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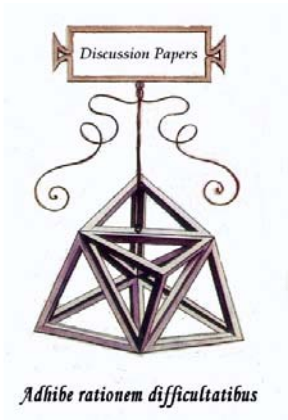

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Notes on Continuous Dynamic Models: the Benhabib-Farmer Condition for Indeterminacy

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Abstract

In these notes, starting from the discrete case, we provide some useful insights on the way in which are derived the necessary (and sometime) sufficient conditions for the solution of a maximum problem developed in continuous time. Moreover, we exploit such conditions to solve the Benhabib-Farmer (1994) model deriving the condition for an indeterminate equilibrium path. Finally, referring to the equilibrium condition of the labour market, we explain the mechanism that in the one-sector optimal growth model leads prophecies to be self-fulfilling.

Keywords: Maximum Problems in Continuous Time, Indeterminate Equilibria, and Self-Fulfilling Prophecies

JEL Classification: E00, E3, 040

1 Introduction

The importance of beliefs and expectations in economics is probably one of the most important element of the Keynesian legacy. After the publication of the General Theory (1936), the idea that pessimistic beliefs of investors may themselves depress the level of economic activity inspired a lot of equilibrium macroeconomic models of the business cycle.

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In 1970s the macroeconomists of rational expectations tried to contrast this argumentation providing a way to endogenise beliefs. If a state of nature is identified by a particular configuration of "fundamentals"[1] and if prices are determined in an Arrow-Debreu (or Walrasian) equilibrium model, then the rational expectations hypothesis seem to suggest that beliefs should be anchored to the same fundamentals[2]. If this were true, then there could be no role for macroeconomic models in which there are self-fulfilling prophecies.

However, the evidence on influence of subjective factors is ample and dates back several centuries[3]. It is hard to neglect that animal spirits[4], consumer sentiment and sunspots[5] activity might spark fluctuations in which prices or quantities change simply because are expected to and price signals convey no structural information.

Nowadays, indeterminate equilibria and self-fulfilling prophecies are consolidated topics of the New Keynesian Economics, usually enclosed in the field of coordination failures (Mankiw and Romer, 1991). The central message of this literature is that the equilibrium path of a model economy is not (at least locally) determinate by the underlying fundamentals. According to the macroeconomists of indeterminacy, there would be a path multiplicity consistent with the unique equilibrium position of the economy. To pin down the actual equilibrium path it would be necessary to specify the forecasting rule used by agents to predict the future (belief function)[6].

The research agenda of indeterminate equilibria has been developed using continuous and discrete dynamic models. An example of the former is the Benhabib-Farmer (1994) model illustrated in these notes. On the other hand, an example of the latter is the recent

[1]By "fundamentals" we mean technology, preferences and endowments.

[2]This was the original point of the rational expectations revolution. It is possible to show that, if there is a unique rational expectations equilibrium, expectations must be a unique function of *fundamentals*.

[3]Azariadis (1981) remembers the Dutch "tulip mania", the South Sea bubble in England and the collapse of the Mississippi Company as three well-documented cases of speculative price movements which historians consider unwarranted by "objective" conditions. More recently, we can mention the bubble of technological equities.

[4]Keynes did not use "animal spirits" to mean self-fulfilling beliefs; instead his view of uncertainty was closer to Frank Knight's concept of an event for which there is too little information to make a frequentist statement about probabilities.

[5]Sunspots is meant to represent "*extrinsic uncertainty*", that is, random phenomena that do not affect tastes, endowments, and production possibilities. Of course, as Javons noted, real-word sunspots may very well be a source of *intrinsic uncertainty* to the economy, affecting, for example, the agricultural production possibilities. However, here we are interested only in a highly stylised version of sunspot activity.

[6]The belief function is the tool used by agents to solve the path multiplicity.

Benhabib-Farmer (2000) model explaining the monetary transmission mechanism. Given the specification of time, the first step moved in these theoretical models is to derive the conditions under which a multiplicity of equilibrium path might occurs. Thereafter, such conditions - usually related to measures of elasticities - are compared to econometric estimations in order to find coherence between theory and reality.

Models developed in discrete time are not very easy to solve. However, they might be easily supported by a belief function selecting the actual equilibrium path followed by the model economy[7]. On the other hand, models developed in continuous time are easier to solve, but they lack for a belief function because there are objective difficulties to find a closed form for the equilibrium paths.

The rules that allow to solve maximum problems in continuous time are quite simple. However they are often applied mechanically. In these notes, starting from the discrete case, we provide some analytical insights explaining how the necessary (and sometime sufficient) first-order conditions for a maximum are derived. Distinguishing between a finite and an infinite horizon, a particular attention is given to the meaning of the transversality condition.

Thereafter, given these essential tools, we solve the Benhabib-Farmer (1994) model deriving the conditions for a multiplicity of equilibrium paths. Furthermore, we explain the underlying mechanism that leads agents' prophecies to be self-fulfilling.

These notes are arranged as follow. Section 2 collects some basic principles helpful to solve maximum problems in continuous time. Section 3 develops the Benhabib-Farmer (1994) model deriving the condition for indeterminacy. Section 4 concludes.

2 Some Basic Principles on Dynamic Programming Problems

Let $f : \Re^{n \times m+1} \rightarrow \Re^n$ be a continuous smooth function. Consider the following system of n first-order differential equations:

$$\dot{x}(t) = f[x(t), u(t), t] \tag{1}$$

$$x(0) = x_0$$

[7]Models developed in discrete time are also easier to simulate with numerical procedures.

where x is an $n \times 1$ vector of state variables, and u is a $m \times 1$ vector of control variables. The vector u may depend on time. If t does not appear in $f(\cdot)$, the system of differential equation is said to be autonomous.

The existence of a solution to (1) is ensured by the Cauchy-Peano theorem, that is

Theorem 1 *Let $\dot{x}(t) = f[x(t), u(t), t]$ be a system of n first-order differential equations, where f is an $\Re^n$-valued function on $\Re^{n\times m+1}$. Suppose that the following conditions hold*

(A1) The function f is continuous on $\Re^{n\times m+1}$

(A2) The partial derivatives $\partial f_i / \partial x_j$ exists and are continuous on $\Re^{n\times m+1}$ for all i and $j = 1, 2, ..., n$.

(A3) The function $u(t)$ is continuous in t

(A4) $(x_0, t_0) \in \Re^{n+1}$

Then there exists a function $\phi(t)$ from some interval (t_1, t_2) containing t_0 into $\Re^n$ such that

(i) The function $\phi(t)$ is continuous on (t_1, t_2)

(ii) $\phi(t_0) = x_0$

(iii) $\phi(t) = f[\phi(t), u(t), t]$, that is, $\phi(t)$ is a solution of the system

(iv) if $\psi(t)$ satisfies (i), (ii), and (iii) above on an interval (s_1, s_2), then $\phi(t) = \psi(t)$ on $(t_1, t_2) \cap (s_1, s_2)$, that is, the solution which satisfies the initial condition is unique.

In these notes, we will be interested in problems of the form

$$\begin{gathered} \max_{u(t)} \int_0^T e^{-\rho t} J[x(t), u(t)]\, dt \\ s.to \\ \dot{x}(t) = f[x(t), u(t)] \quad , \quad x(0) = x_0 \end{gathered} \tag{2}$$

In order to find a solution to problem (2), we define a “Hamiltonian”:

$$H \equiv e^{-\rho t}[J(x, u) + \Psi(t) f(x, u)] \tag{3}$$

where $\Psi(t)$ is a $n \times 1$ vector of co-state variables representing the shadow value of the state variables. Given that the problem (the Hamiltonian) is concave, a set of necessary and sufficient conditions is the following

$$\frac{\partial H}{\partial u} = 0 \quad \text{where } u \in \Re^m \tag{4}$$

Condition (4) suggests to maximise the Hamiltonian with respect to control variables.

$$\frac{\partial \Psi}{\partial t} \equiv \dot{\Psi} = -\frac{\partial H}{\partial x} \quad \text{where } \Psi(t) \in \Re^n \tag{5}$$

Condition (5) leads to a set of n differential equations for the co-state variables.

$$\frac{\partial \Gamma}{\partial t} \equiv \dot{\Gamma} = \frac{\partial H}{\partial \Psi} \quad \text{where } \Gamma(t) \in \Re^n \tag{6}$$

Condition (6) reproduces the original set of differential equations for the state variables.

There are also n initial conditions $\Psi(0)$ to be chosen. When the time horizon considered in the maximization problem is limited, these are fixed by the following transversality condition

$$\lim_{t \to +T} \Psi(t) = 0 \tag{7}$$

A second and sometimes easier method to solve problems as the one expressed in (2) is to define a present value Hamiltonian[8]:

$$\widetilde{H} \equiv J(x,u) + \lambda f(x,u)$$

It is straightforward that $\widetilde{H}$ differs from H by missing the term $e^{-\rho t}$. The rules allowing to find a solution using the present value Hamiltonian $\widetilde{H}$ are the following

$$\frac{\partial \widetilde{H}}{\partial u} = 0 \tag{4'}$$

$$\frac{\partial \lambda}{\partial t} \equiv \dot{\lambda} = \rho\lambda - \frac{\partial \widetilde{H}}{\partial x} \tag{5'}$$

$$\frac{\partial \Gamma}{\partial t} \equiv \dot{\Gamma} = \frac{\partial \widetilde{H}}{\partial \lambda} \tag{6'}$$

$$\lim_{t \to +\infty} e^{-\rho t} \lambda(t) = 0 \tag{7'}$$

Note that conditions (4')-(7') follow from conditions (4)-(7) by defining $\lambda \equiv e^{\rho t}\Psi$. In fact, recognising that $H = e^{-\rho t}\widetilde{H}$, it follows immediately that

[8]The other expression is called current value Hamiltonian.

$$\frac{\partial H}{\partial u} = e^{-\rho t} \frac{\partial \widetilde{H}}{\partial u} = 0 \Rightarrow \frac{\partial \widetilde{H}}{\partial u} = 0 \tag{A}$$

$$\dot{\Psi} = -e^{-\rho t} \frac{\partial \widetilde{H}}{\partial x} \tag{B}$$

By definition, $\lambda \equiv e^{\rho t}\Psi$, therefore, differentiating with respect to time

$$\dot{\lambda} = \rho e^{\rho t}\Psi + e^{\rho t}\dot{\Psi} = \rho\lambda + e^{\rho t}\dot{\Psi}$$

$$\Rightarrow \dot{\Psi} = \left(\dot{\lambda} - \rho\lambda\right) e^{-\rho t}$$

Substituting in (B), it yields

$$\left(\dot{\lambda} - \rho\lambda\right) e^{-\rho t} = -e^{-\rho t} \frac{\partial \widetilde{H}}{\partial x} \quad \text{or}$$

$$\dot{\lambda} = \rho\lambda - \frac{\partial \widetilde{H}}{\partial x} \quad Q.E.D$$

Finally,

$$\dot{\Gamma} = \frac{\partial H}{\partial \Psi} = \frac{\partial \widetilde{H}}{\partial \lambda} \text{ with } \lambda \equiv e^{\rho t}\Psi$$

One may ask where do these rules come from. In order to answer to this question, we consider the following discrete time maximum problem

$$\max_{\{u_t, x_{t+1}\}_t^T} \sum_{t=0}^{T} \left(\frac{1}{1+\rho}\right)^t J\left[x(t), u(t)\right] \tag{8}$$

$$s.to$$

$$\begin{gathered} x(t+1) - x(t) = f\left[x(t), u(t)\right] \\ x(0) = x_0 \end{gathered} \tag{9}$$

This is a finite dimensional problem in which we choose the sequence $\{u_t, x_{t+1}\}_{t=0}^T$ subject to the constraints implied by (9).

We can solve this problem by writing a Lagrangian

$$\begin{aligned} L(\cdot) &= J(x_0,u_0) - \lambda_0 [x_1 - x_0 - f(x_0,u_0)] + \\ &+ \tfrac{1}{1+\rho} \{J(x_1,u_1) - \lambda_1 [x_2 - x_1 - f(x_1,u_1)]\} + \\ &+ \left(\tfrac{1}{1+\rho}\right)^2 \{J(x_2,u_2) - \lambda_2 [x_3 - x_2 - f(x_2,u_2)]\} + ... \\ &... + \left(\tfrac{1}{1+\rho}\right)^T \{J(x_T,u_T) - \lambda_T [x_{T+1} - x_T - f(x_T,u_T)]\} \end{aligned} \tag{10}$$

By discounting the constraints, we are implicitly working towards the present value Hamiltonian. The first-order conditions for (10) take the form

$$\begin{cases} \frac{\partial L(\cdot)}{\partial x_1} = \frac{1}{1+\rho}\frac{\partial J(\cdot)}{\partial x_1} + \frac{\lambda_1}{1+\rho} + \frac{\lambda_1}{1+\rho}\frac{\partial f(\cdot)}{\partial x_1} + \lambda_0 = 0 \\ .... \\ \frac{\partial L(\cdot)}{\partial x_t} = \left(\frac{1}{1+\rho}\right)^t \left[\frac{\partial J(\cdot)}{\partial x_t} + \lambda_t + \lambda_t \frac{\partial f(\cdot)}{\partial x_t}\right] - \left(\frac{1}{1+\rho}\right)^{t-1} \lambda_{t-1} = 0 \\ .... \\ \frac{\partial L(\cdot)}{\partial x_{T+1}} = -\left(\frac{1}{1+\rho}\right)^T \lambda_T = 0 \end{cases} \tag{11}$$

$$\begin{cases} \frac{\partial L(\cdot)}{\partial u_0} = \frac{\partial J(\cdot)}{\partial u_0} + \lambda_0 \frac{\partial f(\cdot)}{\partial u_0} = 0 \\ .... \\ \frac{\partial L(\cdot)}{\partial u_t} = \left(\frac{1}{1+\rho}\right)^t \left[\frac{\partial J(\cdot)}{\partial u_t} + \lambda_t \frac{\partial f(\cdot)}{\partial u_t}\right] = 0 \\ .... \\ \frac{\partial L(\cdot)}{\partial u_T} = \left(\frac{1}{1+\rho}\right)^T \left[\frac{\partial J(\cdot)}{\partial u_T} + \lambda_T \frac{\partial f(\cdot)}{\partial u_T}\right] = 0 \end{cases} \tag{12}$$

$$\begin{cases} \frac{\partial L(\cdot)}{\partial \lambda_0} = -[x_1 - x_0 - f(x_0,u_0)] = 0 \\ .... \\ \frac{\partial L(\cdot)}{\partial \lambda_t} = \left(\frac{1}{1+\rho}\right)^t [x_{t+1} - x_t - f(x_t,u_t)] = 0 \\ .... \\ \frac{\partial L(\cdot)}{\partial \lambda_T} = \left(\frac{1}{1+\rho}\right)^T [x_{T+1} - x_T - f(x_T,u_T)] = 0 \end{cases} \tag{13}$$

For short,

$$\frac{\partial L(\cdot)}{\partial x_t} = 0 \Leftrightarrow \lambda_t - \lambda_{t-1} = \rho\lambda_{t-1} - \frac{\partial J(\cdot)}{\partial x_t} - \lambda_t \frac{\partial f(\cdot)}{\partial x_t} \tag{11'}$$

$$\frac{\partial L(\cdot)}{\partial u_t} = 0 \Leftrightarrow \frac{\partial J(\cdot)}{\partial u_t} + \lambda_t \frac{\partial f(\cdot)}{\partial u_t} = 0 \tag{12'}$$

$$\frac{\partial L(\cdot)}{\partial \lambda_t} = 0 \Leftrightarrow x_{t+1} - x_t = f(x_t,u_t) \tag{13'}$$

Suppose we define $\widetilde{H} \equiv J(x_t, u_t) + \lambda_t f(x_t, u_t)$, then

$$\lambda_t - \lambda_{t-1} = \rho\lambda_{t-1} - \frac{\partial \widetilde{H}(\cdot)}{\partial x_t} \tag{11'}$$

$$\frac{\partial \widetilde{H}}{\partial u_t} = 0 \tag{12'}$$

$$x_{t+1} - x_t = \frac{\partial \widetilde{H}}{\partial \lambda_t} \tag{13'}$$

The transversality condition is simply the first-order condition for the choice of x_{T+1}

$$\frac{\partial L(\cdot)}{\partial x_{T+1}} = 0 \Rightarrow -\left(\frac{1}{1+\rho}\right)^T \lambda_T = 0$$

In continuous time, the conditions derived above imply that to solve the problem

$$\max_u \int_0^T e^{-\rho t} J(x, u)\, dt$$

$$\underset{s.to}{\dot{x} = f(x, u)}$$

we have to

- define $\widetilde{H} \equiv J[x, u] + \lambda f(x, u)$ (present value Hamiltonian)
- $\frac{\partial \widetilde{H}}{\partial u} = 0$
- $\dot{\lambda} = \rho\lambda - \frac{\partial \widetilde{H}(\cdot)}{\partial x}$
- $\dot{x} = f(x, u) = \frac{\partial \widetilde{H}(\cdot)}{\partial \lambda}$
- $e^{-\rho T}\lambda(T) = 0$ (transversality condition)

2.1 Infinite Horizon Problems

Consider the previous discrete problem extended to an infinite horizon

$$\max_{\{u_t, x_{t+1}\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \left(\frac{1}{1+\rho}\right)^t J(x_t, u_t) \tag{14}$$

$$s.to$$

$$\begin{aligned} x_{t+1} - x_t &= f\left(x_t, u_t\right) \\ x\left(0\right) &= x_0 \end{aligned} \tag{15}$$

The additional problems arising in an infinite horizon problem derive from the fact that $J\left(x, u\right)$ may be unbounded. In this case, no optimum may exist. However, for most problems that we find in the optimal growth literature the maximum problem extended in an infinite horizon may by solved applying the first-order conditions from the discrete case, supplementing them with a different transversality condition. Suppose we let

$$L\left(\cdot\right) = \sum_{t=0}^{+\infty}\left(\frac{1}{1+\rho}\right)^t \left\{J\left(x_t, u_t\right) - \lambda_t\left[x_{t+1} - x_t - f\left(x_t, u_t\right)\right]\right\}$$

Now let $\{\widehat{u}_t, \widehat{x}_{t+1}\}_{t=0}^{+\infty}$ be candidate sequences for a solution. Then we evaluate $L(u, x)$ in the neighborhood of $L(\widehat{u}, \widehat{x})$, where $\{u, x\}_{t=0}^{+\infty}$ is an arbitrary sequence.

$$L(u, x) = L(\widehat{u}, \widehat{x}) + L_x\left(x - \widehat{x}\right) + L_u\left(u - \widehat{u}\right) + \Phi(u, x)$$

Consider the terms $L_x\left(x - \widehat{x}\right)$ and $L_u\left(u - \widehat{u}\right)$. Take $L_u\left(u - \widehat{u}\right)$ first. This term has the form

$$\sum_{t=0}^{+\infty}\left(\frac{1}{1+\rho}\right)^t\left[\frac{\partial J\left(\cdot\right)}{\partial u_t} + \lambda_t\frac{\partial f\left(\cdot\right)}{\partial u_t}\right]$$

Since $\frac{\partial J(\cdot)}{\partial u_t} = \lambda_t\frac{\partial f(\cdot)}{\partial u_t}$ for all t, this term is equal to zero, that is

$$L_u\left(u - \widehat{u}\right) = 0$$

Some problems arise with the term $L_x\left(x - \widehat{x}\right)$. This term has the form

$$-\lambda_0 + \frac{1}{1+\rho}\left[\frac{\partial J\left(\cdot\right)}{\partial x_1} + \lambda_1 x_1 - \lambda_1\frac{\partial f\left(\cdot\right)}{\partial x_1}\right] - \frac{\lambda_1}{1+\rho} +$$

$$+\left(\frac{1}{1+\rho}\right)^2\left[\frac{\partial J\left(\cdot\right)}{\partial x_2} + \lambda_2 x_2 - \lambda_2\frac{\partial f\left(\cdot\right)}{\partial x_2}\right] + \ldots\ldots$$

Suppose to consider the finite problem. In this case, $L_x\left(x - \widehat{x}\right)$ would be equal to

$$\sum_{t=0}^{T-1}\left(\frac{1}{1+\rho}\right)^t\left[\frac{\partial J\left(\cdot\right)}{\partial x_t} + \lambda_t + \lambda_t\frac{\partial f\left(\cdot\right)}{\partial x_t} - \left(1+\rho\right)\lambda_{t-1}\right]\left(x_t - \widehat{x}_t\right) -$$

$$-\left(\frac{1}{1+\rho}\right)^{T}\lambda_{T}\left(x_{t}-\widehat{x}_{t}\right)=0$$

In the finite horizon problem, the first-order conditions guarantee that this term is equal to zero. In the infinite horizon problem, it is not enough to set $\lim_{T\to+\infty}\lambda_T=0$, since x_T may be growing too fast.

For most problems that we encounter, the transversality condition

$$\lim_{T\to+\infty}\left(\frac{1}{1+\rho}\right)^{T}\lambda_{T}x_{T}=0$$

is necessary and sufficient.

In the continuous problem, this leads to[9]

$$\lim_{t\to+\infty}e^{-\rho t}\lambda\left(t\right)x\left(t\right)=0$$

Note that this is not the same as $\lim_{t\to+\infty}e^{-\rho t}\lambda\left(t\right)=0$.

3 The Benhabib-Farmer Model in Continuous Time

The Benhabib-Farmer (1994) model developed in continuous time investigates the properties of the one-sector growth model (Ramsey model) assuming increasing returns to scale. The existence of aggregate increasing returns is reconciled with the competitive behaviour of firms using two distinct organisational structures, that is, input externalities and monopolist competition.

The model with input externalities allows for the possibility that in a symmetric equilibrium the social technology might display increasing returns to scale. On the other hand, the model with monopolistic competition proposes a framework which is similar to the one developed by Dixit and Stiglitz (1977). Specifically, there are two sectors: one produces intermediate goods by means of capital and labour, while the other sells finite products. In the former, producers are monopolistic competitors. In the latter, there is perfect competition. Obviously, the possibility of increasing returns to scale holds only in the sector of intermediate goods. Since the dynamic implications of each organisational structure are the same, we develop the simpler version with input externalities.

[9]Sometimes the transversality condition has been interpreted as a non-arbitrage condition (no-Ponzi game condition). In fact, it state that the asymptotic actual value of the state variables has to be zero.

As in the one-sector growth model, in the Benhabib-Farmer (1994) model the infinitely-lived representative agent solves the following problem

$$\max_{c,L} \int_0^{+\infty} e^{-\rho t} \left[\log c - \frac{1}{1+\gamma} L^{1+\gamma} \right] dt \tag{1}$$

$$s.to$$

$$\dot{K} = -\delta K + Y - c \tag{2}$$

The representation of the instantaneous utility in (1) deserves a short comment. If we combine separability between consumption and leisure with a Cobb-Douglas production function, the use of a logarithmic utility function over consumption is the only formulation of preferences that is consistent with a stationary labour supply in a growing economy.

The aggregate technology is given by

$$Y = K^{\alpha} L^{\beta} \quad , \quad \alpha + \beta \geqslant 1 \tag{3}$$

where the inequality $\alpha + \beta \geqslant 1$ allows for the possibility of increasing return to scale at the social level.

We distinguish the individual problem from the aggregate problem using an externality argument. The individual technology is given by

$$Y = AK^{a} L^{b} \quad , \quad a + b = 1 \tag{3'}$$

where $A \equiv \overline{K}^{\alpha - a} \overline{L}^{\beta - b}$ is the productivity parameter (Solow's residual), taken as given by the representative agent. The terms with the upper bar represent, respectively, the aggregate stock of capital and the aggregate labour input.

Whenever it prevails a symmetric equilibrium, that is, when $\overline{K} = K$ and $\overline{L} = L$, the production function reduces to equation (3).

In order to solve the individual problem, we define a present value Hamiltonian

$$H \equiv \log c - \frac{1}{1+\gamma} L^{1+\gamma} - \lambda \left[-\delta K + AK^{a} L^{b} - c \right] \tag{4}$$

In this case, λ is the shadow value of capital.

The first-order conditions for a maximum (see the previous section) are the following

$$\frac{\partial H}{\partial c} = 0 \Rightarrow \frac{1}{c} = \lambda \tag{5}$$

$$\frac{\partial H}{\partial L} = 0 \Rightarrow L^{\gamma} = \frac{\lambda b Y}{L} \tag{6}$$

$$\dot{K} = \frac{\partial H}{\partial \lambda} \Rightarrow \dot{K} = -\delta K + AK^{a}L^{b} - c \tag{7}$$

$$\dot{\lambda} = \rho\lambda - \frac{\partial H(\cdot)}{\partial K} \Rightarrow \rho\lambda + \delta\lambda - \frac{a\lambda Y}{K} \tag{8}$$

$$\lim_{t\to+\infty} e^{-\rho t}\lambda(t)\,K(t) = 0 \tag{9}$$

Conditions (5) and (6) maximise the present value Hamiltonian with respect to control variables. Condition (7) reproduces the original differential equation for the evolution of capital. Condition (8) defines the law of evolution for the co-state variable. Finally, condition (9) is the transversality condition.

Using equations (5), we express the rate of growth of c as function of the rate of growth of λ, that is

$$\frac{\dot{c}}{c} = -\frac{\dot{\lambda}}{\lambda}$$

Substituting in equation (8), it yields

$$\frac{\dot{c}}{c} = \frac{aY}{K} - \rho - \delta \tag{8a}$$

Dividing each member of equation (7) by K we have

$$\frac{\dot{K}}{K} = -\delta + \frac{Y}{K} - \frac{c}{K} \tag{7a}$$

Finally, from equations (3) and (6) we derive

$$Y = \left(\overline{K}^{\alpha-a}\overline{L}^{\beta-b}\right)K^{a}L^{b} \tag{3a}$$

$$cL^{1+\gamma} = bY \tag{6a}$$

In a symmetric equilibrium it holds

$$Y = K^{\alpha}L^{\beta} \tag{3b}$$

$$cL^{1+\gamma} = bY \tag{6b}$$

These equations determine Y and L as function of K and c.

3.1 The Decentralised Solution

Let L^S and L^D be, respectively, the labour supply and demand of the i-th representative family. Let $i \in [0,1]$ index a continuum of identical families. Let K^S and K^D be, respectively, the supply and demand for capital. The i-th family sells L^S units of labour to the market and accumulates K^S units of capital that it rents out to other families. Simultaneously, the i-th family demands L^D units of labour and rents K^D units of capital from other families for the use of the family firm. Using this specifications, the i-th family solves the following problem

$$\max_{c,L^s} \int_0^{+\infty} e^{-\rho t} \left[\log c - \frac{\left(L^S\right)^{1+\gamma}}{1+\gamma} \right] dt \tag{9}$$

$$s.to$$

$$\dot{K^s} = -\delta K^s + Y^s - c + w\left(L^S - L^D\right) + r\left(K^S - K^D\right) \tag{10}$$

$$Y^s = A\left(K^D\right)^a \left(L^D\right)^b \quad , \quad a + b = 1 \tag{11}$$

$$A \equiv \overline{K}^{\alpha - a} \overline{L}^{\beta - b} \tag{12}$$

where w is the real wage and r is the (real) rental rate.

To solve the problem, we define again a present value Hamiltonian

$$\begin{aligned} H = & \left[\log c - \frac{1}{1+\gamma} \left(L^S\right)^{1+\gamma} \right] + \\ & + \lambda \left[-\delta K^s + A\left(K^D\right)^a \left(L^D\right)^b - c + w\left(L^S - L^D\right) + r\left(K^S - K^D\right) \right] \end{aligned} \tag{13}$$

Under this specification, the control variables are L^S, L^D, K^D, and c, the state variable is K^S and the co-state variable is λ. The conditions for a maximum, omitting the transversality condition, are given by

$$\frac{\partial H}{\partial L^S} = 0 \Rightarrow \left(L^S\right)^\gamma = \lambda w \tag{14}$$

$$\frac{\partial H}{\partial L^D} = 0 \Rightarrow Ab\left(K^D\right)^a \left(L^D\right)^{b-1} = w \tag{15}$$

$$\frac{\partial H}{\partial K^D} = 0 \Rightarrow Aa\left(K^D\right)^{a-1} \left(L^D\right)^{b} = r \tag{16}$$

$$\dot{\lambda} = \rho\lambda - \frac{\partial H\left(\cdot\right)}{\partial K^S} \Rightarrow \rho\lambda - \lambda\left(r - \delta\right) \tag{17}$$

$$\dot{K}^s = -\delta K^s + Y^s - c + w\left(L^S - L^D\right) + r\left(K^S - K^D\right) \tag{18}$$

$$\frac{\partial H}{\partial c} = 0 \Rightarrow \frac{1}{c} = \lambda \tag{19}$$

Notice that in a market-clearing equilibrium $\left(L^S - L^D\right) = 0$ and $\left(K^S - K^D\right) = 0$. Conditions (14), (15), and (19) leads to

$$c\left(L^S\right)^{\gamma} = w = Ab\left(K^D\right)^a \left(L^D\right)^{b-1} \tag{20}$$

The left-hand side of equation (20) is the labour supply, while the right-hand side is the labour demand of the individual firm. See figure 1. Notice that the industry labour demand is taken as given by the individual firm.

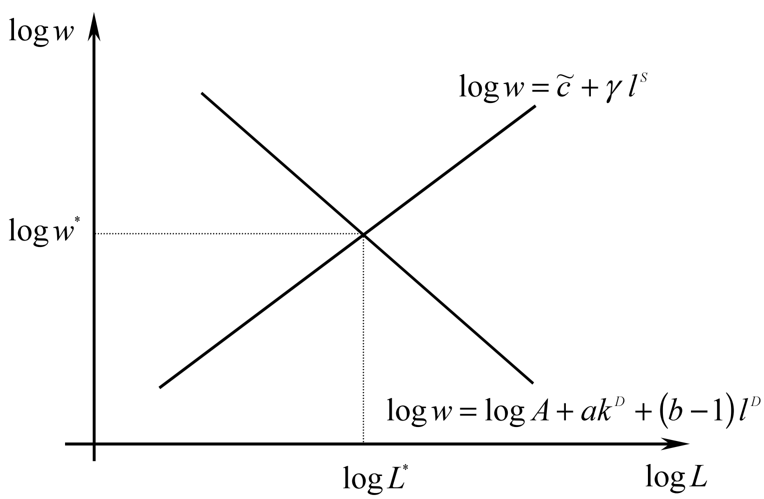

Figure 1: The labour market at the individual level

Taking the logarithms of each member in (20)

$$\widetilde{c} + \gamma l^S = \log w = \log A + \log b + ak^D + (b-1)\, l^D$$

where $\widetilde{c} \equiv \log c$, $k^D \equiv \log K^D$, $l^S \equiv \log L^S$, and $l^D \equiv \log L^D$.

Imposing the conditions for a symmetric equilibrium, that is, $\overline{K} = K$ and $\overline{L} = L$, we have

$$\widetilde{c} + \gamma l^S = \log w = \log \beta + \alpha k^D + (\beta - 1)\, l^D$$

where $\log w = \log \beta + \alpha k^D + (\beta - 1)\, l^D$ is the economy-wide labour demand accounting for externalities. Remember that we assumed $\alpha + \beta \geqslant 1$. See figure 2.

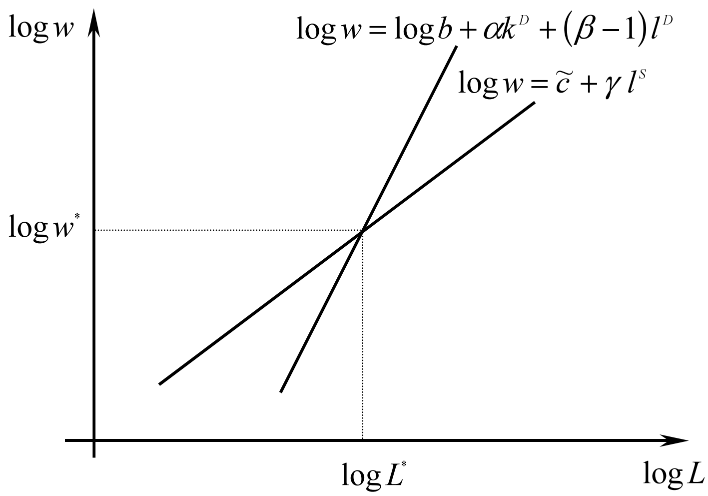

Figure 2: The economy-wide labour market

3.2 Local Dynamics

Using equation (17), (18), and (19) we easily derive

$$\frac{\dot{c}}{c} = \frac{aY}{K} - (\rho + \delta) \tag{21}$$

$$\frac{\dot{K}}{K} = -\delta + \frac{Y}{K} - \frac{c}{K} \tag{22}$$

Let $y \equiv \log Y$, $k \equiv \log K$, and $l \equiv \log L$. Therefore

$$\dot{\widetilde{c}} = ae^{y-k} - \rho - \delta \tag{23}$$

$$\dot{k} = -\delta + e^{y-k} - e^{\widetilde{c}-k}$$

In order to derive an autonomous differential system we have to express y as a function of $\widetilde{c}$ and k. This is possible combining the log-linearisation of the production function and the labour market-clearing equation, that is

$$y = \alpha k + \beta l \tag{24}$$

$$\widetilde{c} + \gamma l = \log b + y - l \tag{25}$$

Putting together equations (24) and (25), we derive

$$\alpha k + \beta l - y = 0$$

$$\widetilde{c} + (1+\gamma)\, l - \log b - y = 0$$

In matrix notation

$$\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} k \\ \widetilde{c} \end{pmatrix} + \begin{bmatrix} -1 & \beta \\ -1 & (1+\gamma) \end{bmatrix} \begin{pmatrix} y \\ l \end{pmatrix} + \begin{pmatrix} 0 \\ -\log b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

It is straightforward to derive

$$\begin{bmatrix} -1 & \beta \\ -1 & (1+\gamma) \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1+\gamma}{\beta-1-\gamma} & \frac{\beta}{1+\gamma-\beta} \\ \frac{1}{\beta-1-\gamma} & \frac{1}{1+\gamma-\beta} \end{bmatrix}$$

Therefore,

$$\begin{pmatrix} y \\ l \end{pmatrix} = \begin{pmatrix} \frac{\beta}{1+\gamma-\beta} \log b \\ \frac{1}{1+\gamma-\beta} \log b \end{pmatrix} - \begin{bmatrix} \frac{(1+\gamma)\alpha}{\beta-1-\gamma} & \frac{\beta}{1+\gamma-\beta} \\ \frac{\alpha}{\beta-1-\gamma} & \frac{1}{1+\gamma-\beta} \end{bmatrix} \begin{pmatrix} k \\ \widetilde{c} \end{pmatrix}$$

$$y = \frac{\beta}{1+\gamma-\beta} \log b - \frac{(1+\gamma)\,\alpha}{\beta-1-\gamma} k - \frac{\beta}{1+\gamma-\beta} \widetilde{c} \tag{D}$$

$$l = \frac{1}{1+\gamma-\beta} \log b - \frac{\alpha}{\beta-1-\gamma} k - \frac{1}{1+\gamma-\beta} \widetilde{c}$$

Subtracting k from each member of equation (D)

$$y - k = -\frac{\beta}{\beta - 1 - \gamma} \log b - \left[1 + \frac{(1+\gamma)\,\alpha}{\beta - 1 - \gamma} \right] k + \frac{\beta}{\beta - 1 - \gamma} \widetilde{c}$$

$$y - k = \phi_0 + \left[\frac{(1+\gamma)\,(1-\alpha) - \beta}{\beta - 1 - \gamma} \right] k + \phi_2 \widetilde{c}$$

$$y - k = \phi_0 + \phi_1 k + \phi_2 \widetilde{c} \tag{25}$$

where the multipliers are given by

$$\phi_0 \equiv -\frac{\beta}{\beta - 1 - \gamma} \log b$$

$$\phi_1 \equiv \frac{(1+\gamma)\,(1-\alpha) - \beta}{\beta - 1 - \gamma}$$

$$\phi_2 \equiv \frac{\beta}{\beta - 1 - \gamma}$$

Using the results in (23), we may write the required pair of autonomous differential equations as

$$\dot{\widetilde{c}} = a e^{\lambda_0 + \lambda_1 k + \lambda_2 \widetilde{c}} - \rho - \delta \tag{26}$$

$$\dot{k} = -\delta + e^{\lambda_0 + \lambda_1 k + \lambda_2 \widetilde{c}} - c^{\widetilde{c} - k} \tag{27}$$

Any trajectory $\{k(t), \widetilde{c}(t)\}$ that solves (26) and (27) subject to the initial condition $k(0) = k_0$ and the transversality condition (9) is an equilibrium path for the model economy. Moreover, the variable k is predetermined since k_0 is given by the initial conditions of the economy while $\widetilde{c}_0$ is free to be determined by the behaviour of the agents.

The Taylor first-order approximation around a generic steady-state is the following:

$$\begin{pmatrix} \dot{\widetilde{c}} \\ \dot{k} \end{pmatrix} = J \begin{pmatrix} \widetilde{c} - \widetilde{c}^{*} \\ k - k^{*} \end{pmatrix}$$

where $J \in \Re^{2\times 2}$ is the Jacobian matrix.

At the steady-state $(\widetilde{c}^*, k^*)$, it holds

$$\dot{\widetilde{c}} = 0 \Rightarrow ae^{\lambda_0+\lambda_1 k^*+\lambda_2 \widetilde{c}^*} = \rho+\delta \Rightarrow e^{\lambda_0+\lambda_1 k^*+\lambda_2 \widetilde{c}^*} = \frac{\rho+\delta}{a}$$

$$\dot{k} = 0 \Rightarrow e^{\lambda_0+\lambda_1 k^*+\lambda_2 \widetilde{c}^*} - e^{\widetilde{c}^*-k^*} = \delta \Rightarrow e^{\widetilde{c}^*-k^*} = \frac{\rho+\delta\,(1-a)}{a}$$

The previous equalities prove that the steady-state exists and it is unique. The Jacobian matrix is the following

$$J \equiv \left[\begin{array}{cc} \phi_2\,(\rho+\delta) & \phi_1\,(\rho+\delta) \\ \frac{\phi_2(\rho+\delta)}{a} - \frac{\rho+\delta(1-a)}{a} & \phi_1\left(\frac{\rho+\delta}{a}\right) + \frac{\rho+\delta(1-a)}{a} \end{array} \right]$$

For brevity,

$$J \equiv \left[\begin{array}{cc} j_{11} & j_{12} \\ j_{21} & j_{22} \end{array} \right]$$

It is well-known that the trace of the Jacobian matrix measures the sum of the eigenvalues and the determinant measures their product. For brevity,

$$\lambda_1 + \lambda_2 = j_{11} + j_{22} = \mathrm{TR}(J)$$

$$\lambda_1 \lambda_2 = j_{11} j_{22} - j_{12} j_{21} = \mathrm{DET}(J)$$

Let be Q the matrix diagonalising J. Then we define

$$\left(\begin{array}{c} z_1 \\ z_2 \end{array} \right) \equiv Q^{-1} \left(\begin{array}{c} \widetilde{c} - \widetilde{c}^* \\ k - k^* \end{array} \right)$$

Hence,

$$\left(\begin{array}{c} \dot{\widetilde{c}} \\ \dot{k} \end{array} \right) = Q\Lambda \left(\begin{array}{c} z_1 \\ z_2 \end{array} \right) \quad \text{where } \Lambda \equiv \left[\begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array} \right]$$

It straightforward to derive that

$$\dot{z}_1 = \lambda_1 z_1 \quad \text{and} \quad \dot{z}_2 = \lambda_2 z_2$$

The decomposition above suggests that the eigenvalues associated to the Jacobian matrix represent the slope of the phase diagram in the stationary points considering a linear transformation of the original variables. See figure 3.

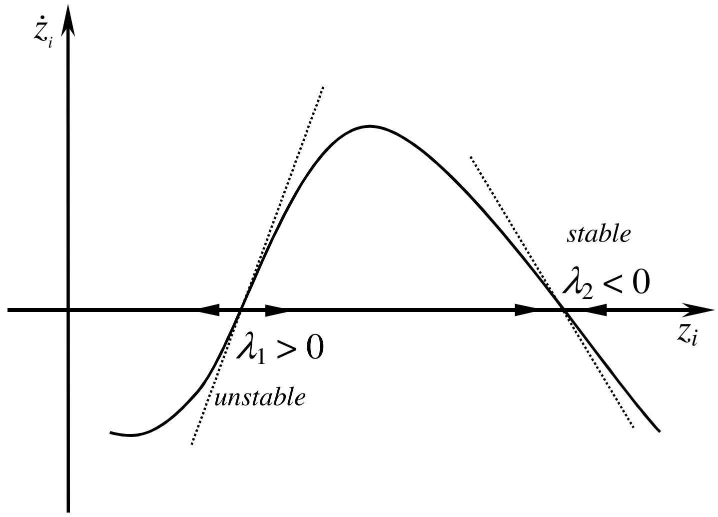

Figure 3: The phase diagram

If λ_1 and $\lambda_2 < 0$, then $\mathrm{TR}(J) < 0$ and $\mathrm{DET}(J) > 0$. On the other hand, if $\lambda_1 < 0$ and $\lambda_2 > 0$, with $\lambda_2 > \lambda_1$, then $\mathrm{TR}(J) > 0$ and $\mathrm{DET}(J) < 0$. Let us derive the expression for the trace and the determinant

$$\mathrm{TR}(J) = \phi_2 (\rho + \delta) + \phi_1 \left(\frac{\rho + \delta}{a} \right) + \frac{\rho + \delta (1 - a)}{a} =$$

$$= \frac{\rho + \delta}{a} (\phi_1 + a\phi_2) + \frac{\rho + \delta (1 - a)}{a}$$

$$\mathrm{DET}(J) = \phi_2 (\rho + \delta) \left[\tfrac{\phi_1 (\rho + \delta)}{a} + \tfrac{\rho + \delta (1 - a)}{a} \right] - \phi_1 (\rho + \delta) \left[\tfrac{\phi_2 (\rho + \delta)}{a} - \tfrac{\rho + \delta (1 - a)}{a} \right] =$$

$$= \frac{(\rho + \delta)}{a} [\rho + \delta (1 - a)] (\phi_1 + \phi_2)$$

What are $(\phi_1 + \phi_2)$ in $\mathrm{DET}(J)$ and $(\phi_1 + a\phi_2)$ in $\mathrm{TR}(J)$?

$$\phi_1 + \phi_2 = \frac{(1 + \gamma)(1 - \alpha) - \beta}{\beta - 1 - \gamma} + \frac{\beta}{\beta - 1 - \gamma} = \frac{(1 + \gamma)(1 - \alpha)}{\beta - 1 - \gamma}$$

$$\phi_1 + a\phi_2 = \frac{(1+\gamma)(1-\alpha) - \beta}{\beta - 1 - \gamma} + \frac{a\beta}{\beta - 1 - \gamma} = \frac{(1+\gamma)(1-\alpha) - \beta + a\beta}{\beta - 1 - \gamma}$$

Whenever there are no externalities $\alpha = a$, therefore

$$\phi_1 + \phi_2 = \frac{(1+\gamma)(1-a)}{-\gamma - (1-\beta)} = \frac{(1+\gamma)(1-a)}{-\gamma - a}$$

$$\phi_1 + a\phi_2 = \frac{(1+\gamma-\beta)(1-\alpha)}{\beta - 1 - \gamma} = -(1-\alpha)$$

We can write $(\phi_1 + a\phi_2)$ as

$$\phi_1 + a\phi_2 = \frac{(1+\gamma)(1-\alpha) + \beta(a-1)}{\beta - 1 - \gamma}$$

Adding and subtracting $(a-1)(-\gamma-1)$

$$\phi_1 + a\phi_2 = \frac{(1+\gamma)(1-\alpha) + \beta(a-1) + (a-1)(-\gamma-1) - (a-1)(-\gamma-1)}{\beta - 1 - \gamma} =$$

$$= (a-1) + \frac{(1+\gamma)(a-\alpha)}{\beta - 1 - \gamma}$$

Therefore,

$$\mathrm{TR}(J) = \frac{\rho+\delta}{a}\left[(a-1) + \frac{(1+\gamma)(a-\alpha)}{\beta-1-\gamma}\right] + \frac{\rho + \delta(1-a)}{a} =$$

$$= \rho + \frac{\rho+\delta}{a}\frac{(1+\gamma)(a-\alpha)}{\beta-1-\gamma}$$

$$\mathrm{DET}(J) = \frac{(\rho+\delta)}{a}\left[\rho + \delta(1-a)\right]\left[\frac{(1+\gamma)(1-\alpha)}{\beta-1-\gamma}\right]$$

Hence,

$$\mathrm{SGN}\left[\mathrm{DET}(J)\right] = \mathrm{SGN}\left[\frac{(1+\gamma)(1-\alpha)}{\beta-1-\gamma}\right]$$

In an economy without externalities

$$\alpha = a < 1 \text{ and } \beta = b < 1$$

Therefore,

$$\mathrm{TR}(J) = \rho \quad \text{and} \quad \mathrm{SGN}\,[\mathrm{DET}(J)] < 0$$

In this case the eigenvalues λ_1 and λ_2 are of opposite sign and the steady-state is a saddle point. In other words, there is a one-dimensional manifold in the $\{k, \widetilde{c}\}$ space with the property that trajectories beginning on this manifold converge to the steady-state, but all other trajectories diverge. Therefore, the equilibrium path will be locally unique in the neighborhood of the steady-state.

On the other hand, whenever

$$\alpha > a \quad \text{and} \quad \beta - 1 - \gamma > 0$$

We have

$$\mathrm{TR}(J) < 0 \quad \text{and} \quad \mathrm{SGN}\,[\mathrm{DET}(J)] > 0$$

In this case there are two negative eigenvalues and the steady-state is a sink. In other words, all the trajectories satisfying (26) and (27) which begin in the neighborhood of $\{k^*, \widetilde{c}^*\}$ converge back to the steady-state. In this case, there will be a continuum of equilibrium paths $\{k(t), \widetilde{c}(t)\}$ indexed by $\widetilde{c}_0$, since any path converging to $\{k^*, \widetilde{c}^*\}$ necessarily satisfies the transversality condition (9). Completely stable steady-states giving rise to a continuum of equilibrium paths will be termed "indeterminate" and in this case we say that the stable manifold has dimension two[10].

Summing up, the condition $\alpha > a$ is necessary for the indeterminacy of the equilibrium path. The condition $\beta - 1 - \gamma > 0$ is also necessary and it is predisposed for an immediate economic rationalization. In fact, it states that the labour demand slopes up more than the labour supply. See figure 3.

3.3 Why Prophecies are Self-Fulfilling?

It is quite striking that the conditions for an indeterminate equilibrium path should lead to precise implications for the labour market outlook. However, if we reflect on the inherent logic underlying the standard Neoclassical growth model, the reasons of such a link are straightforward. In particular, a suggestive explanation has been suggested by Aiyagari (1995).

[10] In this case, all the trajectories could be optimal.

In the standard one-sector Neoclassical growth model, the position of the labour demand schedule is fixed by the stock of capital. On the other hand, the position of the labour supply schedule is fixed by level of consumption. In the absence of shocks, an unique level of consumption determines a unique position for the labour supply schedule. This position, in turn, determines the unique equilibrium level of employment. Therefore, there will be a unique level of output and investment (from the resource constraint), which means a unique level of capital stock for the next period. From this argument, it is clear that the key to indeterminacy is that there can't be a unique position of the labour supply schedule, which means that there can't be a unique level of consumption.

Sometimes it has been argued that optimistic or pessimistic expectations might lead households to spend more or less in consumption. Obviously, this will shift their labour supply schedules. In order to have an equilibrium path driven by self-fulfilling beliefs, these shifts have to lead to labour, output and investment effects that ratify the original optimistic or pessimistic expectations.

How this might happen? Presumably, current income and expectations on future income are what influence consumption most. In order for the households to consume more initially, they have to be optimistic either that current and future labour incomes will be high or that current and future interest rates will be low. In the labour market depicted in figure 1 optimistic expectations leading households to consume more, will lead the labour supply to shift inward. This lowers the current level of employment. Thus, current output and investment are lowered. Thereby, future capital stock and, hence, future employment, output, and so on, are lowered. Furthermore, future interest rates are raised since the capital stock is lowered. These outcomes are clearly inconsistent with the original optimistic expectations.

The above arguments suggest a way in which optimist (or pessimistic) expectations may be self-fulfilling. Consider the labour market depicted in figure 3. In this case, optimistic expectations will shift the labour supply inward. This will raise the employment level and the output. By raising current output, optimistic expectations can also raise the future capital stock and possibly lower interest rates. These effects are consistent with the higher initial consumption. Therefore, the original optimistic expectations are self-validating.

4 Conclusions

The model developed in the previous section has been widely criticised. The degree of increasing returns to scale required to generate an upward sloping labour demand seems to be implausible if compared to empirical evidence (Basu and Fernald, 1997). Furthermore, the picture of the labour market illustrated in figure 3 suggests that supply shocks are related to higher real wage and employment levels. This is at odds with the circumstantial evidence that characterised the period after the oil shocks (Aiyagari, 1995)[11].

There is also another criticism that could be addressed to models with indeterminate equilibrium paths: the assumption of an always-clearing labour market. In other words, there is no role for unemployment in models allowing prophecies to be self-fulfilling. There are some important exceptions to this rule, each of them developed exploiting the transaction approach to unemployment formulated by Pissarides (1990). In particular, we refer to the works proposed by Burda and Weder (2002) and Giammarioli (2003). The former focuses on the complementarity among labour market institutions, the resulting (search) equilibrium unemployment and the propagation of the propagation of business cycles. The latter shows the possibility of an indeterminate equilibrium path whenever the social matching function displays a certain degree of increasing returns to scale with respect to vacancies.

It is well known that search unemployment falls in the category of "frictional" unemployment (Bertola and Caballero, 1994). In fact, in the matching framework, the responsive for unemployment is the absence of a mechanism (say a market) in which the decisions of workers and firms might be coordinated. The task of building models with indeterminate equilibrium paths and involuntary unemployment is still in progress[12].

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