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PAYG pensions and economic cycles: exogenous versus endogenous fertility

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Abstract We analyse the dynamics of an overlapping generations economy with unfunded pay-as-you-go public pensions and myopic expectations by comparing exogenous and endogenous fertility contexts. It is shown that large PAYG pensions may cause endogenous fluctuations in both cases. However, ceteris paribus, when the number of children is an economic decision variable the economy is much more prone to cyclical instability and deterministic chaos emerges.

Keywords Myopic foresight; PAYG pension; Stability; OLG model

JEL Classification C62; H55; J14; J18; J26

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1. Introduction

One of the most fascinating results in the overlapping generations (OLG) literature is the possibility of endogenous fluctuations (Grandmont, 1985; Farmer, 1986; Reichlin, 1986; Azariadis, 1993). Typically, however, cyclical behaviour emerges at low values of the elasticity of substitution in production, in particular for values well below unity. In particular, Farmer (1986) showed, for the CES case, that Hopf bifurcations may occur only when technologies exhibit lower factor substitutability than the Cobb-Douglas function. On the other hand, with myopic foresight, the steady state equilibrium may be oscillatory and even exhibit deterministic complex cycles (Michel and de la Croix, 2000, de la Croix and Michel, 2002; Fanti and Spataro, 2008) when the inter-temporal elasticity of substitution is higher than that of the Cobb-Douglas function.

Therefore, although it is well known that overlapping generations economies with myopic expectations and elasticity of substitution in production and in the utility function relatively low and high respectively (and then necessarily different from that of the Cobb-Douglas functions), may show cyclical (and even chaotic) dynamics, in this paper we show, in the standard case of Cobb-Douglas functions, that the size of pay-as-you-go (PAYG) pensions plays a crucial role in determining the stability of the economy and, in particular, large PAYG pensions causes the appearance of persistent cycles and even deterministic chaos may also emerge when individuals are myopic foresighted.¹

Moreover, the importance of fertility in determining the macroeconomic performances is well established at least starting from the seminal paper by Becker (1960). Therefore, in accord with the recent body of literature arguing that fertility in an economic choice variable rather than an

¹ The debate on the role of PAYG pensions in OLG economies in presence of demographic changes has recently increased due problems of sustainability as well as for optimality purposes (see, e.g., van Groezen et al., 2003; Abio et al., 2004; van Groezen and Meijdam, 2008). However, less attention has been paid to the role of PAYG pensions and endogenous demographic changes on stability, which is the object of this paper.
exogenous event (Becker and Barro, 1988; Barro and Becker, 1989; Becker et al., 1990), we also consider endogenous fertility and compare the dynamical features of this setting with those obtained in the standard exogenous fertility economy. However, since costs of children undoubtedly exist irrespective of whether the size of the family is either based on comparison of economic constraints and incentives or is exogenously given (due, for instance, to religion belief, customs and so on), we assume that child bearing is costly even when fertility is not an economic decision variable. In this paper, therefore, we analyse the dynamical properties of an economy with endogenous fertility and public PAYG pensions and compare them with those obtained in the case of exogenous fertility.

Our results reveal that while the same unique steady state is achieved, the dynamic adjustment processes towards the steady state are dramatically different according to whether or not fertility is endogenously chosen by individuals, and this different dynamic behaviour is due to the presence of public PAYG pensions. Therefore, whether the number of children is the result of a rational choice of individuals rather than an exogenous event significantly matters for stability. In particular, when fertility is endogenously determined, the destabilising effect of PAYG pensions is stronger than when it is exogenously given. The economic reason for this result is the following: the negative impact of public pensions on savings is mitigated by the effect of the individual degree of parsimony in both cases of exogenous and endogenous fertility (see Eqs. 7 and 24 in the sequel of the paper). However, when fertility is endogenous, the degree of parsimony has a negative impact effect also on the demand for children. Therefore, as regards capital accumulation per worker, which is the ratio between savings and fertility, we observe that when fertility is endogenous, a rise in the individual degree of thriftiness negatively affects in the same way savings (through the public pension component) and the number of children, so that both effects exactly cancel each other out on capital accumulation. The final effect, therefore, is that the negative impact effect of public pensions is mitigated by the effect of the individual degree of parsimony.

Note that in the absence of PAYG pensions the dynamic behaviour is the same with exogenous and endogenous fertility.

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pensions on capital accumulation is stronger with endogenous fertility than with exogenous fertility, and thus the destabilising effects of PAYG pensions is more likely to occur in the former case.

The essential message of this paper consists in showing that different dynamic adjustment processes emerge in OLG models with PAYG pensions, depending on whether the number of children is either exogenous or endogenous to the model. Therefore, we argue that to the extent that fertility choices are based on an economic comparison between benefits and costs of children, which is considered going hand to hand with the stages of economic development\(^3\) – as suggested by the home economics literature (see Becker, 1960) –, the more likely large PAYG pensions plays dramatic effects on the stability of the economy.

This paper contributes to three strands of literature: (i) the business cycles analysis framed in the OLG context; (ii) the endogenous population economics, and (iii) the economics with public PAYG pensions, and shows that considering together all these aspects gives rise to some new results which may have interesting policy implications.

The remainder of the paper is organised as follows. In Section 2 (Section 3) we analyse the local stability properties of an OLG economy with exogenous (endogenous) fertility. In Section 4 we show that the dynamic adjustment processes towards the stationary state are different in exogenous and endogenous fertility economies despite the long-run equilibrium is the same, showing, in particular, that the risk of cyclical instability due to large PAYG pensions is higher with endogenous fertility. Section 5 concludes.

2. Exogenous fertility

\(^3\) In fact, as argued by van Groezen et al., (2003, p. 237) “…at least in western countries, people are to a very large extent able to freely choose the number of children they desire. The rate of fertility should therefore be treated as an endogenous variable, that is, as the result of a rational choice which is influenced by economic constraints and incentives. Economic theory can thus help in explaining why the observed decline in the (desired) number of children would occur.”
2.1. Firms

Firms are identical and act competitively on the market. Aggregate production at time $t$ ($Y_t$) takes place according to the Cobb-Douglas technology $Y_t = AK_t^a L_t^{1-a}$, where $K_t$ and $L_t = N_t$ are capital and labour hired by the typical firm, $N_t$ is the number of young workers in the economy, $A > 0$ is a scale parameter and $0 < \alpha < 1$ the output elasticity of capital. Assuming that capital totally depreciates at the end of each period and a unit price of final output, profit maximisation implies that factor inputs are paid their marginal products, that is:

$$r_t = \alpha A k_t^{a-1} - 1, \quad \text{(1)}$$

$$w_t = (1-\alpha) A k_t^a, \quad \text{(2)}$$

where $k_t := K_t / N_t$ is capital per worker, $r_t$ is the interest rate and $w_t$ the wage rate.

2.2. Government

The government redistributes from the young to the old with unfunded public pensions. Therefore, at time $t$ the public pension budget in per worker terms reads as:

$$p_t = \theta w_t n, \quad \text{(3)}$$

the left-hand side being the social security expenditure and the right-hand side the tax receipts, where $p_t > 0$ is the pension benefit, $0 < \theta < 1$ is the fixed contribution rate and $n$ is the exogenous number of children in the whole economy.

2.3. Individuals
Young population of measure $N_t$ grows at a constant rate $n$ and agents are assumed to belong to an overlapping generations economy where life is divided between youth (working period) and old-age (retirement period), as in Diamond (1965).

Only young individuals join the workforce and supply inelastically their whole time endowment (normalised to unity) on the labour market, while receiving a unitary wage income at the competitive rate $w_t$, which is used to consume, to save, to bear children and to finance material consumption of the elderly through the public pension scheme Eq. (3).

Starting from the seminal papers by Diamond (1965), Samuelson (1975) and Deardorff (1976), the economic literature did not consider children costs in models with exogenous fertility. However, it is natural to conjecture that child-rearing activities are costly (either time-based or income-based, or both) even when fertility is not an economic decision variable. In particular, in this paper we assume that the amount of resources needed to care about children is given by a monetary cost $qw_t$ per child, where $0 < q < 1$ is the percentage of child rearing cost on the parents’ working income.\(^4\)

Old individuals are retired and live on (i) the amount of resources saved when young ($s_t$) plus the expected interest accrued at the rate $r_{t+1}$, and (ii) the expected pension benefit $p_{t+1}$.

The representative individual born at $t$ has preferences towards young-aged consumption ($c_{1,t}$) and old-aged consumption ($c_{2,t+1}$), and chooses how much to save out of disposable income in order to maximise the lifetime utility function $U_t$ subject to each period budget constraint, that is

$$\max_{\{s_t\}} U_t = \ln(c_{1,t}) + \beta \ln(c_{2,t+1}),$$  \hspace{1cm} (4)

subject to

$$c_{1,t} + s_t + qw_t n = w_t (1 - \theta)$$

$$c_{2,t+1} = (1 + r_{t+1})s_t + p_{t+1}.$$ \hspace{1cm} (5)

\(^4\) Note that this child cost structure is similar to that adopted, amongst many other, by Wigger (1999) and Boldrin and Jones (2002).
where \(0 < \beta < 1\) is the subjective discount factor.

The saving rate, therefore, is

\[
s_t = \frac{\beta w_t (1 - \theta - qn)}{1 + \beta} - \frac{p_{t+1}'}{(1 + \beta)(1 + r_{t+1}')},
\]

where \(q < (1 - \theta)/n\) must hold to guarantee Eq. (6) to be positive. Now, inserting the one-period-forward pension accounting rule Eq. (3) into Eq. (6) we get:

\[
s_t = \frac{\beta w_t (1 - \theta - qn)}{1 + \beta} - \frac{n\theta}{1 + \beta} \frac{w_{t+1}'}{1 + r_{t+1}'},
\]

From Eq. (7) we see that savings are divided into two components: the \textit{private saving component} (the first term on the right-hand side of Eq. 7) and the \textit{public pension component} (the second term on the right-hand side of Eq. 7).

2.4. Equilibrium

Given the government budget Eq. (3) and knowing that \(N_{t+1} = nN_t\), market-clearing in goods and capital markets is expressed by the equality:

\[
nk_{t+1} = s_t.
\]

Using Eqs. (7) and (8) equilibrium implies:

\[
k_{t+1} = \frac{\beta w_t (1 - \theta - qn)}{(1 + \beta)n} - \frac{\theta}{1 + \beta} \frac{w_{t+1}'}{1 + r_{t+1}'},
\]

The dynamic evolution of capital is different depending on whether individuals are perfect or short sighted. In particular, while in the case of rational expectations the economy does not exhibit any interesting dynamical feature (i.e. the steady state is locally stable and the dynamics is always monotonic), in the case of myopic expectations the dynamics may be non-monotonic and cyclical instability may emerge when the public pension component dominates the private saving component, and this depends on the size of the PAYG scheme as well as on the mutual relationship...
between preference and technological parameters. Therefore, below we concentrate on the case of short-sighted agents to study the complex dynamical properties of this simple economy.

2.5. Exogenous fertility: local stability with myopic foresight

With myopic expectations both the expected interest and wage rates depend on the current value of the stock of capital per worker, that is

\[
\begin{align*}
1 + r^e_{t+1} &= \alpha A k_t^{\alpha - 1} \\
W^e_{t+1} &= (1 - \alpha)A k_t^\alpha.
\end{align*}
\] (10)

Therefore, exploiting Eqs. (1), (2), (9) and (10) the dynamic path of capital accumulation is given by the following non-linear difference equation:

\[
k_{t+1} = \frac{\theta (1 - \theta - qn)(1 - \alpha)A_k}{(1 + \beta)n} k_t^\alpha - \frac{\theta}{1 + \beta} \frac{1 - \alpha}{\alpha} k_t.
\] (11)

Steady state implies \( k_{t+1} = k_t = k^* \). Therefore, the steady state stock of capital (which is the same with both perfect and myopic expectations, see Michel and de la Croix, 2000) is

\[
k^* = \left[ \frac{\beta \alpha (1 - \alpha)A(1 - \theta - qn)}{n(\alpha(1 + \beta) + \theta(1 - \alpha))} \right]^{\frac{1}{1-\alpha}}.
\] (12)

The analysis of Eqs. (11) and (12) gives the following proposition:

**Proposition 1.** The dynamics of an OLG economy with exogenous fertility, PAYG pensions and myopic foresighted individuals is the following.

(1) Let \( 0 < \alpha < \alpha_4 \) hold. Then \( \theta_1 < \theta_2 < 1 \), and

(1.1) if \( 0 < \theta < \theta_1 \), the dynamics of capital is monotonic and convergent to \( k^* \);

(1.2) if \( \theta_1 < \theta < \theta_2 \), the dynamics of capital is oscillatory and convergent to \( k^* \);
(1.3) If \( \theta = \theta_2 \), a flip bifurcation emerges;

(1.4) If \( \theta_2 < \theta < 1 \), the dynamics of capital is oscillatory and divergent to \( k^\ast \).

(2) Let \( \alpha < \alpha_2 \) hold. Then \( \theta_1 < 1 \), \( \theta_2 > 1 \), and

(2.1) If \( 0 < \theta < \theta_2 \), the dynamics of capital is monotonic and convergent to \( k^\ast \);

(2.2) If \( \theta_1 < \theta < 1 \), the dynamics of capital is oscillatory and convergent to \( k^\ast \).

(3) Let \( \alpha_2 < \alpha < 1 \) hold. Then \( \theta_2 > \theta_1 > 1 \) and the dynamics of capital is monotonic and convergent to \( k^\ast \) for any \( 0 < \theta < 1 \),

where

\[
\theta_1 = \theta_1(\alpha, \beta) := \frac{\alpha^2(1 + \beta)}{(1 - \alpha)^2},
\]

\[
\theta_2 = \theta_2(\alpha, \beta) := \frac{\alpha(1 + \alpha)(1 + \beta)}{(1 - \alpha)^2} = \theta_1 \frac{1 + \alpha}{\alpha},
\]

\[
\alpha_2 = \alpha_2(\beta) := -\frac{1 + \sqrt{1 + \beta^2}}{\beta} > 0, \quad 0 < \alpha_2 < 1/2,
\]

\[
\alpha_4 = \alpha_4(\beta) := \frac{(3 + \beta) + \sqrt{\beta^2 + 10\beta + 9}}{2\beta} > 0, \quad 0 < \alpha_4 < 1/3.
\]

**Proof.** Differentiating Eq. (11) with respect to \( k \), and using Eq. (12) we get:

\[
\frac{\partial k_{t+1}}{\partial k_t} \bigg|_{k_t = k} = \alpha - \frac{\theta}{1 + \beta} \cdot \frac{(1 - \alpha)^2}{\alpha}.
\]

Monotonic and non-monotonic dynamics with exogenous fertility
From Eq. (17), the condition \( \left. \frac{\partial k_{t+1}}{\partial k_t} \right|_{k_t = \bar{k}} < 0 \) implies

\[
\alpha - \frac{\theta}{1 + \beta} \cdot \frac{(1 - \alpha)^2}{\alpha} > 0 \iff \theta > \theta_1, \tag{18}
\]

so that \( \theta_1 < 1 \) (\( \theta_1 > 1 \)) for any \( 0 < \alpha < \alpha_2 \) (\( \alpha_2 < \alpha < 1 \)). Moreover, \( \theta_1 < 1 \) if and only if \( \alpha_1 < \alpha < \alpha_2 \), where \( \alpha_i = \alpha_i(\beta) := \frac{-1 - \sqrt{1 + \beta}}{\beta} < 0 \) and \( 0 < \alpha_2 < 1/2 \) is given by Eq. (15). Since \( \alpha_1 < 0 \) it is ruled out.

Now, \( \left. \frac{\partial k_{t+1}}{\partial k_t} \right|_{k_t = \bar{k}} < 1 \) gives

\[
\alpha - \frac{\theta}{1 + \beta} \cdot \frac{(1 - \alpha)^2}{\alpha} < 1 \iff \theta > \frac{\alpha(1 + \beta)}{1 - \alpha}. \tag{19}
\]

Therefore, in the case of monotonic behaviour, \( 0 < \left. \frac{\partial k_{t+1}}{\partial k_t} \right|_{k_t = \bar{k}} < 1 \) and thus the dynamics of capital is always convergent to the steady state.

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The condition \( \left. \frac{\partial k_{t+1}}{\partial k_t} \right|_{k_t = \bar{k}} > -1 \) implies

\[
\alpha - \frac{\theta}{1 + \beta} \cdot \frac{(1 - \alpha)^2}{\alpha} > -1 \iff \theta < \theta_2, \tag{20}
\]

where \( \theta_2 > \theta_1 \), so that \( \theta_2 < 1 \) (\( \theta_2 > 1 \)) for any \( 0 < \alpha < \alpha_4 \) (\( \alpha_4 < \alpha < 1 \)), with \( \alpha_4 < \alpha_2 \). Moreover, \( \theta_2 < 1 \) if and only if \( \alpha_3 < \alpha < \alpha_4 \), where \( \alpha_3 = \alpha_3(\beta) := \frac{-\left(3 + \beta\right) - \sqrt{\beta^2 + 10\beta + 9}}{2\beta} < 0 \) and \( 0 < \alpha_4 < 1/3 \) is given by Eq. (16). Since \( \alpha_3 < 0 \) it is ruled out.

Therefore,
(i) if \( 0 < \alpha < \alpha_4 \) then \( \theta_1 < \theta_2 < 1 \) and (1.1) \( 0 < \frac{\partial k_{t+1}}{\partial k_t} \bigg|_{k=k^*} < 1 \) for any \( 0 < \theta < \theta_4 \), (1.2)

\[-1 < \frac{\partial k_{t+1}}{\partial k_t} \bigg|_{k=k^*} < 0 \text{ for any } \theta_1 < \theta < \theta_2, \] (1.3) \( \frac{\partial k_{t+1}}{\partial k_t} \bigg|_{k=k^*} = -1 \) if and only if \( \theta = \theta_2 \), and (1.4)

\[\frac{\partial k_{t+1}}{\partial k_t} \bigg|_{k=k^*} < -1 \text{ for any } \theta_2 < \theta < 1. \] This proves point (1);

(ii) if \( \alpha_4 < \alpha < \alpha_2 \) then \( \theta_1 < 1, \theta_2 > 1 \) and (2.1) \( 0 < \frac{\partial k_{t+1}}{\partial k_t} \bigg|_{k=k^*} < 1 \) for any \( 0 < \theta < \theta_4 \) and

\[-1 < \frac{\partial k_{t+1}}{\partial k_t} \bigg|_{k=k^*} < 0 \text{ for any } \theta_1 < \theta < 1. \] This proves point (2);

(iii) if \( \alpha_2 < \alpha < 1 \) then \( \theta_2 > \theta_1 > 1 \) and 0 \( < \frac{\partial k_{t+1}}{\partial k_t} \bigg|_{k=k^*} < 1 \) for any \( 0 < \theta < 1. \) This proves point (3). Q.E.D.

3. Endogenous fertility

In this section we assume that individuals also draw utility from the number of children they wish to raise and, hence, choose fertility on the basis of a rational comparison between benefits and costs of children (i.e., the so-called weak form of altruism, see Zhang and Zhang, 1998). The model is outlined below.

3.1. Government

The public pension budget is now

\[p_t = \theta w_n_t, \] (21)
where \( n_t \) is the endogenous number of children at time \( t \).

### 3.2. Individuals

Individuals draw utility from material consumption over the life cycle and the number of children, see Eckstein and Wolpin (1985) and Galor and Weil (1996). The representative individual entering the working period at \( t \) solves the following problem:

\[
\max_{\{t_n, c_n\}} U_t(c_{1,t}, c_{2,t+1}, n_t) = \ln(c_{1,t}) + \beta \ln(c_{2,t+1}) + \phi \ln(n_t),
\]

subject to Eqs. (5), where \( \phi > 0 \) captures the parents’ taste for children.

The constrained maximisation of Eq. (22) gives (upon substitution of the one-period forward pension accounting rule Eq. 21 for \( p_{t+1} \)) the demand for children and the saving rate, respectively:

\[
n_t = \frac{\phi w_t (1 - \theta)}{(1 + \beta + \phi)q w_t - \phi \theta \frac{w_{t+1}}{1 + r_{t+1}}},
\]

\[
s_t = \frac{w_t (1 - \theta)}{(1 + \beta + \phi)q w_t - \phi \theta \frac{w_{t+1}}{1 + r_{t+1}}} \left( \beta q w_t - \phi \theta \frac{w_{t+1}}{1 + r_{t+1}} \right).
\]

### 3.3. Equilibrium

Given the government budget Eq. (21) and knowing that \( N_{t+1} = n_t N_t \), equilibrium in goods and capital market is

\[
n_t k_{t+1} = s_t.
\]

Using Eqs. (23)-(25) the equilibrium condition reads as:

\[
k_{t+1} = \frac{\beta}{\phi} q w_t - \theta \frac{w_{t+1}}{1 + r_{t+1}}.
\]
3.4. Endogenous fertility: local stability with myopic foresight

Exploiting Eqs. (1), (2), (10) and (26) the dynamic path of capital accumulation with short-sighted agents and endogenous fertility is given by the following non-linear difference equation:

\[ k_{t+1} = \frac{\beta}{\phi} q(1-\alpha)Ak_t^\alpha - \theta \frac{1-\alpha}{\alpha} k_t. \] (27)

Steady state implies

\[ k^* = \left\{ \frac{q\beta \alpha(1-\alpha)A}{\phi[\alpha + \theta(1-\alpha)]} \right\}^{\frac{1}{1-\theta}}. \] (28)

The analysis of Eqs. (27) and (28) gives the following proposition:

**Proposition 2.** The dynamics of an OLG economy with endogenous fertility, PAYG pensions and myopic foresighted individuals is the following.

1. Let \( 0 < \alpha < 1/3 \) hold. Then \( \theta_3 < \theta < 1 \), and
   
   (1.1) if \( 0 < \theta < \theta_3 \), the dynamics of capital is monotonic and convergent to \( k^* \);

   (1.2) if \( \theta_3 < \theta < \theta_4 \), the dynamics of capital is oscillatory and convergent to \( k^* \);

   (1.3) if \( \theta = \theta_4 \), a flip bifurcation emerges;

   (1.4) if \( \theta_4 < \theta < 1 \), the dynamics of capital is oscillatory and divergent to \( k^* \).

2. Let \( 1/3 < \alpha < 1/2 \) hold. Then \( \theta_3 < \theta < 1 \), and
   
   (2.1) if \( 0 < \theta < \theta_3 \), the dynamics of capital is monotonic and convergent to \( k^* \);

   (2.2) if \( \theta_3 < \theta < 1 \), the dynamics of capital is oscillatory and convergent to \( k^* \).
(3) Let $1/2 < \alpha < 1$ hold. Then $\theta_3 > \theta_4 > 1$, and the dynamics of capital is monotonic and convergent to $k^*$ for any $0 < \theta < 1$.

where

$$\theta_3 = \theta_3(\alpha) := \frac{\alpha^2}{(1-\alpha)^2},$$  \hspace{1cm} (29)

$$\theta_4 = \theta_4(\alpha) := \frac{\alpha(1+\alpha)}{(1-\alpha)^2} = \theta_1 \frac{1+\alpha}{\alpha}. \hspace{1cm} (30)$$

**Proof.** Differentiating Eq. (27) with respect to $k_t$ and using Eq. (28) gives:

$$\left. \frac{\partial k_{t+1}}{\partial k_t} \right|_{k=k^*} = \alpha - \frac{1-\alpha}{\alpha}. \hspace{1cm} (31)$$

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Therefore, $\left. \frac{\partial k_{t+1}}{\partial k_t} \right|_{k=k^*} > 0$ yields

$$\alpha - \theta \frac{(1-\alpha)^2}{\alpha} > 0 \Leftrightarrow \theta < \theta_3,$$  \hspace{1cm} (32)

so that $\theta_3 < 1$ ($\theta_4 > 1$) for any $0 < \alpha < 1/2$ ($1/2 < \alpha < 1$).

Now, $\left. \frac{\partial k_{t+1}}{\partial k_t} \right|_{k=k^*} < 1$ yields

$$\alpha - \theta \frac{(1-\alpha)^2}{\alpha} < 1 \Rightarrow \theta > \frac{\alpha}{1-\alpha}. \hspace{1cm} (33)$$
Therefore, in the case of monotonic behaviour, \(0 < \frac{\partial k_{t+1}}{\partial k_t} \bigg|_{k'=k} < 1\) and the dynamics of capital is always convergent to the steady state.

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Moreover, \(\frac{\partial k_{t+1}}{\partial k_t} \bigg|_{k'=k'} > -1\) yields

\[
\alpha - \theta \left(1-\alpha\right)^2 \frac{\alpha}{\alpha} < -1 \Rightarrow \theta < \theta_4,
\]

where \(\theta_4 > \theta_3\), so that \(\theta_4 < 1\) for any \(0 < \alpha < 1/3\) and \((1/3 < \alpha < 1)\).

Therefore,

(i) if \(0 < \alpha < 1/3\) then \(\theta_3 < \theta < 1\) and \((1.1)\) \(0 < \frac{\partial k_{t+1}}{\partial k_t} \bigg|_{k'=k'} < 1\) for any \(0 < \theta < \theta_3\), \((1.2)\)

\(-1 < \frac{\partial k_{t+1}}{\partial k_t} \bigg|_{k'=k'} < 0\) for any \(\theta_3 < \theta < \theta_4\), \((1.3)\) \(\frac{\partial k_{t+1}}{\partial k_t} \bigg|_{k'=k'} = -1\) if and only if \(\theta = \theta_4\), and \((1.4)\)

\(\frac{\partial k_{t+1}}{\partial k_t} \bigg|_{k'=k'} < -1\) for any \(\theta_4 < \theta < 1\). This proves point (1);

(ii) if \(1/3 < \alpha < 1/2\) then \(\theta_3 < 1\), \(\theta_4 > 1\) and \((2.1)\) \(0 < \frac{\partial k_{t+1}}{\partial k_t} \bigg|_{k'=k'} < 1\) for any \(0 < \theta < \theta_3\) and

\(2.2) -1 < \frac{\partial k_{t+1}}{\partial k_t} \bigg|_{k'=k'} < 0\) for any \(\theta_3 < \theta < 1\). This proves point (2);

(iii) if \(1/2 < \alpha < 1\) then \(\theta_4 > \theta_3 > 1\) and \(0 < \frac{\partial k_{t+1}}{\partial k_t} \bigg|_{k'=k'} < 1\) for any \(0 < \theta < 1\). This proves point (3). Q.E.D.
4. Exogenous versus endogenous fertility with myopic expectations

In order to compare exogenous and endogenous fertility economies we now assume that the same number of children is raised in both contexts. We find that when PAYG pensions exist, the same unique positive steady-state is achieved but with two different dynamic adjustment processes.\(^5\)

Moreover, the risk of cyclical instability due to large PAYG pensions is higher with endogenous fertility.

**Proposition 3.** If the number of children is the same in both exogenous and endogenous fertility economies, then the same unique positive steady state is achieved.

**Proof.** Assume that in the long run fertility is the same in both contexts, that is \(n = n^*\). Then

\[
n = \frac{\phi(1-\alpha)A(k^*)^\alpha (1-\theta)}{(1+\beta+\phi)q(1-\alpha)A(k^*)^\alpha - \phi \theta \frac{1-\alpha}{\alpha} k^*}. \tag{35}
\]

Substituting Eq. (35) into Eq. (12) to eliminate \(n\) and solving for \(k^*\) we get Eq. (28). Q.E.D.

**Proposition 4.** (1) The existence of public PAYG pensions causes a higher risk of cyclical instability when fertility is endogenously determined rather than when it is an exogenous event. Moreover, (2) while in an economy with endogenous fertility the individual degree of thriftiness is stability-neutral, when fertility is exogenous it acts as an economic stabiliser.

\(^5\) Of course, when PAYG pensions are absent and the number of children is the same in both exogenous and endogenous fertility economies, the dynamic paths of capital accumulation and the steady states stock of capital in these scenarios coincide.
Proof. When fertility is exogenous (endogenous), cyclical instability emerges only in the case $0 < \alpha < \alpha_2$ when $\theta_2 < \theta < 1$ ($0 < \alpha < 1/3$ when $\theta_4 < \theta < 1$). Since $\alpha_4 < 1/3$ and $\theta_4 < \theta_2 := \theta_4(1 + \beta)$. This proves point (1). Moreover, $\frac{\partial \theta_2}{\partial \beta} = \theta_4 > 0$ and $\frac{\partial \theta_4}{\partial \beta} = 0$. This proves point (2). Q.E.D.

Comparison of Eqs. (11) and (27) makes clear the reason why cyclical instability more likely emerges when individuals choose endogenously the number of children. When fertility is exogenous, in fact, a rise in the individual degree of thriftiness reduces the relative weight of the public pension component on savings and capital accumulation (see Eq. 11) and this, therefore, tends to mitigate the negative impact effect of a higher contribution rate on the accumulation of capital. In contrast, when fertility is endogenous, the subjective discount factor does not affect the public pension component (see Eq. 27) because it negatively affects in the same way both savings and the demand for children and, hence it is neutral on the public pension component of the dynamic path of capital accumulation. Therefore, the cyclical unstable region is larger than with endogenous fertility.

To illustrate the different dynamic adjustment processes in both exogenous and endogenous fertility economies with PAYG pensions, we take the following parameter set: $A = 10$, $\alpha = 0.20$, $\beta = 0.40$ and $k_0 = 0.568$ (the initial condition) for both economies. For simplicity, we assume stationary population in the model with exogenous fertility, i.e. $n = 1$. Then we choose $\phi = 0.30$ and calibrate $q = 0.11243$ to obtain $n^* = 1$ in the model with endogenous fertility. This parameter set generates the following flip bifurcation values of the contribution rate: $\theta_2 = 0.525$ (exogenous fertility) and $\theta_4 = 0.375$ (endogenous fertility). Then we assume $\theta = 0.46$. Despite the steady state stock of capital is the same in both economies, $k^* = 0.3404$, the dynamic adjustment processes are dramatically different. In particular, endogenous fertility is a source of cyclical instability.
The cobweb depicted in Figure 1 shows the stable non-monotonic trajectory towards the stationary state when fertility is exogenous. In contrast, Figure 2 shows the unstable non-monotonic trajectory in the case of endogenous fertility. The difference is explained by the stabilising effect exerted by the subjective discount factor in reducing the relative weight of the public pension component in the PAYG-exogenous-fertility economy, which is instead absent in the PAYG-endogenous-fertility economy. In fact, when fertility is an economic decision variable, individuals accumulate less capital because the negative effect of PAYG pensions on the public pension component of the phase map Eq. (27) are not mitigated by the coefficient $\beta$, and this also causes a reduction in the transitional rate of economic growth. Therefore, to the extent that the steady state equilibrium is stable, the capital stock more rapidly approaches its stationary state when fertility is exogenously given than when it is endogenously determined to the model.

![Diagram](image)

**Figure 1.** (Exogenous fertility). A pictorial view of cyclical stability.
Figure 2. (Endogenous fertility). A pictorial view of cyclical instability.

Since the focus of this paper is on the steady state effects of public pensions, in Figure 3 we only sketch a numerical illustration to show that the cyclical behaviour depicted in Figure 2 as regards the endogenous fertility economy is chaotic.

Define the right-hand side of Eq. (27) as \( f(k_t) \). As known, the standard version of the Li and Yorke’s (1975) theorem predicts that a sufficient condition for the occurrence of deterministic chaos is \( f(3) > k_0 > f(1) > f(2) \), where \( f(\cdot) \) indicates the number of iterates of the function \( f \), i.e. period three implies topological chaos (for an example of an application of the Li and Yorke’s theorem in a one-dimensional OLG economy see, amongst others, Zhang, 1999).
Figure 3. A pictorial view of three iterations that satisfy the Li and Yorke’s (1975) theorem in the model with endogenous fertility.

Figure 3 displays three iterations of the phase map $f(k_i)$ that satisfy the Li and Yorke’s theorem. Assume that the initial condition is $k_0 = 0.568$. Then we find $f(1) = 0.529$, $f(2) = 0.081$, $f(3) = 0.576$. Therefore, the non-monotonic dynamics of Figure 2 is chaotic and the following result holds.

**Result 1.** *Ceteris paribus, in an economy pay-as-you-go public pensions, the endogenisation of fertility rates gives rise to deterministic chaos.*

Result 1 may have interesting policy implications given the importance of fertility on economic growth and development (see, e.g., Becker, 1960; Becker and Barro, 1988; Barro and Becker, 1989; Becker et al., 1990). In fact, the more people tend to acquire a rational wisdom of the choice of the number of children to raise, and this especially regards developed countries, the more likely the financing of public PAYG pensions creates high risks of cyclical instability.
5. Conclusions

In this paper we analysed the dynamics of a Cobb-Douglas overlapping generations economy with public PAYG pensions by comparing exogenous and endogenous fertility settings. Despite large PAYG pensions may generate cyclical instability irrespective of whether the number of children is either exogenously given or endogenously determined, in the latter case the economy is much more prone to be exposed to endogenous fluctuations even if the stationary state is the same. This because with endogenous fertility individuals save less and the relative weight of the public pension component on capital accumulation is higher in that case, ceteris paribus as regards the key parameters of the problem. Therefore, the dynamic adjustment processes are different due to the endogenous fertility hypothesis, and when the steady-state equilibrium is stable the transitional rate of economic growth is lower in that case.

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