathbf{s}_{T_1}\right].\end{aligned}$$

This gives ammissibility and so the claim in this case.

If this case do not hold then there are times $t \geq 0$ where the number of zero components changes. However, due to the above argument, if the $j-$th component is positive at a certain time t then it remains always positive

$$\mathbf{s}_t^T\mathbf{e}_j > 0 \Rightarrow \mathbf{s}_s^T\mathbf{e}_j > 0 \quad \forall s > t.$$

This implies that any changes in the number of zero components comes when a zero component become positive and this means that there is a finite sequence of times $0 \le t_1 < ... < t_k = T_1$ where some zero components become strictly positive (i.e. immediately after). If $k = 1$ we are in the previous case. Let $k = 2$ and, for simplicity, $t_1 = 0$. We want to show that the strategy

$$\tilde{\mathbf{x}}_t = \begin{cases} \alpha_1 \mathbf{x}_t & t \in \left[0, \frac{T_1}{2}\right] \\ \alpha_2 \mathbf{x}_{t+\frac{T_1}{2}} & t > \frac{T_1}{2} \end{cases}$$

for suitable $\alpha_1, \alpha_2 > 0$ is admissible and brings the first component to be positive immediately after $\frac{T_1}{2}$.

Concerning the positivity of the first component we have, for $t > \frac{T_1}{2}$

$$\begin{aligned} \tilde{\mathbf{s}}_t &= e^{-\delta_{\mathbf{z}} t} \bar{\mathbf{s}} + \int_0^t e^{-\delta_{\mathbf{z}}(t-s)} \tilde{\mathbf{x}}_s^T \left[\mathbf{B} - \delta \mathbf{A}\right] ds \\ &= e^{-\delta_{\mathbf{z}} t} \bar{\mathbf{s}} + \alpha_1 \int_0^{\frac{T_1}{2}} e^{-\delta_{\mathbf{z}}(t-s)} \mathbf{x}_s^T \left[\mathbf{B} - \delta \mathbf{A}\right] ds + \alpha_2 \int_{\frac{T_1}{2}}^t e^{-\delta_{\mathbf{z}}(t-s)} \mathbf{x}_{s+\frac{T_1}{2}}^T \left[\mathbf{B} - \delta \mathbf{A}\right] ds \end{aligned}$$

and, in particular, since by the construction $\bar{\mathbf{s}}^T \mathbf{e}_1 = 0$ and $\int_0^{\frac{T_1}{2}} e^{-\delta_{\mathbf{z}}(t-s)} \mathbf{x}_s^T [\mathbf{B} - \delta \mathbf{A}$ 0 we have, for $t > \frac{T_1}{2}$,

$$\begin{aligned} \tilde{\mathbf{s}}_t^T \mathbf{e}_1 &= \alpha_2 \int_{\frac{T_1}{2}}^t e^{-\delta_{\mathbf{z}}(t-s)} \mathbf{x}_{s+\frac{T_1}{2}}^T \left[\mathbf{B} - \delta \mathbf{A}\right] \mathbf{e}_1 ds \\ &= \alpha_2 \int_{T_1}^{t+\frac{T_1}{2}} e^{-\delta_{\mathbf{z}}\left(t+\frac{T_1}{2}-s_1\right)} \mathbf{x}_{s_1}^T \left[\mathbf{B} - \delta \mathbf{A}\right] \mathbf{e}_1 ds_1. \end{aligned}$$

Now we know that $\int_{T_1}^{t+\frac{T_1}{2}} e^{-\delta_{\mathbf{z}}\left(t+\frac{T_1}{2}-s_1\right)} \mathbf{x}_{s_1}^T \left[\mathbf{B} - \delta \mathbf{A}\right] \mathbf{e}_1 ds_1 = \mathbf{s}_{t+\frac{T_1}{2}}^T \mathbf{e}_1 > 0$ for any $t > \frac{T_1}{2}$ (thanks also to the above reduction argument). If we would have

$$\int_{T_1}^{t+\frac{T_1}{2}} e^{-\delta_{\mathbf{z}}\left(t+\frac{T_1}{2}-s_1\right)} \mathbf{x}_{s_1}^T \left[\mathbf{B} - \delta \mathbf{A}\right] \mathbf{e}_1 ds_1 = 0 \text{ for every } t > \frac{T_1}{2}$$

then this would mean that (thanks to the density of step functions in L^2 of a bounded interval)

$$e^{-\delta_{\mathbf{z}}\left(t+\frac{T_1}{2}-s_1\right)} \mathbf{x}_{s_1}^T \left[\mathbf{B} - \delta \mathbf{A}\right] \mathbf{e}_1 ds_1 = 0, \quad \forall s_1 > T_1.$$

this would imply that

$$\mathbf{x}_{s_1}^T\left[\mathbf{B}-\delta\mathbf{A}\right]\mathbf{e}_1=0,\quad \forall s_1>T_1$$

so that also $\int_{T_1}^{t+\frac{T_1}{2}} e^{-\delta_{\mathbf{z}}\left(t+\frac{T_1}{2}-s_1\right)}\mathbf{x}_{s_1}^T\left[\mathbf{B}-\delta\mathbf{A}\right]\mathbf{e}_1 ds_1=0$ for $t>\frac{T_1}{2}$ and this is a contradiction.

Concerning admissibility it is clear that $\tilde{\mathbf{x}}_s\geq\mathbf{0}$ by its definition. Let us show that, for suitable choice of the parameters, we have $\tilde{\mathbf{s}}_t\geq\tilde{\mathbf{x}}_t^T\mathbf{A}$ for $t>0$. This is obvious for $t\leq\frac{T_1}{2}$ since we already know that $(\mathbf{x},0)$ is admissible, and so $(\alpha_1\mathbf{x},0)$ is admissible thanks to the structure of the set of admissible trajectories (see on this [4]). For $t>\frac{T_1}{2}$ we have

$$\begin{aligned}
\tilde{\mathbf{s}}_t-\tilde{\mathbf{x}}_t^T\mathbf{A} &= e^{-\delta_{\mathbf{z}}t}\overline{\mathbf{s}}+\alpha_1\int_0^{\frac{T_1}{2}} e^{-\delta_{\mathbf{z}}(t-s)}\mathbf{x}_s^T\left[\mathbf{B}-\delta\mathbf{A}\right]ds\\
&\quad+\alpha_2\int_{\frac{T_1}{2}}^{t} e^{-\delta_{\mathbf{z}}(t-s)}\mathbf{x}_{s+\frac{T_1}{2}}^T\left[\mathbf{B}-\delta\mathbf{A}\right]ds-\alpha_2\mathbf{x}_{t+\frac{T_1}{2}}^T\mathbf{A}\\
&= e^{-\delta_{\mathbf{z}}\left(t-\frac{T_1}{2}\right)}\tilde{\mathbf{s}}_{\frac{T_1}{2}}+\alpha_2\left[\int_{\frac{T_1}{2}}^{t} e^{-\delta_{\mathbf{z}}(t-s)}\mathbf{x}_{s+\frac{T_1}{2}}^T\left[\mathbf{B}-\delta\mathbf{A}\right]ds-\mathbf{x}_{t+\frac{T_1}{2}}^T\mathbf{A}\right]\\
&= e^{-\delta_{\mathbf{z}}\left(t-\frac{T_1}{2}\right)}\tilde{\mathbf{s}}_{\frac{T_1}{2}}+\alpha_2\left[\int_{T_1}^{t+\frac{T_1}{2}} e^{-\delta_{\mathbf{z}}\left(t+\frac{T_1}{2}-s_1\right)}\mathbf{x}_{s_1}^T\left[\mathbf{B}-\delta\mathbf{A}\right]ds_1-\mathbf{x}_{t+\frac{T_1}{2}}^T\mathbf{A}\right]
\end{aligned}$$

Now recall that, for $t_1>T_1$

$$\mathbf{s}_{t_1}=e^{-\delta_{\mathbf{z}}(t-T_1)}\mathbf{s}_{T_1}+\int_{T_1}^{t_1} e^{-\delta_{\mathbf{z}}(t_1-s)}\mathbf{x}_{s_1}^T\left[\mathbf{B}-\delta\mathbf{A}\right]ds_1$$

so that taking $t_1=t+\frac{T_1}{2}$ above

$$\begin{aligned}
&\alpha_2\left[\int_{T_1}^{t+\frac{T_1}{2}} e^{-\delta_{\mathbf{z}}\left(t+\frac{T_1}{2}-s_1\right)}\mathbf{x}_{s_1}^T\left[\mathbf{B}-\delta\mathbf{A}\right]ds_1-\mathbf{x}_{t+\frac{T_1}{2}}^T\mathbf{A}\right]\\
&=\alpha_2\left[\mathbf{s}_{t+\frac{T_1}{2}}-e^{-\delta_{\mathbf{z}}\left(t-\frac{T_1}{2}\right)}\mathbf{s}_{T_1}-\mathbf{x}_{t+\frac{T_1}{2}}^T\mathbf{A}\right]
\end{aligned}$$

so

$$\begin{aligned}\tilde{\mathbf{s}}_t - \tilde{\mathbf{x}}_t^T\mathbf{A} &= e^{-\delta_{\mathbf{z}}\left(t-\frac{T_1}{2}\right)}\tilde{\mathbf{s}}_{\frac{T_1}{2}} + \alpha_2\left[\int_{T_1}^{t+\frac{T_1}{2}} e^{-\delta_{\mathbf{z}}\left(t+\frac{T_1}{2}-s_1\right)}\mathbf{x}_{s_1}^T\left[\mathbf{B}-\delta\mathbf{A}\right]ds_1 - \mathbf{x}_{t+\frac{T_1}{2}}^T\mathbf{A}\right]\\ &= e^{-\delta_{\mathbf{z}}\left(t-\frac{T_1}{2}\right)}\tilde{\mathbf{s}}_{\frac{T_1}{2}} + \alpha_2\left[\mathbf{s}_{t+\frac{T_1}{2}} - e^{-\delta_{\mathbf{z}}\left(t-\frac{T_1}{2}\right)}\mathbf{s}_{T_1} - \mathbf{x}_{t+\frac{T_1}{2}}^T\mathbf{A}\right]\\ &= e^{-\delta_{\mathbf{z}}\left(t-\frac{T_1}{2}\right)}\left(\tilde{\mathbf{s}}_{\frac{T_1}{2}} - \alpha_2\mathbf{s}_{T_1}\right) + \alpha_2\left[\mathbf{s}_t - \mathbf{x}_t^T\mathbf{A}\right]\end{aligned}$$

Now by construction $\tilde{\mathbf{s}}_{\frac{T_1}{2}}$ and $\mathbf{s}_{T_1}$ have the same zero components, so we can choose $\alpha_2 > 0$ so that $\tilde{\mathbf{s}}_{\frac{T_1}{2}} - \alpha_2\mathbf{s}_{T_1} \geq 0$. This gives the claim for $k = 2$. The case $k > 2$ can be treated exactly in the same way.

Proof of 2). It is exactly the same proof as above for every $j \neq 1$.

Proof of 3). From the points 1) and 2) we know how to find strategies that brings one component to be positive. To brings all components to be positive one needs at most $k \leq n$ steps where k is the number of zero components of $\bar{\mathbf{s}}$. Then fix $t_0 > 0$ and apply the strategy to brings the first components to be positive on $\left[0, \frac{t_0}{k}\right]$ and so in k steps all components become positive at most at t_0. ■

Proposition D.2 *Let Assumptions II..2, II..3 hold true. Assume also that Assumption II..5 does not hold. Then the truncated problem (if the input-output matrices are not empty) satisfies II..2, II..3.*

Proof. For the proof we will consider first the case where only one commodity is always zero so the truncation algorithm consists only in deleting one column and the rows where such column has nonzero elements.

The fact that Assumption II..2 still holds in the truncated problem is easy. In fact if the j-th commodity is always zero for every admissible strategy, then the input matrix $\mathbf{C}$ in the truncated problem is obtained from the original matrix $\mathbf{A}$ deleting the j-th column and all the rows i such that $a_{ij} > 0$. If, by contradiction, the k-th row of $\mathbf{C}$ is zero this means that the k-th row of $\mathbf{A}$ have all elements zero except for a_{kj}. But this means that such row should have been deleted in the truncation process, a contradiction.

In the general case, if $\mathbf{C}$ has a zero row (say the i-th), then this means that in the i-th row of $\mathbf{A}$ there is at least a nonzero element belonging to a column (say the j-th) corresponding to a zero component. But $a_{ij} > 0$ would implie that the i-th row would be deleted in truncation process. This gives again a contradiction.

More difficult is to prove that Assumption II..3 still holds. We argue by contradiction. Assume that in the truncated problem the output matrix $\mathbf{D}$ has at least one row zero, let one of them be the k-th row. This means that in such row all elements of the original output matrix $\mathbf{B}$ where zero except for b_{kj}. Moreover such row was not cancelled in the truncation so this means that $a_{kj} = 0$ (i.e. the commodity j is not used in the k-th production process). Consider now a strategy ($\hat{\mathbf{x}}$,0) such that, in time t_0 it brings all commodities, except for the j-th, to be positive (this is possible thanks to point 2 of Proposition D.1) But in this case the production strategy

$$\begin{cases} \hat{\mathbf{x}}_t & t \in [0, t_0] \\ \mathbf{x}_t = \alpha \mathbf{e}_k & t > t_0 \end{cases}$$

is admissible for small α (and small time) and brings the j-th commodity to be nonzero immediately after t_0. In fact

$$\mathbf{x}^T \mathbf{A} = \alpha \mathbf{e}_k^T \mathbf{A} = \alpha \left(a_{k1}, ..., a_{kn} \right)$$

and since $\mathbf{s}_{t_0}^T \mathbf{e}_i > 0$ for $i \neq j$, $\mathbf{s}_{t_0}^T \mathbf{e}_j = 0$, $a_{kj} = 0$ we can choose α such that

$$\alpha \left(a_{k1}, ..., a_{kn} \right) \leq \frac{\mathbf{s}_{t_0}^T}{2}$$

($\alpha \leq \min \left\{ \frac{\mathbf{s}_{t_0}^T \mathbf{e}_i}{2 a_{ki}}, fori \neq j \right\}$). Then, recalling

$$\begin{aligned} \mathbf{s}_t^T &= e^{-\delta_{\mathbf{z}} t} \mathbf{s}_{t_0}^T + \int_{t_0}^{t} e^{-\delta_{\mathbf{z}}(t-s)} \mathbf{x}_s^T \left(\mathbf{B} - \delta \mathbf{A} \right) ds \\ &= e^{-\delta_{\mathbf{z}} t} \mathbf{s}_{t_0}^T + \alpha \frac{1 - e^{-\delta_{\mathbf{z}} t}}{\delta_{\mathbf{z}}} \mathbf{e}_k^T \left(\mathbf{B} - \delta \mathbf{A} \right) \end{aligned}$$

it is clear that for small α and small time $\mathbf{s}_t^T \mathbf{e}_i > 0$ for $\iota \neq j$ since (recall that $\mathbf{e}_k^T \mathbf{B} = \mathbf{0}$ and $\mathbf{e}_k^T \mathbf{A} = (a_{k1}, ..., a_{kn})$)

$$\mathbf{s}_t^T \mathbf{e}_i = e^{-\delta_{\mathbf{z}} t} \mathbf{s}_{t_0}^T \mathbf{e}_i - \frac{1 - e^{-\delta_{\mathbf{z}} t}}{\delta_{\mathbf{z}}} \alpha \delta a_{ki}.$$

Moreover, since $\bar{\mathbf{s}}^T \mathbf{e}_j = 0$, $b_{kj} > 0$, $a_{kj} = 0$ we have

$$\mathbf{s}_t^T \mathbf{e}_j = e^{-\delta_{\mathbf{z}} t} \mathbf{s}_{t_0}^T \mathbf{e}_j + \frac{1 - e^{-\delta_{\mathbf{z}} t}}{\delta_{\mathbf{z}}} \alpha b_{kj}$$

which gives the claim.

In the general case we argue again by contradiction. Assume that the k-th row of the output truncated matrix is zero. This means that the k-th process produces only commodities that are always zero in the system (let such commodities be indexed by $j_1, ..., j_p$). Moreover we have that

$$a_{kj_1} = a_{kj_2} = ... = a_{kj_p} = 0$$

otherwise the k-th row would be deleted in the truncation process. This means that the k-th process do uses as inputs only commodities that are available from the beginning, (i.e. such that $\bar{\mathbf{s}}^T \mathbf{e}_i > 0$) or that can be made available at any small time t_0 (due to the controllability results of Proposition D.1) Then one can apply first for small time a strategy that bring positive all components that are not always zero, and after this the constant strategy $\mathbf{x}_t = \alpha \mathbf{e}_k$. Arguing as above it can be easily seen that such strategy brings one of the asbsent commodities to be present and this is again a contradiction. ■

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