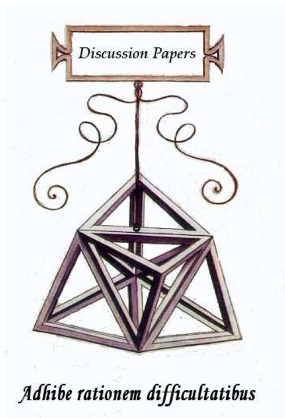
Discussion Papers

Collana di

E-papers del Dipartimento di Scienze Economiche – Università di Pisa



Luciano Fanti and Luca Gori

PAYG pensions and economic cycles: exogenous versus endogenous fertility

Discussion Paper n. 75

2008

Discussion Paper **n. 75: presentato Dicembre 2008**

Luciano Fanti

Department of Economics, University of Pisa

Via Cosimo Ridolfi, 10, I–56124 Pisa (PI), Italy

e-mail address: lfanti@ec.unipi.it

tel.: +39 050 22 16 369

fax: +39 050 22 16 384

Luca Gori

Department of Economics, University of Pisa

Via Cosimo Ridolfi, 10, I–56124 Pisa (PI), Italy

e-mail address: luca.gori@ec.unipi.it

tel.: +39 050 22 16 212

fax: +39 050 22 16 384

PAYG pensions and economic cycles: exogenous versus endogenous fertility

Luciano Fanti* **and Luca Gori****

Department of Economics, University of Pisa, Via Cosimo Ridolfi, 10, I–56124 Pisa (PI), Italy

Abstract We analyse the dynamics of an overlapping generations economy with unfunded pay-as-you-go public pensions and myopic expectations by comparing exogenous and endogenous fertility contexts. It is shown that large PAYG pensions may cause endogenous fluctuations in both cases. However, *ceteris paribus*, when the number of children is an economic decision variable the economy is much more prone to cyclical instability and deterministic chaos emerges.

Keywords Myopic foresight; PAYG pension; Stability; OLG model

JEL Classification C62; H55; J14; J18; J26

* *E-mail address*: lfanti@ec.unipi.it; tel.: +39 050 22 16 369; fax: +39 050 22 16 384.

** Corresponding author. *E-mail address*: luca.gori@ec.unipi.it; tel.: +39 050 22 16 212; fax: +39 050 22 16 384.

1. Introduction

One of the most fascinating results in the overlapping generations (OLG) literature is the possibility of endogenous fluctuations (Grandmont, 1985; Farmer, 1986; Reichlin, 1986; Azariadis, 1993). Typically, however, cyclical behaviour emerges at low values of the elasticity of substitution in production, in particular for values well below unity. In particular, Farmer (1986) showed, for the CES case, that Hopf bifurcations may occur only when technologies exhibit lower factor substitutability than the Cobb-Douglas function. On the other hand, with myopic foresight, the steady state equilibrium may be oscillatory and even exhibit deterministic complex cycles (Michel and de la Croix, 2000, de la Croix and Michel, 2002; Fanti and Spataro, 2008) when the inter-temporal elasticity of substitution is higher than that of the Cobb-Douglas function.

Therefore, although it is well known that overlapping generations economies with myopic expectations and elasticity of substitution in production and in the utility function relatively low and high respectively (and then necessarily different from that of the Cobb-Douglas functions), may show cyclical (and even chaotic) dynamics, in this paper we show, in the standard case of Cobb-Douglas functions, that the size of pay-as-you-go (PAYG) pensions plays a crucial role in determining the stability of the economy and, in particular, large PAYG pensions causes the appearance of persistent cycles and even deterministic chaos may also emerge when individuals are myopic foresighted.[1]

Moreover, the importance of fertility in determining the macroeconomic performances is well established at least starting from the seminal paper by Becker (1960). Therefore, in accord with the recent body of literature arguing that fertility in an economic choice variable rather than an

[1] The debate on the role of PAYG pensions in OLG economies in presence of demographic changes has recently increased due problems of sustainability as well as for optimality purposes (see, e.g., van Groezen et al., 2003; Abio et al., 2004; van Groezen and Meijdam, 2008). However, less attention has been paid to the role of PAYG pensions and endogenous demographic changes on stability, which is the object of this paper.

exogenous event (Becker and Barro, 1988: Barro and Becker, 1989; Becker et al., 1990), we also consider endogenous fertility and compare the dynamical features of this setting with those obtained in the standard exogenous fertility economy. However, since costs of children undoubtedly exist irrespective of whether the size of the family is either based on comparison of economic constraints and incentives or is exogenously given (due, for instance, to religion belief, customs and so on), we assume that child bearing is costly even when fertility is not an economic decision variable. In this paper, therefore, we analyse the dynamical properties of an economy with endogenous fertility and public PAYG pensions and compare them with those obtained in the case of exogenous fertility.

Our results reveal that while the same unique steady state is achieved, the dynamic adjustment processes towards the steady state are dramatically different according to whether or not fertility is endogenously chosen by individuals, and this different dynamic behaviour is due to the presence of public PAYG pensions.[2] Therefore, whether the number of children is the result of a rational choice of individuals rather than an exogenous event significantly matters for stability. In particular, when fertility is endogenously determined, the destabilising effect of PAYG pensions is stronger than when it is exogenously given. The economic reason for this result is the following: the negative impact of public pensions on savings is mitigated by the effect of the individual degree of parsimony in both cases of exogenous and endogenous fertility (see Eqs. 7 and 24 in the sequel of the paper). However, when fertility is endogenous, the degree of parsimony has a negative impact effect also on the demand for children. Therefore, as regards capital accumulation per worker, which is the ratio between savings and fertility, we observe that when fertility is endogenous, a rise in the individual degree of thriftiness negatively affects in the same way savings (through the public pension component) and the number of children, so that both effects exactly cancel each other out on capital accumulation. The final effect, therefore, is that the negative impact effect of public

[2] Note that in the absence of PAYG pensions the dynamic behaviour is the same with exogenous and endogenous fertility.

pensions on capital accumulation is stronger with endogenous fertility than with exogenous fertility, and thus the destabilising effects of PAYG pensions is more likely to occur in the former case.

The essential message of this paper consists in showing that different dynamic adjustment processes emerge in OLG models with PAYG pensions, depending on whether the number of children is either exogenous or endogenous to the model. Therefore, we argue that to the extent that fertility choices are based on an economic comparison between benefits and costs of children, which is considered going hand to hand with the stages of economic development[3] – as suggested by the home economics literature (see Becker, 1960) –, the more likely large PAYG pensions plays dramatic effects on the stability of the economy.

This paper contributes to three strands of literature: (*i*) the business cycles analysis framed in the OLG context; (*ii*) the endogenous population economics, and (*iii*) the economics with public PAYG pensions, and shows that considering together all these aspects gives rise to some new results which may have interesting policy implications.

The remainder of the paper is organised as follows. In Section 2 (Section 3) we analyse the local stability properties of an OLG economy with exogenous (endogenous) fertility. In Section 4 we show that the dynamic adjustment processes towards the stationary state are different in exogenous and endogenous fertility economies despite the long-run equilibrium is the same, showing, in particular, that the risk of cyclical instability due to large PAYG pensions is higher with endogenous fertility. Section 5 concludes.

2. Exogenous fertility

---

[3] In fact, as argued by van Groezen et al., (2003, p. 237) "…at least in western countries, people are to a very large extent able to freely choose the number of children they desire. The rate of fertility should therefore be treated as an endogenous variable, that is, as the result of a rational choice which is influenced by economic constraints and incentives. Economic theory can thus help in explaining why the observed decline in the (desired) number of children would occur."

2.1. Firms

Firms are identical and act competitively on the market. Aggregate production at time t (Y_t) takes place according to the Cobb-Douglas technology $Y_t = AK_t^{\alpha} L_t^{1-\alpha}$, where K_t and $L_t = N_t$ are capital and labour hired by the typical firm, N_t is the number of young workers in the economy, $A > 0$ is a scale parameter and $0 < \alpha < 1$ the output elasticity of capital. Assuming that capital totally depreciates at the end of each period and a unit price of final output, profit maximisation implies that factor inputs are paid their marginal products, that is:

$$r_t = \alpha A k_t^{\alpha-1} - 1, \tag{1}$$

$$w_t = (1-\alpha) A k_t^{\alpha}, \tag{2}$$

where $k_t := K_t / N_t$ is capital per worker, r_t is the interest rate and w_t the wage rate.

2.2. Government

The government redistributes from the young to the old with unfunded public pensions. Therefore, at time t the public pension budget in per worker terms reads as:

$$p_t = \theta w_t n, \tag{3}$$

the left-hand side being the social security expenditure and the right-hand side the tax receipts, where $p_t > 0$ is the pension benefit, $0 < \theta < 1$ is the fixed contribution rate and n is the exogenous number of children in the whole economy.

2.3. Individuals

Young population of measure N_t grows at a constant rate n and agents are assumed to belong to an overlapping generations economy where life is divided between youth (working period) and old-age (retirement period), as in Diamond (1965).

Only young individuals join the workforce and supply inelastically their whole time endowment (normalised to unity) on the labour market, while receiving a unitary wage income at the competitive rate w_t, which is used to consume, to save, to bear children and to finance material consumption of the elderly through the public pension scheme Eq. (3).

Starting from the seminal papers by Diamond (1965), Samuelson (1975) and Deardorff (1976), the economic literature did not consider children costs in models with exogenous fertility. However, it is natural to conjecture that child-rearing activities are costly (either time-based or income-based, or both) even when fertility is not an economic decision variable. In particular, in this paper we assume that the amount of resources needed to care about children is given by a monetary cost $q w_t$ per child, where $0 < q < 1$ is the percentage of child rearing cost on the parents' working income.[4]

Old individuals are retired and live on (*i*) the amount of resources saved when young (s_t) plus the expected interest accrued at the rate r^e_{t+1}, and (*ii*) the expected pension benefit p^e_{t+1}.

The representative individual born at t has preferences towards young-aged consumption ($c_{1,t}$) and old-aged consumption ($c_{2,t+1}$), and chooses how much to save out of disposable income in order to maximise the lifetime utility function U_t subject to each period budget constraint, that is

$$\max_{\{s_t\}} U_t = \ln(c_{1,t}) + \beta \ln(c_{2,t+1}), \tag{4}$$

subject to

$$\begin{aligned} c_{1,t} + s_t + q w_t n &= w_t (1-\theta) \\ c_{2,t+1} &= (1 + r^e_{t+1}) s_t + p^e_{t+1} \end{aligned}, \tag{5}$$

[4] Note that this child cost structure is similar to that adopted, amongst many other, by Wigger (1999) and Boldrin and Jones (2002).

where $0 < \beta < 1$ is the subjective discount factor.

The saving rate, therefore, is

$$s_t = \frac{\beta w_t(1-\theta-qn)}{1+\beta} - \frac{p^e_{t+1}}{(1+\beta)(1+r^e_{t+1})}. \tag{6}$$

where $q < (1-\theta)/n$ must hold to guarantee Eq. (6) to be positive. Now, inserting the one-period-forward pension accounting rule Eq. (3) into Eq. (6) we get:

$$s_t = \frac{\beta w_t(1-\theta-qn)}{1+\beta} - \frac{n\theta}{1+\beta} \cdot \frac{w^e_{t+1}}{1+r^e_{t+1}}. \tag{7}$$

From Eq. (7) we see that savings are divided into two components: the *private saving component* (the first term on the right-hand side of Eq. 7) and the *public pension component* (the second term on the right-hand side of Eq. 7).

2.4. Equilibrium

Given the government budget Eq. (3) and knowing that $N_{t+1} = nN_t$, market-clearing in goods and capital markets is expressed by the equality:

$$nk_{t+1} = s_t. \tag{8}$$

Using Eqs. (7) and (8) equilibrium implies:

$$k_{t+1} = \frac{\beta w_t(1-\theta-qn)}{(1+\beta)n} - \frac{\theta}{1+\beta} \cdot \frac{w^e_{t+1}}{1+r^e_{t+1}}. \tag{9}$$

The dynamic evolution of capital is different depending on whether individuals are perfect or short sighted. In particular, while in the case of rational expectations the economy does not exhibit any interesting dynamical feature (i.e. the steady state is locally stable and the dynamics is always monotonic), in the case of myopic expectations the dynamics may be non-monotonic and cyclical instability may emerge when the public pension component dominates the private saving component, and this depends on the size of the PAYG scheme as well as on the mutual relationship between preference and technological parameters. Therefore, below we concentrate on the case of short-sighted agents to study the complex dynamical properties of this simple economy.

2.5. Exogenous fertility: local stability with myopic foresight

With myopic expectations both the expected interest and wage rates depend on the current value of the stock of capital per worker, that is

$$\begin{cases} 1+r^{e}{}_{t+1}=\alpha A k_t^{\alpha-1} \\ w^{e}{}_{t+1}=(1-\alpha)A k_t^{\alpha} \end{cases}. \tag{10}$$

Therefore, exploiting Eqs. (1), (2), (9) and (10) the dynamic path of capital accumulation is given by the following non-linear difference equation:

$$k_{t+1}=\frac{\beta(1-\theta-qn)(1-\alpha)A}{(1+\beta)n}k_t^{\alpha}-\frac{\theta}{1+\beta}\cdot\frac{1-\alpha}{\alpha}k_t. \tag{11}$$

Steady state implies $k_{t+1}=k_t=k^*$. Therefore, the steady state stock of capital (which is the same with both perfect and myopic expectations, see Michel and de la Croix, 2000) is

$$k^*=\left\{\frac{\beta\alpha(1-\alpha)A(1-\theta-qn)}{n[\alpha(1+\beta)+\theta(1-\alpha)]}\right\}^{\frac{1}{1-\alpha}}. \tag{12}$$

The analysis of Eqs. (11) and (12) gives the following proposition:

Proposition 1. *The dynamics of an OLG economy with exogenous fertility, PAYG pensions and myopic foresighted individuals is the following.*

(1) Let $0<\alpha<\alpha_4$ *hold. Then* $\theta_1<\theta_2<1$*, and*

(1.1) if $0<\theta<\theta_1$*, the dynamics of capital is monotonic and convergent to* k^**;*

(1.2) if $\theta_1<\theta<\theta_2$*, the dynamics of capital is oscillatory and convergent to* k^**;*

(1.3) if $\theta = \theta_2$*, a flip bifurcation emerges;*

(1.4) if $\theta_2 < \theta < 1$*, the dynamics of capital is oscillatory and divergent to* k^**.*

(2) Let $\alpha_4 < \alpha < \alpha_2$ *hold. Then* $\theta_1 < 1$*,* $\theta_2 > 1$*, and*

(2.1) if $0 < \theta < \theta_1$*, the dynamics of capital is monotonic and convergent to* k^**;*

(2.2) if $\theta_1 < \theta < 1$*, the dynamics of capital is oscillatory and convergent to* k^**.*

(3) Let $\alpha_2 < \alpha < 1$ *hold. Then* $\theta_2 > \theta_1 > 1$ *and the dynamics of capital is monotonic and convergent to* k^* *for any* $0 < \theta < 1$*,*

where

$$\theta_1 = \theta_1(\alpha, \beta) := \frac{\alpha^2 (1+\beta)}{(1-\alpha)^2}, \tag{13}$$

$$\theta_2 = \theta_2(\alpha, \beta) := \frac{\alpha(1+\alpha)(1+\beta)}{(1-\alpha)^2} = \theta_1 \frac{1+\alpha}{\alpha}, \tag{14}$$

$$\alpha_2 = \alpha_2(\beta) := \frac{-1+\sqrt{1+\beta}}{\beta} > 0, \quad 0 < \alpha_2 < 1/2, \tag{15}$$

$$\alpha_4 = \alpha_4(\beta) := \frac{-(3+\beta)+\sqrt{\beta^2 + 10\beta + 9}}{2\beta} > 0, \quad 0 < \alpha_4 < 1/3. \tag{16}$$

Proof. Differentiating Eq. (11) with respect to k_t and using Eq. (12) we get:

$$\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t = k^*} = \alpha - \frac{\theta}{1+\beta} \cdot \frac{(1-\alpha)^2}{\alpha}. \tag{17}$$

Monotonic and non-monotonic dynamics with exogenous fertility

From Eq. (17), the condition $\frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} \gtrless 0$ implies

$$\alpha - \frac{\theta}{1+\beta}\cdot\frac{(1-\alpha)^2}{\alpha} \gtrless 0 \Leftrightarrow \theta \lessgtr \theta_1, \tag{18}$$

so that $\theta_1 < 1$ ($\theta_1 > 1$) for any $0 < \alpha < \alpha_2$ ($\alpha_2 < \alpha < 1$). Moreover, $\theta_1 < 1$ if and only if $\alpha_1 < \alpha < \alpha_2$, where $\alpha_1 = \alpha_1(\beta) := \frac{-1-\sqrt{1+\beta}}{\beta} < 0$ and $0 < \alpha_2 < 1/2$ is given by Eq. (15). Since $\alpha_1 < 0$ it is ruled out.

Now, $\frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} < 1$ gives

$$\alpha - \frac{\theta}{1+\beta}\cdot\frac{(1-\alpha)^2}{\alpha} < 1 \Rightarrow \theta > -\frac{\alpha(1+\beta)}{1-\alpha}. \tag{19}$$

Therefore, in the case of monotonic behaviour, $0 < \frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} < 1$ and thus the dynamics of capital is always convergent to the steady state.

Non-monotonic dynamics: stability analysis with exogenous fertility

The condition $\frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} \gtrless -1$ implies

$$\alpha - \frac{\theta}{1+\beta}\cdot\frac{(1-\alpha)^2}{\alpha} \gtrless -1 \Rightarrow \theta \lessgtr \theta_2, \tag{20}$$

where $\theta_2 > \theta_1$, so that $\theta_2 < 1$ ($\theta_2 > 1$) for any $0 < \alpha < \alpha_4$ ($\alpha_4 < \alpha < 1$), with $\alpha_4 < \alpha_2$. Moreover, $\theta_2 < 1$ if and only if $\alpha_3 < \alpha < \alpha_4$, where $\alpha_3 = \alpha_3(\beta) := \frac{-(3+\beta)-\sqrt{\beta^2+10\beta+9}}{2\beta} < 0$ and $0 < \alpha_4 < 1/3$ is given by Eq. (16). Since $\alpha_3 < 0$ it is ruled out.

Therefore,

(i) if $0<\alpha<\alpha_4$ then $\theta_1<\theta_2<1$ and (1.1) $0<\frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*}<1$ for any $0<\theta<\theta_1$, (1.2) $-1<\frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*}<0$ for any $\theta_1<\theta<\theta_2$, (1.3) $\frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*}=-1$ if and only if $\theta=\theta_2$, and (1.4) $\frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*}<-1$ for any $\theta_2<\theta<1$. This proves point (1);

(ii) if $\alpha_4<\alpha<\alpha_2$ then $\theta_1<1$, $\theta_2>1$ and (2.1) $0<\frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*}<1$ for any $0<\theta<\theta_1$ and (2.2) $-1<\frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*}<0$ for any $\theta_1<\theta<1$. This proves point (2);

(iii) if $\alpha_2<\alpha<1$ then $\theta_2>\theta_1>1$ and $0<\frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*}<1$ for any $0<\theta<1$. This proves point (3). **Q.E.D.**

3. Endogenous fertility

In this section we assume that individuals also draw utility from the number of children they wish to raise and, hence, choose fertility on the basis of a rational comparison between benefits and costs of children (i.e., the so-called weak form of altruism, see Zhang and Zhang, 1998). The model is outlined below.

3.1. Government

The public pension budget is now

$$p_t=\theta w_t n_t, \tag{21}$$

where n_t is the endogenous number of children at time t.

3.2. Individuals

Individuals draw utility from material consumption over the life cycle and the number of children, see Eckstein and Wolpin (1985) and Galor and Weil (1996). The representative individual entering the working period at t solves the following problem:

$$\max_{\{s_t, n_t\}} U_t(c_{1,t}, c_{2,t+1}, n_t) = \ln(c_{1,t}) + \beta \ln(c_{2,t+1}) + \phi \ln(n_t), \tag{22}$$

subject to Eqs. (5), where $\phi > 0$ captures the parents' taste for children.

The constrained maximisation of Eq. (22) gives (upon substitution of the one-period forward pension accounting rule Eq. 21 for p_{t+1}) the demand for children and the saving rate, respectively:

$$n_t = \frac{\phi w_t (1-\theta)}{(1+\beta+\phi) q w_t - \phi\theta \dfrac{w^e_{t+1}}{1+r^e_{t+1}}}, \tag{23}$$

$$s_t = \frac{w_t(1-\theta)}{(1+\beta+\phi) q w_t - \phi\theta \dfrac{w^e_{t+1}}{1+r^e_{t+1}}} \left(\beta q w_t - \phi\theta \frac{w^e_{t+1}}{1+r^e_{t+1}} \right). \tag{24}$$

3.3. Equilibrium

Given the government budget Eq. (21) and knowing that $N_{t+1} = n_t N_t$, equilibrium in goods and capital market is

$$n_t k_{t+1} = s_t. \tag{25}$$

Using Eqs. (23)-(25) the equilibrium condition reads as:

$$k_{t+1} = \frac{\beta}{\phi} q w_t - \theta \frac{w^e_{t+1}}{1+r^e_{t+1}}. \tag{26}$$

3.4. Endogenous fertility: local stability with myopic foresight

Exploiting Eqs. (1), (2), (10) and (26) the dynamic path of capital accumulation with short-sighted agents and endogenous fertility is given by the following non-linear difference equation:

$$k_{t+1} = \frac{\beta}{\phi} q(1-\alpha) A k_t^{\alpha} - \theta \frac{1-\alpha}{\alpha} k_t . \tag{27}$$

Steady state implies

$$k^* = \left\{ \frac{q\beta\alpha(1-\alpha)A}{\phi[\alpha+\theta(1-\alpha)]} \right\}^{\frac{1}{1-\alpha}} . \tag{28}$$

The analysis of Eqs. (27) and (28) gives the following proposition:

Proposition 2. *The dynamics of an OLG economy with endogenous fertility, PAYG pensions and myopic foresighted individuals is the following.*

(1) Let $0 < \alpha < 1/3$ *hold. Then* $\theta_3 < \theta_4 < 1$*, and*

(1.1) if $0 < \theta < \theta_3$*, the dynamics of capital is monotonic and convergent to* k^**;*

(1.2) if $\theta_3 < \theta < \theta_4$*, the dynamics of capital is oscillatory and convergent to* k^**;*

(1.3) if $\theta = \theta_4$*, a flip bifurcation emerges;*

(1.4) if $\theta_4 < \theta < 1$*, the dynamics of capital is oscillatory and divergent to* k^**.*

(2) Let $1/3 < \alpha < 1/2$ *hold. Then* $\theta_3 < 1$*,* $\theta_4 > 1$*, and*

(2.1) if $0 < \theta < \theta_3$*, the dynamics of capital is monotonic and convergent to* k^**;*

(2.2) if $\theta_3 < \theta < 1$*, the dynamics of capital is oscillatory and convergent to* k^**.*

(3) Let $1/2<\alpha<1$ *hold. Then* $\theta_4>\theta_3>1$*, and the dynamics of capital is monotonic and convergent to* k^* *for any* $0<\theta<1$*,*

where

$$\theta_3=\theta_3(\alpha):=\frac{\alpha^2}{(1-\alpha)^2}, \tag{29}$$

$$\theta_4=\theta_4(\alpha):=\frac{\alpha(1+\alpha)}{(1-\alpha)^2}=\theta_3\frac{1+\alpha}{\alpha}. \tag{30}$$

Proof. Differentiating Eq. (27) with respect to k_t and using Eq. (28) gives:

$$\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}=\alpha-\theta\frac{(1-\alpha)^2}{\alpha}. \tag{31}$$

Monotonic and non-monotonic dynamics with endogenous fertility

Therefore, $\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}\underset{<}{\overset{>}{-}}0$ yields

$$\alpha-\theta\frac{(1-\alpha)^2}{\alpha}\underset{<}{\overset{>}{-}}0\Leftrightarrow\theta\underset{>}{\overset{<}{-}}\theta_3, \tag{32}$$

so that $\theta_3<1$ ($\theta_3>1$) for any $0<\alpha<1/2$ ($1/2<\alpha<1$).

Now, $\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}<1$ yields

$$\alpha-\theta\frac{(1-\alpha)^2}{\alpha}<1\Rightarrow\theta>-\frac{\alpha}{1-\alpha}. \tag{33}$$

Therefore, in the case of monotonic behaviour, $0<\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}<1$ and the dynamics of capital is always convergent to the steady state.

Non-monotonic dynamics: stability analysis with endogenous fertility

Moreover, $\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}\gtreqless -1$ yields

$$\alpha-\theta\frac{(1-\alpha)^2}{\alpha}\gtreqless -1\Rightarrow \theta\lesseqgtr\theta_4, \tag{34}$$

where $\theta_4>\theta_3$, so that $\theta_4<1$ ($\theta_4>1$) for any $0<\alpha<1/3$ ($1/3<\alpha<1$).

Therefore,

(i) if $0<\alpha<1/3$ then $\theta_3<\theta_4<1$ and (1.1) $0<\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}<1$ for any $0<\theta<\theta_3$, (1.2) $-1<\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}<0$ for any $\theta_3<\theta<\theta_4$, (1.3) $\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}=-1$ if and only if $\theta=\theta_4$, and (1.4) $\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}<-1$ for any $\theta_4<\theta<1$. This proves point (1);

(ii) if $1/3<\alpha<1/2$ then $\theta_3<1$, $\theta_4>1$ and (2.1) $0<\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}<1$ for any $0<\theta<\theta_3$ and (2.2) $-1<\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}<0$ for any $\theta_3<\theta<1$. This proves point (2);

(iii) if $1/2<\alpha<1$ then $\theta_4>\theta_3>1$ and $0<\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}<1$ for any $0<\theta<1$. This proves point (3). **Q.E.D.**

4. Exogenous versus endogenous fertility with myopic expectations

In order to compare exogenous and endogenous fertility economies we now assume that the same number of children is raised in both contexts. We find that when PAYG pensions exist, the same unique positive steady-state is achieved but with two different dynamic adjustment processes.[5] Moreover, the risk of cyclical instability due to large PAYG pensions is higher with endogenous fertility.

Proposition 3. *If the number of children is the same in both exogenous and endogenous fertility economies, then the same unique positive steady state is achieved.*

Proof. Assume that in the long run fertility is the same in both contexts, that is $n = n^*$. Then

$$n = \frac{\phi(1-\alpha)A(k^*)^\alpha(1-\theta)}{(1+\beta+\phi)q(1-\alpha)A(k^*)^\alpha - \phi\theta\frac{1-\alpha}{\alpha}k^*}. \tag{35}$$

Substituting Eq. (35) into Eq. (12) to eliminate n and solving for k^* we get Eq. (28). **Q.E.D.**

Proposition 4. *(1) The existence of public PAYG pensions causes a higher risk of cyclical instability when fertility is endogenously determined rather than when it is an exogenous event. Moreover, (2) while in an economy with endogenous fertility the individual degree of thriftiness is stability-neutral, when fertility is exogenous it acts as an economic stabiliser.*

[5] Of course, when PAYG pensions are absent and the number of children is the same in both exogenous and endogenous fertility economies, the dynamic paths of capital accumulation and the steady states stock of capital in these scenarios coincide.

Proof. When fertility is exogenous (endogenous), cyclical instability emerges only in the case $0<\alpha<\alpha_4$ when $\theta_2<\theta<1$ ($0<\alpha<1/3$ when $\theta_4<\theta<1$). Since $\alpha_4<1/3$ and $\theta_4<\theta_2:=\theta_4(1+\beta)$. This proves point (1). Moreover, $\frac{\partial\theta_2}{\partial\beta}=\theta_4>0$ and $\frac{\partial\theta_4}{\partial\beta}=0$. This proves point (2). **Q.E.D.**

Comparison of Eqs. (11) and (27) makes clear the reason why cyclical instability more likely emerges when individuals choose endogenously the number of children. When fertility is exogenous, in fact, a rise in the individual degree of thriftiness reduces the relative weight of the public pension component on savings and capital accumulation (see Eq. 11) and this, therefore, tends to mitigate the negative impact effect of a higher contribution rate on the accumulation of capital. In contrast, when fertility is endogenous, the subjective discount factor does not affect the public pension component (see Eq. 27) because it negatively affects in the same way both savings and the demand for children and, hence it is neutral on the public pension component of the dynamic path of capital accumulation. Therefore, the cyclical unstable region is larger than with endogenous fertility.

To illustrate the different dynamic adjustment processes in both exogenous and endogenous fertility economies with PAYG pensions, we take the following parameter set: $A=10$, $\alpha=0.20$, $\beta=0.40$ and $k_0=0.568$ (the initial condition) for both economies. For simplicity, we assume stationary population in the model with exogenous fertility. i.e. $n=1$. Then we choose $\phi=0.30$ and calibrate $q=0.11243$ to obtain $n^*=1$ in the model with endogenous fertility. This parameter set generates the following flip bifurcation values of the contribution rate: $\theta_2=0.525$ (exogenous fertility) and $\theta_4=0.375$ (endogenous fertility). Then we assume $\theta=0.46$. Despite the steady state stock of capital is the same in both economies, $k^*=0.3404$, the dynamic adjustment processes are dramatically different. In particular, endogenous fertility is a source of cyclical instability.

The cobweb depicted in Figure 1 shows the stable non-monotonic trajectory towards the stationary state when fertility is exogenous. In contrast, Figure 2 shows the unstable non-monotonic trajectory in the case of endogenous fertility. The difference is explained by the stabilising effect exerted by the subjective discount factor in reducing the relative weight of the public pension component in the PAYG-exogenous-fertility economy, which is instead absent in the PAYG-endogenous-fertility economy. In fact, when fertility is an economic decision variable, individuals accumulate less capital because the negative effect of PAYG pensions on the public pension component of the phase map Eq. (27) are not mitigated by the coefficient β, and this also causes a reduction in the transitional rate of economic growth. Therefore, to the extent that the steady state equilibrium is stable, the capital stock more rapidly approaches its stationary state when fertility is exogenously given than when it is endogenously determined to the model.

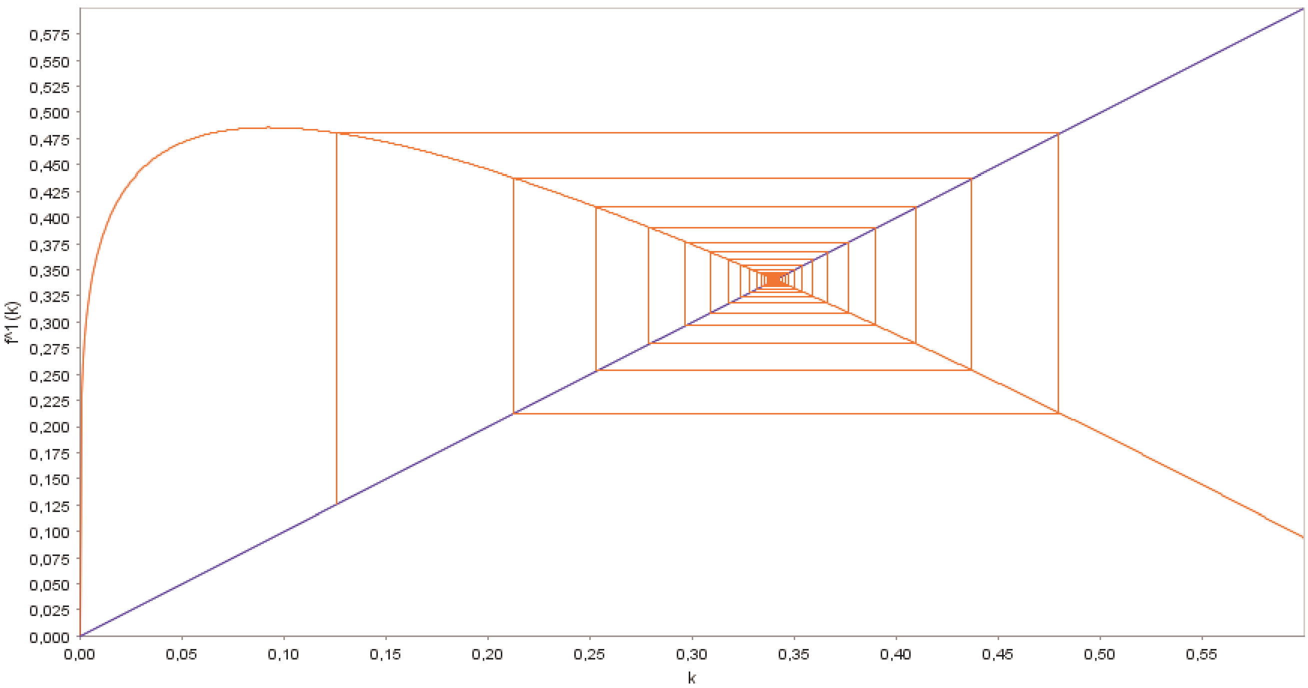


Figure 1. (Exogenous fertility). A pictorial view of cyclical stability.

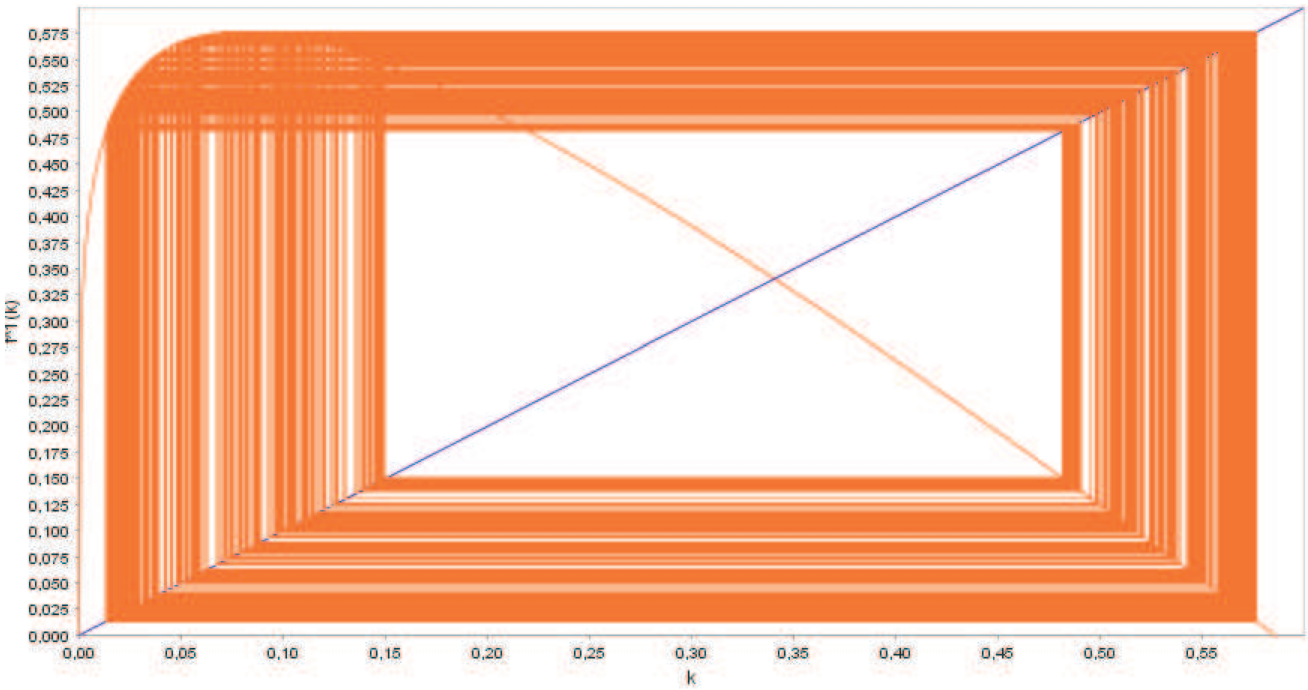


Figure 2. (Endogenous fertility). A pictorial view of cyclical instability.

Since the focus of this paper is on the steady state effects of public pensions, in Figure 3 we only sketch a numerical illustration to show that the cyclical behaviour depicted in Figure 2 as regards the endogenous fertility economy is chaotic.

Define the right-hand side of Eq. (27) as $f(k_t)$. As known, the standard version of the Li and Yorke's (1975) theorem predicts that a sufficient condition for the occurrence of deterministic chaos is $f(3) > k_0 > f(1) > f(2)$, where $f(.)$ indicates the number of iterates of the function f, i.e. period three implies *topological chaos* (for an example of an application of the Li and Yorke's theorem in a one-dimensional OLG economy see, amongst others, Zhang, 1999).

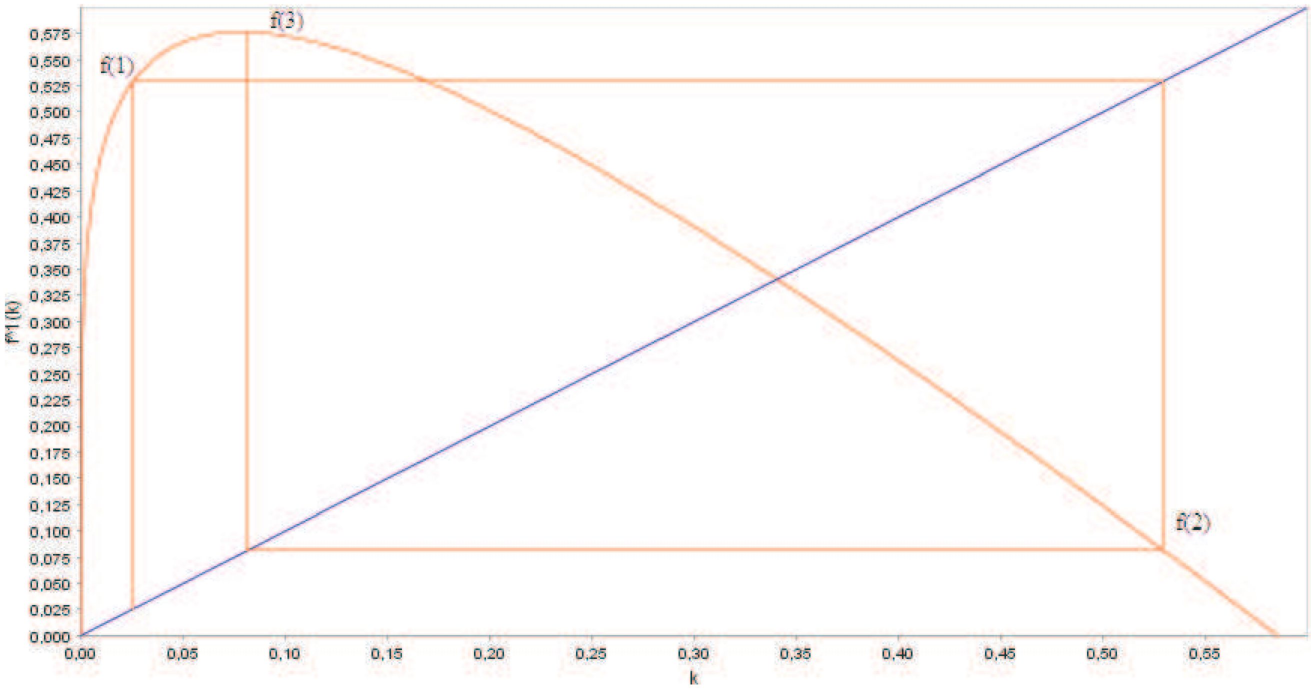


Figure 3. A pictorial view of three iterations that satisfy the Li and Yorke's (1975) theorem in the model with endogenous fertility.

Figure 3 displays three iterations of the phase map $f(k_t)$ that satisfy the Li and Yorke's theorem. Assume that the initial condition is $k_0 = 0.568$. Then we find $f(1) = 0.529$, $f(2) = 0.081$, $f(3) = 0.576$. Therefore, the non-monotonic dynamics of Figure 2 is chaotic and the following result holds.

Result 1. *Ceteris paribus, in an economy pay-as-you-go public pensions, the endogenisation of fertility rates gives rise to deterministic chaos.*

Result 1 may have interesting policy implications given the importance of fertility on economic growth and development (see, e.g., Becker, 1960; Becker and Barro, 1988: Barro and Becker, 1989; Becker et al., 1990). In fact, the more people tend to acquire a rational wisdom of the choice of the number of children to raise, and this especially regards developed countries, the more likely the financing of public PAYG pensions creates high risks of cyclical instability.

5. Conclusions

In this paper we analysed the dynamics of a Cobb-Douglas overlapping generations economy with public PAYG pensions by comparing exogenous and endogenous fertility settings. Despite large PAYG pensions may generate cyclical instability irrespective of whether the number of children is either exogenously given or endogenously determined, in the latter case the economy is much more prone to be exposed to endogenous fluctuations even if the stationary state is the same. This because with endogenous fertility individuals save less and the relative weight of the public pension component on capital accumulation is higher in that case, *ceteris paribus* as regards the key parameters of the problem. Therefore, the dynamic adjustment processes are different due to the endogenous fertility hypothesis, and when the steady-state equilibrium is stable the transitional rate of economic growth is lower in that case.

References

Abio, G., Mahieu, G., Patxot, C., 2004. On the optimality of PAYG pension systems in an endogenous fertility setting. Journal of Pension Economics and Finance 3, 35–62.

Azariadis, C., 1993. Intertemporal Macroeconomics, Blackwell Publishers.

Barro, R.J., Becker, G.S., 1989. Fertility choice in a model of economic growth. Econometrica 57, 481–501.

Becker, G.S., 1960. An economic analysis of fertility. In Demographic and Economic Change in Developing Countries. National Bureau Committee for Economic Research, Ed., Princeton, NJ: Princeton University Press.

Becker, G.S., Barro, R.J., 1988. A reformulation of the economic theory of fertility. Quarterly Journal of Economics 103, 1–25.

Becker, G.S., Murphy, K.M., Tamura, R., 1990. Human capital, fertility and growth. Journal of Political Economy 98, S12–S37.

Boldrin, M., Jones, L.E., 2002. Mortality, fertility and saving in a Malthusian economy. Review of Economic Dynamics 5, 775–814.

Deardorff, A.V., 1976. The optimum growth rate for population: Comment. International Economic Review 17 (2), 510–515.

Diamond, P., 1965. National debt in a neoclassical growth model. American Economic Review 55, 1126–1150.

Eckstein, Z., Wolpin, K.I., 1985. Endogenous fertility and optimal population size. Journal of Public Economics 27, 93–106.

Farmer, R.E., 1986. Deficits and cycles. Journal of Economic Theory 40, 77–86.

Fenge, R., Meier, V., 2005. Pensions and fertility incentives. Canadian Journal of Economics 38, 28–48.

Galor, O., Weil, D.N., 1996. The gender gap, fertility, and growth. American Economic Review 86, 374–387.

Grandmont, J.M., 1985. On endogenous business cycles. Econometrica 53, 995–1045.

Groezen, B. van, Meijdam, L., 2008. Growing old and staying young: population policy in an ageing closed economy. Journal of Population Economics 21, 573–588.

Groezen, B. van, Leers, T., Meijdam, L., 2003. Social security and endogenous fertility: pensions and child allowances as Siamese twins. Journal of Public Economics 87, 233–251.

Li, T.Y., Yorke, J.A., 1975. Period three implies chaos. American Mathematical Monthly 82, 985–992.

Michel, P., de La Croix, D., 2000. Myopic and perfect foresight in the OLG model. Economics Letters 67, 53–60.

Reichlin, P., 1986. Equilibrium cycles in an overlapping generations economy with production. Journal of Economic Theory 40, 89–102.

Samuelson, P.A., 1975. The optimum growth rate of population. International Economic Review 16 (3), 531–538.

Wagener, A., 2003. Equilibrium dynamics with different types of pay-as-you-go pension schemes. Economics Bulletin 8, 1–12.

Wigger, B.U., 1999. Pay-as-you-go financed public pensions in a model of endogenous growth and fertility. Journal of Population Economics 12, 625–640.

Zhang, J., 1999. Environmental sustainability, nonlinear dynamics and chaos. Economic Theory 14, 489–500.

Zhang, J., Zhang, J., 1998. Social Security, Intergenerational Transfers, and Endogenous Growth. Canadian Journal of Economics 31, 1225–1241.

Discussion Papers - Dipartimento Scienze Economiche – Università di Pisa

1. Luca Spataro, Social Security And Retirement Decisions In Italy, (luglio 2003)
2. Andrea Mario Lavezzi, Complex Dynamics in a Simple Model of Economic Specialization, (luglio2003)
3. Nicola Meccheri, Performance-related-pay nel pubblico impiego: un'analisi economica, (luglio 2003)
4. Paolo Mariti, The BC and AC Economics of the Firm, (luglio- dicembre 2003)
5. Pompeo Della Posta, Vecchie e nuove teorie delle aree monetarie ottimali, (luglio 2003)
6. Giuseppe Conti, Institutions locales et banques dans la formation et le développement des districts industriels en Italie, (luglio 2003)
7. F. Bulckaen - A. Pench - M. Stampini, Evaluating Tax Reforms without utility measures : the performance of Revenue Potentialities, (settembre 2003, revised June 2005)
8. Luciano Fanti - Piero Manfredi, The Solow’s model with endogenous population: a neoclassical growth cycle model (settembre 2003)
9. Piero Manfredi - Luciano Fanti, Cycles in dynamic economic modelling (settembre 2003)
10. Gaetano Alfredo Minerva, Location and Horizontal Differentiation under Duopoly with Marshallian Externalities (settembre 2003)
11. Luciano Fanti - Piero Manfredi, Progressive Income Taxation and Economic Cycles: a Multiplier-Accelerator Model (settembre 2003)
12. Pompeo Della Posta, Optimal Monetary Instruments and Policy Games Reconsidered (settembre 2003)
13. Davide Fiaschi - Pier Mario Pacini, Growth and coalition formation (settembre 2003)
14. Davide Fiaschi - Andre Mario Lavezzi, Nonlinear economic growth; some theory and cross-country evidence (settembre 2003)
15. Luciano Fanti , Fiscal policy and tax collection lags: stability, cycles and chaos (settembre 2003)
16. Rodolfo Signorino- Davide Fiaschi, Come scrivere un saggio scientifico:regole formali e consigli pratici (settembre 2003)
17. Luciano Fanti, The growth cycle and labour contract lenght (settembre 2003)
18. Davide Fiaschi , Fiscal Policy and Welfare in an Endogenous Growth Model with Heterogeneous Endowments (ottobre 2003)
19. Luciano Fanti, Notes on Keynesian models of recession and depression (ottobre 2003)
20. Luciano Fanti, Technological Diffusion and Cyclical Growth (ottobre 2003)
21. Luciano Fanti - Piero Manfredi, Neo-classical labour market dynamics, chaos and the Phillips Curve (ottobre 2003)
22. Luciano Fanti - Luca Spataro, Endogenous labour supply and Diamond's (1965) model: a reconsideration of the debt role (ottobre 2003)

23. Giuseppe Conti, Strategie di speculazione, di sopravvivenza e frodi bancarie prima della grande crisi (novembre 2003)

24. Alga D. Foschi, The maritime container transport structure in the Mediterranean and Italy (dicembre 2003)

25. Davide Fiaschi - Andrea Mario Lavezzi, On the Determinants of Growth Volatility: a Nonparametric Approach (dicembre 2003)

26. Alga D. Foschi, Industria portuale marittima e sviluppo economico negli Stati Uniti (dicembre 2003)

27. Giuseppe Conti - Alessandro Polsi, Elites bancarie durante il fascismo tra economia regolata ed autonomia (gennaio 2004)

28. Annetta Maria Binotti - Enrico Ghiani, Interpreting reduced form cointegrating vectors of incomplete systems. A labour market application (febbraio 2004)

29. Giuseppe Freni - Fausto Gozzi - Neri Salvadori, Existence of Optimal Strategies in linear Multisector Models (marzo 2004)

30. Paolo Mariti, Costi di transazione e sviluppi dell'economia d'impresa (giugno 2004)

31. Domenico Delli Gatti - Mauro Gallegati - Alberto Russo, Technological Innovation, Financial Fragility and Complex Dynamics (agosto 2004)

32. Francesco Drago, Redistributing opportunities in a job search model: the role of self-confidence and social norms (settembre 2004)

33. Paolo Di Martino, Was the Bank of England responsible for inflation during the Napoleonic wars (1897-1815)? Some preliminary evidence from old data and new econometric techniques (settembre 2004)

34. Luciano Fanti, Neo-classical labour market dynamics and uniform expectations: chaos and the "resurrection" of the Phillips Curve (settembre 2004)

35. Luciano Fanti – Luca Spataro, Welfare implications of national debt in a OLG model with endogenous fertility (settembre 2004)

36. Luciano Fanti – Luca Spataro, The optimal fiscal policy in a OLG model with endogenous fertility (settembre 2004)

37. Piero Manfredi – Luciano Fanti, Age distribution and age heterogeneities in economic profiles as sources of conflict between efficiency and equity in the Solow-Stiglitz framework (settembre 2004)

38. Luciano Fanti – Luca Spataro, Dynamic inefficiency, public debt and endogenous fertility (settembre 2004)

39. Luciano Fanti – Luca Spataro, Economic growth, poverty traps and intergenerational transfers (ottobre 2004)

40. Gaetano Alfredo Minerva, How Do Cost (or Demand) Asymmetries and Competitive Pressure Shape Trade Patterns and Location? (ottobre 2004)

41. Nicola Meccheri, Wages Behaviour and Unemployment in Keynes and New Keynesians Views. A Comparison (ottobre 2004)

42. Andrea Mario Lavezzi - Nicola Meccheri, Job Contact Networks, Inequality and Aggregate Output (ottobre 2004)

43. Lorenzo Corsini - Marco Guerrazzi, Searching for Long Run Equilibrium Relationships in the Italian Labour Market: a Cointegrated VAR Approach (ottobre 2004)

44. Fabrizio Bulckaen - Marco Stampini, Commodity Tax Reforms In A Many Consumers Economy: A Viable Decision-Making Procedure (novembre 2004)

45. Luzzati T. - Franco A. (2004), “Idrogeno fonti rinnovabili ed eco-efficienza: quale approccio alla questione energetica?”

46. Alga D. Foschi , “The coast port industry in the U.S.A: a key factor in the process of economic growth” (dicembre 2004)

47. Alga D. Foschi , “A cost – transit time choice model: monomodality vs. intermodality” (dicembre 2004)

48. Alga D. Foschi , “Politiques communautaires de soutien au short sea shipping (SSS)” (dicembre 2004)

49. Marco Guerrazzi, Intertemporal Preferences, Distributive Shares, and Local Dynamics (dicembre 2004)

50. Valeria Pinchera, “Consumo d’arte a Firenze in età moderna. Le collezioni Martelli, Riccardi e Salviati nel XVII e XVIII secolo” (dicembre 2004)

51. Carlo Casarosa e Luca Spataro, “Propensione aggregata al risparmio, rapporto ricchezza-reddito e distribuzione della ricchezza nel modello del ciclo di vita "egualitario": il ruolo delle variabili demografiche” (aprile 2005)

52. Alga D. Foschi – Xavier Peraldi – Michel Rombaldi, “Inter – island links in Mediterranean Short Sea Shipping Networks” (aprile 2005)

53. Alga D. Foschi (2005), “Lo shipping, la cantieristica ed i porti nell’industria marittima” (aprile 2005)

54. Marco Guerrazzi, “Notes on Continuous Dynamic Models: the Benhabib-Farmer Condition for Indeterminacy” (settembre 2005)

55. Annetta Binotti e Enrico Ghiani, "Changes of the aggregate supply conditions in Italy: a small econometric model of wages and prices dynamics" (settembre 2005)

56. Tommaso Luzzati, “Leggere Karl William Kapp (1910-1976) per una visione unitaria di economia, società e ambiente” (dicembre 2005)

57. Lorenzo Corsini (2006), “Firm's Entry, Imperfect Competition and Regulation”

58. Mario Morroni (2006), “Complementarities among capability, transaction and scale-scope considerations in determining organisational boundaries”

59. Mario Morroni (2006), “Innovative activity, substantive uncertainty and the theory of the firm”

60. Akos Dombi (2006), "Scale Effects in Idea-Based Growth Models: a Critical Survey"

61. Binotti Annetta Maria e Ghiani Enrico (2006), “La politica economica di breve periodo e lo sviluppo dei primi modelli mocroeconometrici in Italia: dalla vicenda ciclica degli anni ’60 alla prima crisi petrolifera”

62. Fioroni Tamara (2006), “Life Expectancy, Health Spending and Saving”

63. Alga D. Foschi (2006), “La concentrazione industriale per i sistemi di trasporto sostenibile: un caso di successo nel Mediterraneo orientale”

64. Alga D. Foschi (2006), “La concentrazione industriale per i sistemi di trasporto sostenibile”

65. Maurizio Lisciandra (2007), “The Role of Reciprocating Behaviour in Contract Choice”

66. Luciano Fanti e Luca Spataro (2007), “Poverty traps and intergenerational transfers”

67. Luciano Fanti and Luca Spataro (2007), "Neoclassical OLG growth and underdeveloped, developing and developed countries"

68. Luciano Fanti and Luca Gori (2007), "Economic growth and welfare in a simple neoclassical OLG model with minimum wage and consumption taxes"

69. Carlo Brambilla and Giandomenico Piluso (2008), "Italian investment and merchant banking up to 1914: Hybridising international models and practices"

70. Luciano Fanti and Luca Gori (2008), "Fertility and regulated wages in an OLG model of neoclassical growth: Pensions and old age support"

71. Luciano Fanti and Luca Gori (2008), "Neoclassical economic growth and lifetime welfare in a simple OLG model with unions"

72. Nicola Meccheri (2008), "A note on noncompetes, bargaining and training by firms"

73. Lorenzo Corsini e Elisabetta Olivieri (2008), "Technological Change and the Wage Differential between Skilled and Unskilled Workers: Evidence from Italy"

74. Luciano Fanti e Luca Gori (2008), "'Backyard' technology and regulated wages in a neoclassical OLG growth model"