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Neoclassical OLG growth and underdeveloped, developing and developed countries

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Neoclassical OLG growth and underdeveloped, developing and developed countries

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Abstract

In this paper we show that the neoclassical standard OLG growth model, under low substitution in preferences and technology, may generate three stable steady states. In particular we show the richness of the dynamical roles played by the intertemporal substitution parameter.

The novelty of our work is that a three stable equilibria world may exist, which allows to reinterpret in a new light the evidence of three groups of countries: underdeveloped, developing and developed.

Our theoretical results may have far-reaching implications on the debate on convergence, on the corresponding growth empirics and on the policies for escaping from poverty traps.

Keywords: poverty traps, elasticity of substitution, OLG model

JEL: D51, D9, O12

1. Introduction

There is renewed interest about the relation between economic growth and poverty traps\(^1\). In particular, the literature has mainly focused on the role of multiple equilibria and corresponding poverty traps in explaining the different long run performance of rich and poor countries\(^2\).

Our model is much related with this strand of literature. Precisely, we show that low intertemporal substitution in consumption (IES henceforth) and low technical factor substitution can generate a world with three attractive long run income levels, corresponding, loosely speaking, to the usual threefold classification of countries:

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\(^1\) Poverty traps are analyzed by, among many others, see Galor (1996), Azariadis (1996), Mookherjee and Ray (2001).

\(^2\) Recent econometric works have produced evidence which is compatible with the above theoretical results. In particular Durlauf-Jouhnson (1995) and Fiaschi-Lavezzi (2003) provide evidence supporting the existence of multiple equilibria. For instance: Durlauf-Johnson conclude that “these features illustrate the compatibility of growth rate behavior with a multiple steady-state perspective” (p. 378).
underdeveloped, developing and developed countries. Hence, our result extends the class of growth models producing multiple steady states.

The work is organized as follows: in next section we present the main results of the OLG model (described in Appendix) with respect to the existence of the equilibria. Third section discusses the stability and the conditions leading to three-equilibria world. Numerical simulations and concluding remarks will end the paper.

2- The role of IES on the steady states

The traditional neoclassical OLG model, adopting CIES preferences and CES technology boils down in the following dynamic equation (see Appendix):

\[ k_{t+1} = \frac{1}{(1+n)(1+\beta^\sigma (1+r_{t+1})^{\sigma-1})}\left[\beta^\sigma (1+r_{t+1})^{\sigma-1} w_t\right]. \]

Existence and number of the steady states can be studied via the implicit equation

\[ k - \frac{1}{(1+n)(1+\beta^\sigma (1+r(k))^{\sigma-1})}\left[\beta^\sigma (1+r(k))^{\sigma-1} w(k)\right] = 0, \]

which may deliver multiple equilibria and, consequently, generate the poverty trap problem.

Focusing on the case of low technical substitution case (\(\rho > 0\)), we get the following possible outcomes:

a1) one equilibrium: the trivial one (\(k=0\), locally stable;

a2) two equilibria: \(k=0\) (locally stable) and one tangent equilibrium point (this is obviously a very special case);

a3) three equilibria: \(k=0\) (stable), one positive intermediate locally unstable and one high locally stable but, due to the strong nonlinearity of the model, the intermediate one is potentially endowed with a stable attractive region.

a4) three equilibria: \(k=0\) (stable), and other two (intermediate and high) stable.

Examples of Cases a3 and a4 are depicted in Fig. 1.

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1 Empirical results on the value of IES are controversial. For example, Hall (1988) for the US founded that there is no strong evidence that such an elasticity is positive. In contrast with Hall, other studies have suggested higher values (e.g. Hansen and Singleton (1982), Mulligan (2004)).

2 The discussion on whether the actual value of the elasticity of substitution (\(\theta = 1/(1+\rho)\)) between labor and capital is lower than 1 or not is a long debated one. The case of \(\theta < 1\) has been recognized of significant empirical relevance since long time. For example, Solow (1958) argued for a \(\theta\) around 2/3 and Lucas (1969) around 0.6. Finally Rowthorn (1999a) reports the estimates of \(\theta\) from 33 econometric studies, according to which the overall median of the summary values (median of the medians) is equals to 0.58 (and only in 7 cases the elasticity is greater than 0.8).

3 For economy of space we omit the formal analysis of the existence conditions of equilibria. A graphical exposition gathering all the possible outcomes is provided in Fig. 1. For a deeper insight on this point see de la Croix and Michel (2000).

4 As known, the condition for local stability of the capital intensity equilibrium is the following:

\[ \left| \frac{dk_{t+1}}{dk_t} \right| = \left| \frac{f_w k_{t+1} s_{nt}}{(1+n) - f_w s_{r+1}} \right| < 1. \]
Figure 1. Accumulation loci as functions of IES (Parameters: $A=10$, $\rho=7$, $\alpha=0.3$, $\beta=0.7$, $n=0.8$).

Focusing the analysis on the role of the IES parameter in determining the level of the steady state capital intensity (in the cases $a3$-$a4$), we can summarize the results in the following proposition:

**Proposition 1:** The effect of an increase of the IES on the steady state capital intensity is given by

$$
\frac{dk}{d\sigma} = \frac{s_\sigma}{(1+n) + f(k_s - s_r)},
$$

(\text{where } s_\sigma \text{ is the derivative of savings with respect to } \sigma) \text{ which is ambiguous. Precisely: 1) as for the rich equilibrium, if } s_\sigma < 0 \text{ (} s_\sigma > 0 \text{) locally holds, such an equilibrium is decreased (increased) by an increase of the IES; 2) as for the intermediate equilibrium, if } s_\sigma < 0 \text{ (} s_\sigma > 0 \text{) locally holds, such an equilibrium is increased (decreased) by an increase of the IES. In particular, from eq. (A1) of Appendix,}

$$
s_\sigma = \frac{w\beta^\sigma (1+r)^{\sigma+1}}{[\beta^\sigma (1+r)^\sigma + (1+r)]^{2}} [\log \beta + \log(1+r)] \leq 0 \quad \text{iff} \quad \beta \leq \frac{1}{1+r};
$$

[3] (Proof is omitted for brevity and available on request).

It is worth noting that the sign of $s_\sigma$ depends on $r(k)$ and so such a sign may be different at different values of capital stock and, thus, at different long run equilibria. If this occurs, it may well be the case that an increase of IES makes both equilibria (rich and intermediate ones) move in the same direction. In order to better understand this point, we note that the crucial condition determining the sign of $s_\sigma$ and thus the
direction of the effect a change in the IES on the long run equilibria can be expressed by the following implicit function:

$$J = \beta - \frac{1}{1 + r(\Phi)}$$

where \( \Phi \) is the set of parameters of the model. In particular, the sign of \( J \) depends on the level of the capital stock which is present in the economy. A clear example is presented in Fig. 1: when \( \sigma = 0.25 \), \( J \) happens to be positive around the intermediate equilibrium (i.e. \( s_\sigma > 0 \)) and negative around the rich equilibrium (i.e. \( s_\sigma < 0 \)), so that both equilibria are increased by a rise in the IES parameter as expected from Proposition 1.

Equation (4) allows for an economic intuition of the effect of the IES parameter on the long run equilibria: suppose that \( \beta < 1/(1+r) \) (i.e. \( s_\sigma < 0 \)): an increase of the IES has an impact effect of decreasing savings and thus capital accumulation; this, in turn, progressively causes an increase of interest rate and a reduction in wages. Now, if \( s_r \) is sufficiently small (or, better, negative), then savings and capital will further decrease until a new, lower long run equilibrium is reached. If, on the contrary, \( s_r \) is strongly negative, then the interest rate effect can revert the impact effect, so that the steady state capital intensity will be higher than the previous one.

Symmetrically, when \( \beta > 1/(1+r) \) (i.e. \( s_\sigma > 0 \)), an increase of the IES, on impact, will increase savings and next period capital intensity. Thus, next period’s interest rate will be lower and wages higher. Now, if \( s_r \) is sufficiently small, then savings will further increase and the economy will converge to a higher capital intensity equilibrium; on the contrary, if \( s_r \) is strongly negative, then the interest rate change can revert the impact effect, such that the new long run capital intensity will be lower than the previous one.

Moreover, IES not only determines the level of the equilibria, as above shown, but may affect the stability of these equilibria.

To recognise this, preliminary we note that, as shown in Fig. 1 above for a given set of parameters, the accumulation locus can be backward bending within a certain interval of \( k_r \).

More precisely, the latter occurrence requires:

1) $$\frac{dk_{t+1}}{dk_t} = -\frac{f_k'(k_s)w}{(1+n) - f_k'(s_r)s_r} < 0; \text{ where } f_k', \ s_w \text{ and } s_r \text{ are, respectively, the second derivative of the production function with respect to capital, the derivative of savings with respect to wage and interest rate;}$$
2) since the numerator is always positive, necessary condition for inequality above to hold is \( s_r < 0 \Leftrightarrow \sigma < 1 \), while sufficient condition is that the denominator is negative, that is \( s_r < \frac{1+n}{f''} \).

Consequently, the intermediate equilibrium can be stable (unstable) when
\[
-1 < \frac{dk_{t+1}}{dk_t} < 0, \quad \left( \frac{dk_{t+1}}{dk_t} \leq -1 \right).
\]

3- The stability analysis, the role of IES and the world with three stable equilibria

We now briefly discuss the way in which changes in IES are crucial for the stability of the intermediate equilibria. The following proposition gives sufficient conditions for determining the sign of \( \frac{d}{d\sigma} \left( \frac{dk_{t+1}}{dk_t} \right) \) at the intermediate equilibrium.

**Proposition 2:** Provided that the accumulation locus is backward bending around the intermediate equilibrium and either 1) \( \varepsilon_{f''_r,k} > 0 \) or 2) \( \varepsilon_{f''_r,k} < 0 \) and \( |\varepsilon_{s_r,k}| > |\varepsilon_{f''_r,k}| \) holds:

i) if \( s_\sigma > 0 \) and \( \varepsilon_{s_r,\sigma} \leq \varepsilon_{s_w,\sigma} \) then \( \frac{d}{d\sigma} \left( \frac{dk_{t+1}}{dk_t} \right) < 0; \)

ii) if \( s_\sigma < 0 \) and \( \varepsilon_{s_r,\sigma} \geq \varepsilon_{s_w,\sigma} \) then \( \frac{d}{d\sigma} \left( \frac{dk_{t+1}}{dk_t} \right) > 0. \)

where \( \varepsilon_{x,y} = -\frac{\partial x}{\partial y} \) as the elasticity of variable \( x \) with respect to \( y. \)

(proofs available on request).

Hence, if conditions sub i) are satisfied, an increase of the IES works for destabilizing the intermediate equilibrium; conversely, if conditions sub ii) hold, then an increase of this parameter works for stabilizing the intermediate equilibrium.

From Proposition 2 it descends that, under certain conditions, which the numerical simulation in next section confirms to be rather robust, and for sufficiently low values of \( \sigma \), such that the intermediate equilibrium is stable, the case of economies sticking in the middle of the path between richness and poorness may arise.

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7 This case may emerge when the IES parameter is lower than 1, so that the substitution effect is dominated by the income effect, and therefore, a rise in the rate of interest has a negative effect on savings: a rise of remuneration of savings leads to more consumption today.
Therefore the following remark holds:

**Remark 1**: the case of low IES preferences and low intertemporal elasticity of technical substitution enriches the range of attracting long run equilibria of the basic OLG economy, because the case of three long run stable equilibria may occur. Therefore, the consequences for the economic growth analysis are striking: besides poor and rich countries, the case of economies sticking at intermediate long run growth regimes may occur. The currently developing countries might remain entrapped at the current level of development or even be caught by a lower level of income if IES decreases at a sufficiently low value.

4- Numerical simulations

Since conditions shown from Proposition 2 are much complex and, in any case, have poor economic intuition, we now turn to simulation exercises in which we evaluate the predictions of the our theoretical model under a specific but plausible set of parameters. In particular, we focus on the role of the IES parameter in determining both the level and the stability of the long run equilibria.

We show the occurrence of “three stable equilibria” (k_0 =0, k_1= intermediate, k_2= high) for three different values of the IES parameter and for two different sets of the other parameters. Let us start from a situation (case I with σ=0.1) in which the intermediate equilibrium is stable. In this case an increase of σ unambiguously reduces both steady state levels of capital. Note that, as expected, by Proposition 1 this effect depends strongly on the sign of s_σ.

**Table 1**: Changes of the two positive capital equilibria, of the sign of J and of slope of the accumulation locus when IES changes for two parametric sets.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>σ</th>
<th>Positive long run capital intensity equilibria</th>
<th>Sign(J)</th>
<th>( \frac{dk_{i+1}}{dk_i} )</th>
</tr>
</thead>
</table>
| **Case I**: A=30, α=0.4, ρ=7, β=0.4, n=0.8 | 0.1 | k_1=1.032  
  k_2=8.55 | + | -0.966  
  0.15*10^{-5} |
| | 0.2 | k_1=1.00  
  k_2=8.14 | + | -1.38  
  0.22*10^{-5} |
| | 0.3 | k_1=0.977  
  k_2=7.73 | - | -2.16  
  0.32*10^{-5} |
| **Case II**: A=10, α=0.4, | 0.1 | k_1=1.084  
  k_2=2.661 | - | -0.798  
  1*10^{-4} |
| | 0.2 | k_1=1.078  
  k_2=2.565 | + | -0.98  
  2*10^{-4} |

---

8 Note that the derivative of the accumulation locus calculated at the equilibrium is -0.966 -see last column, first row.

9 For economy of space we do not report here the values of the zero equilibrium.
\[ \begin{array}{|c|c|c|c|} \hline \rho = 11, \\ \beta = 0.5, \\ n = 0.9 & 0.3 & k_1 = 1.071 & + \\ k_2 = 2.470 & - & -1.25 \times 10^{-4} \\ \hline \end{array} \]

However, as for stability, the increase of \( \sigma \) generates different effects, depending on whether the rich or the intermediate equilibrium is considered. More precisely, the rise in the IES parameter raises the absolute value of the slope of the accumulation locus; however, while this change does not produce qualitative consequences on the stability of the high equilibrium, it generates local instability of the intermediate equilibrium. On this respect we note that in the table above only cases sub i) of Proposition 2 are reported.

Also for another set of parameters (Case II), the same qualitative results of Case I are preserved; in particular, in such a case, for higher values of the IES (e.g. \( \sigma = 0.2 \)) the world with three stable equilibria arises. Therefore, simulations show that the case unveiled in this paper is rather robust to different parameter configurations.

By defining \( \sigma^* \) as the solution for \( \frac{dk_1}{dt} = -1^{10} \), in the following table a taxonomy of the possible long run results of this economy when the IES parameter varies is proposed:

<table>
<thead>
<tr>
<th>( \sigma \leq \sigma^* \leq 1 )</th>
<th>( \sigma \geq \sigma^* \leq 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Properties of equilibria</td>
<td>Three (locally asymptotically stable) equilibria</td>
</tr>
<tr>
<td>Economic interpretation of the steady-state scenarios</td>
<td>Three groups of countries: underdeveloped, developing and developed</td>
</tr>
</tbody>
</table>

### 5- Conclusions

In this paper we show that when both technical substitution and intertemporal elasticity of consumption substitution are sufficiently low, besides steadily poor and rich countries, a case of persistently intermediate living standards’ countries arises.

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\(^{10}\) For example, under the parametric configuration of Case II of our simulations, \( \sigma^* = 0.21 \).
So, the observed different stages of development in the real world, namely underdeveloped, developing and developed countries may correspond to three long run stable income growth equilibria rather than the three different stages of a trajectory converging to the same long run income growth. As a consequence, it is possible that developing countries, instead of being engaged in a catching up process towards the rich countries are steadily converging to their own intermediate equilibrium.

If so, any slow down affecting developing countries\textsuperscript{11} can be explained not much through a concept of conditional convergence, or as an example of the failure of the concept of absolute convergence, but, rather, through the existence of an intermediate stable growth by which these countries could have been attracted, as a consequence of a low IES and coordination failure of agents’ expectations.

Our results, although merely theoretical, can shed new light on the debate on convergence, on the corresponding growth empirics and on the policies for escaping from poverty traps.

References


\textsuperscript{11} Or alternatively, any difference in economic growth in these countries, such as the one featuring some Asian countries.


**APPENDIX: The model setup**

The closed economy (a la Dimond (1965)) is populated by two -period living individuals, who maximize a utility function $U$, defined over $c_1$ and $c_2$, that is consumption in the first and second period of life; hence, in their youth the $N_t$ agents born in period $t$ must choose how much to save out of their wage, with fixed labor supply normalized to 1. The population grows at a constant growth rate, $n$.

### 2.1 Households

By assuming a CIES utility function, individuals solve the following problem

\[
\text{Max } U_t = u(c_{1t}) + \beta u(c_{2t+1}) \quad \text{sub } c_{1t} + \frac{c_{2t+1}}{1+E_t(r_{t+1})} = w_t,
\]

where $u = \frac{c^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}$, with $\sigma > 0$ and $\sigma \neq 1$, $\beta$ the intertemporal discount rate (strictly positive and lower than 1) and $E_t$ is the expectation operator as for time $t$. We assume rational expectations (perfect foresight), so that: $E_t r_{t+1} = r_{t+1}$. First order conditions deliver the following results:

\[
s_t = w_t - c_t = \frac{\beta^\sigma (1+r_{t+1})^{\sigma-1} w_t^\sigma}{1+\beta^\sigma (1+r_{t+1})^{\sigma-1}}, \quad s_t > 0.
\]

[A1]

Recall that $s_t \geq 0$ iff $\sigma \geq 1$ (e.g. de la Croix and Michel (2002), section 1).

### 2.2 The production sector

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12 Diamond’s model with CIES preferences and CES technology has been ketch by de la Croix and Michel (2000).
We assume a competitive market with identical firms owning CRS technology: thus, capital and labor are remunerated according to their marginal productivities. Moreover, due to the homogeneity of degree one of the aggregate production function \( F(K,L) \) and defining \( k = \frac{K}{L} \) and \( f(k) \equiv \frac{F(K,L)}{L} = F(k,1) \), it follows that:

\[
F'_k = f'_k = r_t \quad \text{and} \quad F'_L = f - f'_k L = w_t,
\]

where \( r_t \) and \( w_t \) are the interest rate and the wage in period \( t \), and the subscript of the derivative functions \( F' \) and \( f' \) indicates the derivation variable. In our model we use a CES production function of the following type:

\[
F(K,L) = A\left[\alpha K^{-\rho} + (1-\alpha)L^{-\rho}\right]^{\frac{1}{\rho}}, \quad \rho > -1, \ \rho \neq 0, \ A > 0, \ \alpha \in (0,1)
\]

[A3]

which, in per worker terms, has the form:

\[
f(k) = A\left[\alpha k^{-\rho} + (1-\alpha)\right]^{\frac{1}{\rho}}, \quad \text{with}
\]

[A4]

\[
f'_k = \alpha A^{-\rho} \left(f_k\right)^{\rho} \quad \text{and} \quad f - f'_k L = (1-\alpha)A^{-\rho} f_k^{\rho}.
\]

[A5]

Finally the equation providing the market clearing condition, whereby \( K_{t+1} = S_t \), in per capita terms gives (1) in the main text.
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