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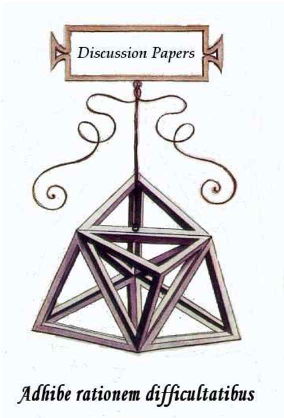

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Discussion Paper n. 122

2011

Discussion Paper **n. 122: presentato Settembre 2011**

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Stability analysis in a Bertrand duopoly with different product quality and heterogeneous expectations

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Abstract We study the local stability properties of a duopoly game with price competition, different product quality and heterogeneous expectations. We show that the Nash equilibrium can loose stability through a flip bifurcation when the consumer's type range increases. This result occurs irrespective of whether the high(low)-quality firm has either bounded rational (naïve) or naïve (bounded rational) expectations about the price that should be set in the future by the rival to maximise profits. Therefore, although, on the one hand, an increase in the consumer's types range increases profits, on the other hand, it contributes to reduce the parametric stability region of the unique interior equilibrium. Moreover, we show that the stability region is larger when the high-quality firm has naïve expectations and the low-quality firm has bounded rational expectations. This implies that when the expectations formation mechanism of the high-quality firm becomes more complicated than the naïve one, and, in particular, it follows the mechanism proposed by Dixit (1986), the stability of the Nash equilibrium in a duopoly market with price competition becomes under increasing strain.

Keywords Bifurcation; Different product quality; Duopoly; Heterogeneous players; Price competition

JEL Classification C62; D43; L13; L15

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1. Introduction

As is known firms usually supply differentiated products in the market, so that consumers face a large domain of varieties of goods and services. Moreover, it may be assumed that these products can unambiguously be ranked along some quality ladder. The focus of the present study is to analyse the stability issue in a duopoly game with price competition by assuming that each firm produces products of different, namely high and low, quality.

As is known, the Nash equilibrium in a duopoly with standard linear demand and cost functions is stable if expectations of every firm are of the "naïve" type (i.e., each firm expects that the price set by the rival to maximise profits in the future will equal the current period price level),[1] as shown by Theocharis (1960). However, if expectations of one or both firms are those of the type suggested by the most part of the recent dynamic oligopoly literature, see, for instance, Dixit (1986) (that is, firms are assumed to be bounded rational and increase/decrease their outputs/price today depending on information given by the marginal profit in the previous period, according to a certain speed of adjustment), then the equilibrium in a duopoly market with standard linear demand and cost functions may be destabilised when the speed of adjustment is fairly high, and complex dynamics can also occur, as shown by a burgeoning literature (e.g., Puu 1991, 1998; Kopel, 1996; Zhang et al. 2007, 2009; Tramontana, 2010). In particular, the stability issue in standard duopoly models without different product quality has been analysed, amongst many others, by Agiza and Elsadany (2003, 2004), Zhang et al. (2007) and Tramontana (2010) as regards quantity competition, and Zhang et al. (2009) and Fanti and Gori (2011) as regards price competition.

However, at the best of our knowledge, the stability analysis in a duopoly with price competition, in which firms provide products or services of different (say, high and low) quality, has not been so far tackled on. In this paper, therefore, we aim to fill this gap by investigating two different cases: (*i*) the high-quality firm has bounded rational expectations and the low quality firm has naïve expectations; (*ii*) the low-quality firm has bounded rational expectations and the high-quality firm has naïve expectations. In particular, we focus on the dynamical role played by the existence of product qualitative varieties, and we pose the following question: how the degree of market competition, represented either by the product quality level chosen by firms or the consumers' types range, affects stability outcomes when the expectations formation mechanism is heterogeneous between the two firms?

We can summarize the main results of the paper as follows: (1) a decrease in the consumers' types range tends to destabilise the unique Nash equilibrium of the economy; (2) the firm's product quality differential does not matter for stability. This means that only the degree of competition by the side of consumers matters for stability.

Moreover, we show that the Nash equilibrium can loose stability exclusively through a period-doubling bifurcation and the parametric stability region is larger

[1] As is known, in a duopoly each firm must forecast the behaviour of the competitor in order to make the optimal output choice. It seems more realistic to consider mechanisms through which players form their expectations on the decisions of the competitors rather than assuming that firms are able to perfectly infer the choices of the other firms. Cournot (1838) was the first to introduce naive expectations in a duopoly market.

when the high-quality firm has naïve expectations and the low-quality firm has bounded rational expectations than when the high-quality firm has bounded rational expectations and the low-quality firms has naïve expectations.

The remainder of the paper is organised as follows. Section 2 introduces the duopoly model with price competition. Section 3 studies the conditions under which the unique Nash equilibrium of the economy can loose stability in the case of heterogeneous expectations about the price to be set in the future period to maximise profits by both high-quality and low-quality firms. Section 4 concludes.

2. The model[2]

We assume that: (1) there exist two firms in the market, namely firm H and firm L, providing products (or services) of different quality to the customers; (2) it is unanimously believed that products (or services) of firm H are of a higher quality than those of firm L; (3) the average cost of production is not affected by quality, and we set it to be equal to zero without loss of generality.

Consumers are identified by the parameter $\phi \in [a,b]$, where $0 \leq a < b$, which is uniformly distributed with density equal to 1. Preferences (U) of consumer ϕ are described by the following expected utility function:

$$U_i(\phi, p_i) = \phi u_i - p_i, \quad i = \{H, L\}, \tag{1}$$

where u_H and u_L, with $u_H > u_L$, represent two indexes that capture the different quality perceived by consumers of products and services provided by firms H and L, and p_i is the price that consumers pay to buy the product (or service) provided by the ith firm. Moreover, in order to guarantee that the two firms can set strictly positive prices at an interior equilibrium, we assume that:[3]

$$a \in \left[0, \frac{b}{2}\right). \tag{2}$$

Let $\bar{\phi}$ be an index that identifies the consumer who is indifferent between purchasing products of high or low quality from firms H and L at the price p_H and p_L, respectively. Such an index is obtained by equating $U_H(\phi, p_H) = U_L(\phi, p_L)$. Then, solving the equation

$$\phi u_H - p_H = \phi u_L - p_L, \tag{3}$$

for ϕ we get:

$$\bar{\phi} = \frac{p_H - p_L}{u_H - u_L}. \tag{4}$$

The demand functions to firms H and L are respectively given by:

$$D_H(p_H, p_L) = b - \bar{\phi} = b - \frac{p_H - p_L}{u_H - u_L}, \tag{5.1}$$

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[2] Note that in this paper we follow the standard model outlined by Tarola et al. (2011), which refers to a banking duopoly in which two banks provide differentiated services to the customers. However they only focus on the issue of entry of a third bank in the market in a static context, while abstracting from any stability issues.

[3] This will be clear from Eqs. (13) and (26) in the sequel of the paper. Note that this condition also guarantees that outputs and profits of both firms are positive at an interior equilibrium.

$$D_L(p_H, p_L) = \bar{\phi} - a = \frac{p_H - p_L}{u_H - u_L} - a. \tag{5.2}$$

The profit functions are therefore determined by:

$$\Pi_H(p_H, p_L) = p_H\left(b - \frac{p_H - p_L}{u_H - u_L}\right), \tag{6.1}$$

$$\Pi_L(p_H, p_L) = p_L\left(\frac{p_H - p_L}{u_H - u_L} - a\right). \tag{6.2}$$

The maximization of Eqs. (6.1) and (6.2) with respect to p_H and p_L gives the following marginal profits:

$$\frac{\partial \Pi_H}{\partial p_H} = b - \frac{2p_H - p_L}{u_H - u_L}, \tag{7.1}$$

$$\frac{\partial \Pi_L}{\partial p_L} = \frac{p_H - 2p_L}{u_H - u_L} - a. \tag{7.2}$$

Therefore, the reaction- or best-reply functions of firms H and L are determined by setting Eqs. (7.1) and (7.2) to zero and solving for p_H and p_L, respectively, that is:

$$\frac{\partial \Pi_H}{\partial p_H} = 0 \Leftrightarrow p_H(p_L) = \frac{1}{2}\left[p_L + b(u_H - u_L)\right], \tag{8.1}$$

$$\frac{\partial \Pi_L}{\partial p_L} = 0 \Leftrightarrow p_L(p_H) = \frac{1}{2}\left[p_H - a(u_H - u_L)\right]. \tag{8.2}$$

In the next section we assume that firms H and L have heterogeneous (namely, bounded rational and naïve) expectations and study the local stability properties of the Nash equilibrium.

3. Equilibrium and local stability analysis with heterogeneous players

3.1. Case $\boxed{BH/NL}$

In this section we study the local stability properties of the Nash equilibrium when firm H has bounded rational expectations (BH) and firm L has naïve expectations (NL) about the prices to be set in the future to maximise profits.

Therefore, firm H uses information on its profit at time $t = 0,1,2,...$ to increase or decrease prices at time $t+1$ according to the following mechanism (see Dixit, 1986):

$$p_{H,t+1} = p_{H,t} + \alpha_H p_{H,t} \frac{\partial \Pi_{H,t}}{\partial p_{H,t}}, \tag{9}$$

where $\alpha_H > 0$ is a coefficient that captures the speed of adjustment of firm H's price with respect to a marginal change in its profits when $p_{H,t}$ varies. Therefore, $p_{H,t+1}$ is increased or decreased depending on whether current marginal profits are positive of negative, respectively.

Using Eq. (9), and knowing that firm L has naïve expectations, the two-dimensional system that characterises the dynamics of this simple duopolistic market is the following:

$$\begin{cases} p_{H,t+1} = p_{H,t} + \alpha_H p_{H,t} \dfrac{\partial \Pi_{H,t}}{\partial p_{H,t}} \\ p_{L,t+1} = p_{L,t} \end{cases}. \tag{10}$$

Using Eq. (7.1) to substitute out for $\partial \Pi_{H,t} / \partial p_{H,t}$ into the first equation of (10), and Eq. (8.2) to substitute out into the right-hand side of the second equation of (10), we get:

$$\begin{cases} p_{H,t+1} = p_{H,t} + \alpha_H p_{H,t} \left(b - \dfrac{2 p_{H,t} - p_{L,t}}{u_H - u_L} \right) \\ p_{L,t+1} = \dfrac{1}{2} \left[p_{H,t} - a(u_H - u_L) \right] \end{cases}. \tag{11}$$

Equilibrium implies that $p_{H,t+1} = p_{H,t} = p_H$ and $p_{L,t+1} = p_{L,t} = p_L$. Therefore, the dynamic system defined by (11) can be reduced to:

$$\begin{cases} \alpha_H p_H \left(b - \dfrac{2 p_H - p_L}{u_H - u_L} \right) = 0 \\ p_L - \dfrac{1}{2} \left[p_H - a(u_H - u_L) \right] = 0 \end{cases}. \tag{12}$$

The unique non-negative fixed point $E_{BH/NL} = \left(p^*_H, p^*_L \right)$ of the dynamic system defined by Eq. (11) is determined by the following non-negative solution of Eq. (12), that is:

$$E_{BH/NL} = \left(p^*_H, p^*_L \right) = \left(\frac{1}{3}(2b - a)(u_H - u_L), \frac{1}{3}(b - 2a)(u_H - u_L) \right). \tag{13.1}$$

From (13.1) we note that in equilibrium $p^*_H - p^*_L = \frac{1}{3}(a + b)(u_H - u_L)$, so that $\bar{\phi} = \frac{a+b}{3}$. Furthermore, the outputs and profits at the equilibrium point are given by:

$$D^*_H = \frac{2b - a}{3}, \quad D^*_L = \frac{b - 2a}{3}, \tag{13.2}$$

$$\Pi^*_H = \frac{(2b - a)^2 (u_H - u_L)}{9}, \quad \Pi^*_L = \frac{(b - 2a)^2 (u_H - u_L)}{9}. \tag{13.3}$$

Notice that profits of both firms reduce when: (*i*) the quality differential, $u_H - u_L$, reduces and (*ii*) for every b, the value of the parameter a increases.

In order to investigate the local stability properties of the Nash equilibrium (13.1) of the two-dimensional system (11), we build on the Jacobian matrix J evaluated at the equilibrium point $E_{BH/NL}$, that is:

$$J_{BH/NL} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \begin{pmatrix} 1 - \dfrac{2}{3}\alpha_H (2b - a) & \dfrac{1}{3}\alpha_H (2b - a) \\ \dfrac{1}{2} & 0 \end{pmatrix}. \tag{14}$$

whose trace and determinant are respectively given by:

$$T := Tr(J_{BH/NL}) = J_{11} + J_{22} = 1 - \frac{2}{3}\alpha_H (2b - a), \tag{15}$$

$$D := Det(J_{BH/NL}) = J_{11} J_{22} - J_{12} J_{21} = -\frac{1}{6}\alpha_H (2b - a). \tag{16}$$

Therefore, the characteristic polynomial of (14) can be written as follows:

$$G(\lambda)=\lambda^2-T\lambda+D, \quad (17)$$

with its discriminant being determined by $Q:=T^2-4D=\left[1-\frac{2}{3}\alpha_H(2b-a)\right]^2+\frac{1}{6}\alpha_H(2b-a)>0$. Since the discriminant is positive, the existence of complex eigenvalues of $J_{BH/NL}$ is prevented.

As is known, bifurcation theory describes the way the topological features of a dynamic system (such as the number of stationary points or their stability) vary as some parameter values change. In particular, for the system in two dimensions determined by (11), the stability conditions ensuring that both eigenvalues remain within the unit circle are:

$$\begin{cases} (i) \quad F:=1+T+D=\dfrac{12-5\alpha_H(2b-a)}{6}>0 \\ (ii) \quad TC:=1-T+D=\dfrac{1}{2}\alpha_H(2b-a)>0 \\ (iii) \quad H:=1-D=\dfrac{6+\alpha_H(2b-a)}{6}>0 \end{cases}. \quad (18)$$

The violation of any single inequality in (18), with the other two being simultaneously fulfilled leads to: (*i*) a flip or period-doubling bifurcation (a real eigenvalue that passes through -1) when $F=0$; (*ii*) a fold or transcritical bifurcation (a real eigenvalue that passes through $+1$) when $TC=0$; (*iii*) a Neimark-Sacker bifurcation (i.e., the modulus of a complex eigenvalue pair that passes through 1) when $H=0$, namely $Det(J)=1$ and $|Tr(J)|<2$.

While from (18) it is clear that conditions (*ii*) and (*iii*) are always fulfilled for any $a\in[0,b/2)$, condition (*i*) can be violated. Since the dynamical role played by the speed of adjustments α (when at least one of the two players has bounded rational expectations) on the stability of a fixed point in duopoly models has widely been investigated in economic literature (see, e.g., Bischi and Naimzada, 1999; Agiza and Elsadany, 2003; Zhang et al., 2007; 2009; Tramontana, 2010), and since the present model shows results that accord with those of the existing literature as regards the role played by such a parameter, we now concentrate on the dynamic effects of the existence of different quality of products and consumers' types, which are the main features that distinguish our model from the preceding dynamic oligopoly literature. In particular, in what follows we choose a as the parameter of interest. In other words, for any given value of α_H and b, we let a vary within its domain if definition defined by Eq. (2), and study the stability properties of Nash equilibrium $E_{BH/NL}$ in such a case.

Now, define

$$a^F{}_{BH/NL}=\frac{10\alpha_H b-12}{5\alpha_H}, \quad (19)$$

$$\alpha^F{}_1=\frac{6}{5b}, \quad (20)$$

$$\alpha^F{}_2=\frac{24}{15b}, \quad (21)$$

as the flip bifurcation value of a in the case BH/NL, where $\lim_{\alpha\to+\infty} a^F{}_{BH/NL}=2b$, and two threshold values of the speed of adjustment α, where $\alpha^F{}_2>\alpha^F{}_1$, which are the roots for α obtained by equating Eq. (19) to zero and $b/2$, respectively, i.e., the boundaries of the domain of definition of a. Then, from (18)-(21) we have the following proposition.

Proposition 1. *(1) Let* $0<\alpha<\alpha^F{}_1$ *hold. Then, the Nash equilibrium* $E_{BH/NL}$ *of the two-dimensional system (11) is locally asymptotically stable for any* $a\in[0,b/2)$. *(2) Let* $\alpha^F{}_1\le\alpha<\alpha^F{}_2$ *hold. Then, starting from a situation in which the Nash equilibrium* $E_{BH/NL}$ *is locally asymptotically stable, it looses stability through a flip or period-doubling bifurcation when the parameter* a *is reduced to* $a=a^F{}_{BH/NL}$. *(3) Let* $\alpha\ge\alpha^F{}_2$ *hold. Then, the Nash equilibrium* $E_{BH/NL}$ *is locally unstable.*

Proof. Since $a^F{}_{BH/NL}<0$ for any $0<\alpha<\alpha^F{}_1$, then $F>0$ for any $a\in[0,b/2)$. This proves point (1). Since $0\le a^F{}_{BH/NL}<b/2$ for any $\alpha^F{}_1\le\alpha<\alpha^F{}_2$, then $F>0$ for any $b/2>a>a^F{}_{BH/NL}$, $F=0$ if and only if $a=a^F{}_{BH/NL}$ and $F<0$ for any $a^F{}_{BH/NL}>a>0$. This proves point (2). Since $a^F{}_{BH/NL}\ge b/2$ for any $\alpha\ge\alpha^F{}_2$, then $F<0$ for any $a\in[0,b/2)$. This proves point (3). **Q.E.D.**

3.2. Case $\boxed{NH/BL}$

In this section we assume that firm H has naïve expectations (NH), while firm L has bounded rational expectations (BL) and then only the latter firm uses information on its current profit to increase or decrease prices at time $t+1$ according to the adjustment process:

$$p_{L,t+1}=p_{L,t}+\alpha_L p_{L,t}\frac{\partial\Pi_{L,t}}{\partial p_{L,t}}, \tag{22}$$

where $\alpha_L>0$, while firm. Therefore, the two-dimensional system that characterises the dynamics of the economy becomes the following:

$$\begin{cases} p_{H,t+1}=p_{H,t} \\ p_{L,t+1}=p_{L,t}+\alpha_L p_{L,t}\dfrac{\partial\Pi_{L,t}}{\partial p_{L,t}} \end{cases}. \tag{23}$$

Using Eq. (7.2) to substitute out for $\partial\Pi_{L,t}/\partial p_{L,t}$ into the second equation of (23), and Eq. (8.1) to substitute out into the right-hand side of the first equation of (23), we get:

$$\begin{cases} p_{H,t+1}=\dfrac{1}{2}\left[p_{L,t}+b(u_H-u_L)\right] \\ p_{L,t+1}=p_{L,t}+\alpha_L p_{L,t}\left(\dfrac{p_{H,t}-2p_{L,t}}{u_H-u_L}-a\right) \end{cases}. \tag{24}$$

Equilibrium implies that $p_{H,t+1}=p_{H,t}=p_H$ and $p_{L,t+1}=p_{L,t}=p_L$. Then, the dynamic system defined by (24) can be reduced to:

$$\begin{cases} p_H-\dfrac{1}{2}\left[p_L+b(u_H-u_L)\right]=0 \\ \alpha_L p_L\left(\dfrac{p_H-2p_L}{u_H-u_L}-a\right)=0 \end{cases}. \tag{25}$$

The unique non-negative fixed point $E_{NH/BL} = \left(p^*_H, p^*_L\right)$ of the dynamic system defined by Eq. (24) is determined by the following non-negative solution of Eq. (25), that is:

$$E_{NH/BL} = \left(p^*_H, p^*_L\right) = \left(\frac{1}{3}(2b-a)(u_H - u_L), \frac{1}{3}(b-2a)(u_H - u_L)\right). \tag{26}$$

It should be noted that the different type of expectations formation is not relevant for the equilibrium outcomes of prices, outputs and profits of firms H and L, while playing a role in determining different stability conditions, as can be seen below. Indeed, by comparing Eqs. (13.1) and (26) it is easy to see that $E_{BH/NL} = E_{NH/BL}$.

In order to investigate the local stability properties of the Nash equilibrium (26) of the two-dimensional system (24), we build on the Jacobian matrix J evaluated at $E_{NH/BL}$, that is:

$$J_{NH/BL} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{3}\alpha_L(b-2a) & 1-\frac{2}{3}\alpha_L(b-2a) \end{pmatrix}. \tag{27}$$

whose trace and determinant are given by:

$$T := Tr\left(J_{NH/BL}\right) = J_{11} + J_{22} = 1 - \frac{2}{3}\alpha_L(b-2a), \tag{28}$$

$$D := Det\left(J_{NH/BL}\right) = J_{11}J_{22} - J_{12}J_{21} = -\frac{1}{6}\alpha_L(b-2a). \tag{29}$$

Therefore, the characteristic polynomial of (27) is:

$$Z(\lambda) = \lambda^2 - T\lambda + D, \tag{30}$$

with its discriminant being determined by $P := T^2 - 4D = \left[1 - \frac{2}{3}\alpha_L(b-2a)\right]^2 + \frac{1}{6}\alpha_L(b-2a) > 0$. Since the discriminant is positive, the existence of complex eigenvalues of $J_{NH/BL}$ is prevented.

The stability conditions for the system in two-dimension (24) are the following:

$$\begin{cases} (i) \quad F := 1 + T + D = \dfrac{12 - 5\alpha_L(b-2a)}{6} > 0 \\ (ii) \quad TC := 1 - T + D = \dfrac{1}{2}\alpha_L(b-2a) > 0 \\ (iii) \quad H := 1 - D = \dfrac{6 + \alpha_L(b-2a)}{6} > 0 \end{cases}. \tag{31}$$

Now, define

$$a^F_{NH/BL} = \frac{5\alpha_L b - 12}{10\alpha_L}, \tag{32}$$

$$\alpha^F_3 = \frac{12}{5b}, \tag{33}$$

as the flip bifurcation value of a in the case NH/BL and a threshold value of the speed of adjustment α, which is the unique root for α obtained by equating Eq. (32) to zero. Note that in this case $a^F_{NH/BL} \to b/2$ when $\alpha \to +\infty$, and $\alpha^F_3 > \alpha^F_2 > \alpha^F_1$. Then, from (31)-(33) we have the following proposition.

Proposition 2. *(1) Let* $0<\alpha<\alpha^{F}{}_{3}$ *hold. Then, the Nash equilibrium* $E_{NH/BL}$ *of the two-dimensional system (24) is locally asymptotically stable for any* $a\in[0,b/2)$. *(2) Let* $\alpha\geq\alpha^{F}{}_{3}$ *hold. Then, starting from a situation in which the Nash equilibrium* $E_{NH/BL}$ *is locally asymptotically stable, it looses stability through a flip or period-doubling bifurcation when the parameter* a *is reduced to* $a=a^{F}{}_{NH/BL}$.

Proof. Since $a^{F}{}_{NH/BL}<0$ for any $0<\alpha<\alpha^{F}{}_{3}$, then $F>0$ for any $a\in[0,b/2)$. This proves point (1). Since $0\leq a^{F}{}_{BH/NL}<b/2$ for any $\alpha\geq\alpha^{F}{}_{3}$, then $F>0$ for any $b/2>a>a^{F}{}_{NH/BL}$, $F=0$ if and only if $a=a^{F}{}_{NH/BL}$ and $F<0$ for any $a^{F}{}_{NH/BL}>a>0$. This proves point (2). **Q.E.D.**

Notice that the stability boundaries in both cases (i.e., the conditions stated in 18 and 31) depend on the speed of adjustment, α_L and α_H, and only on the parameters that reveal the differences in the consumers' types range, a and b, while being independent of the product quality differential, u_H-u_L.

Moreover, extensive numerical simulations (not reported here for economy of space) reveal the occurrence of periodic cycles, as predicted by Propositions 1 and 2, but complex dynamics do not occur within the economically meaningful domain of definition of the parameter that identifies the consumer's tastes as regards the quality of products of varieties H and L.

We now compare the stability-instability regions in the cases BH/NL and NH/BL by assuming that $\alpha_H=\alpha_L=\alpha$. The results are summarised in the proposition that follows.

Proposition 3. *The (parametric) stability region of the Nash equilibrium* $E_{BH/NL}=E_{NH/BL}$ *is larger when firms have NH/BL expectations than when they have BH/NL expectations.*

Proof. Since $a^{F}{}_{NH/BL}<a^{F}{}_{BH/NL}$ for any $\alpha>\alpha^{F}{}_{1}$, as can easily be ascertained from Eqs. (19) and (32), Proposition 3 follows. **Q.E.D.**

Figure 1 shows, in the (α,a) plane, that the stability regions under NH/BL are larger than under BH/NL. It is also interesting to note, ceteris paribus as regards the speed of adjustment, that when the economy enters the region of unfeasible trajectories in the case BH/NL, it is still "strongly" stable in the case NH/BL, ceteris paribus as regards the value of the speed of adjustment.

This result emphasizes the perils for market stability resulting from the high quality firm having expectations more elaborated than the simple naïve ones.

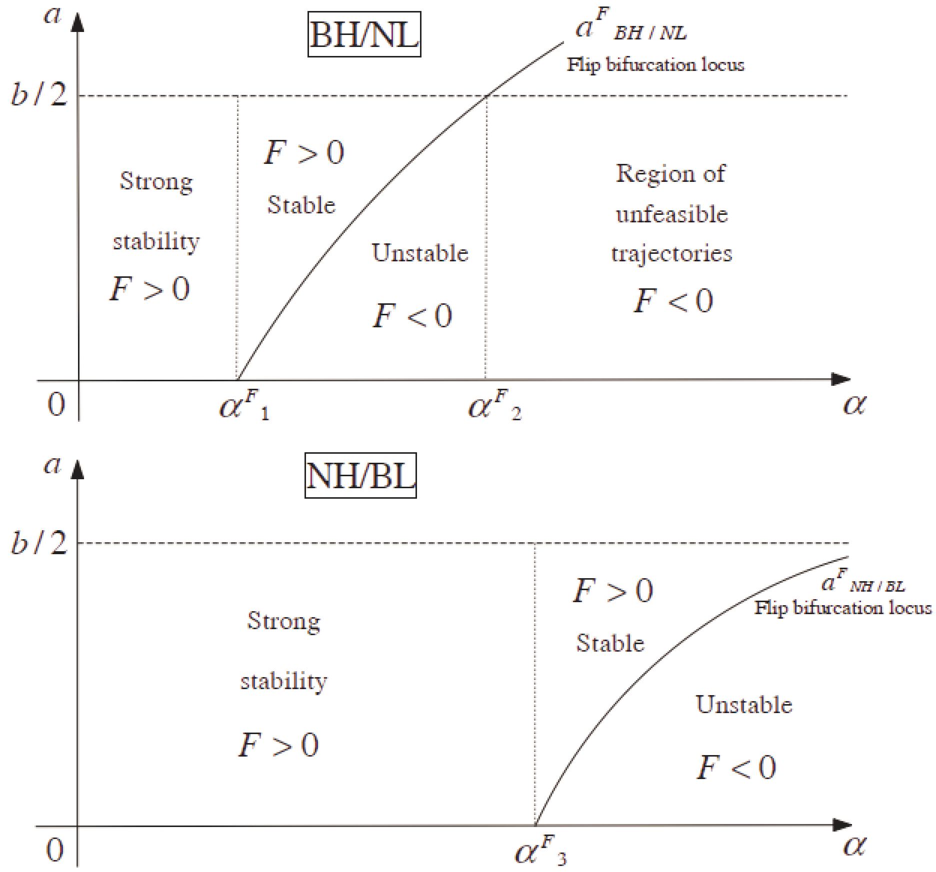

Figure 1. Stability-instability regions in the (α, a) plane under BH/NL and NH/BL.

The following remarks summarise the main findings of the paper.

Remark 1. *The local stability properties of the unique Nash equilibrium* $E_{BH/NL} = E_{NH/BL}$ *crucially depend on the size of the consumers' type, as captured by the parameter* a*, while being not affected by the levels of product quality chosen by firms, as captured by the parameters* u_H *and* u_L*. The more similar the consumers' tastes as regards product quality (low values of* a*), the more likely a duopoly market is (locally) unstable.*

Remark 2. *Ceteris paribus as regards the speed of adjustment* α*, the more similar the consumers' tastes as regards product quality (low values of* a*), the more likely a duopoly market is (locally) unstable when firms have* BH/NL *expectations than when they have* NH/BL *expectations.*

4. Conclusions

This study has originated by the increasing interest for a refined analysis in the nonlinear oligopoly literature (see. e.g., Bischi et al., 2010). The novelty of the present paper is the analysis of the local stability properties in a duopoly game with price competition, different product quality, heterogeneous expectations and different consumers' types.

We showed that a decrease in the range of consumer's preferences with respect to differences in product quality, tends to destabilise the market equilibrium through a period-doubling bifurcation. However, several numerical experiments (not reported in the paper) revealed that nothing more complicated than periodic cycles may occur within the economically meaningful domain of definition of the parameters that captures the consumers' types range, so that complex dynamics cannot be observed.

We note interestingly that the stability boundaries are independent of the product quality differential, which only affect equilibrium profits. By contrast, a twofold role is played by the consumer's type range: i.e., the more similar the consumers' tastes as regards product quality, the higher profits and the lower the parametric stability region.

Moreover, we showed that the expectations formation mechanisms matter for stability of the Nash equilibrium. In particular, when the high-quality firm has naïve expectations and the low-quality firm has bounded rational expectations, the parametric region of local stability of the fixed point is larger than when the high-quality firm has bounded rational expectations and the low-quality firm has naïve expectations. This implies that when the expectations of the high quality firms become more complicated than the naïve ones, as for instance supposed by Dixit (1986), the stability of the Nash equilibrium becomes under increasing strain.

To sum up, our results show that the Nash equilibrium of a duopoly market with different product quality, different consumers' type and heterogeneous expectations is more likely to be stable when the relative degree of consumers' tastes as regards product quality is high and the high-quality firm has naïve expectations.

Finally, it is worth to note that when both high- and low-quality firms have bounded rational expectations, then the local stability properties as well as the global dynamics that emerges after the loss of stability of the Nash equilibrium are completely different (for instance, a quasi-periodic route to chaos emerges), and are the object of a companion paper.

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