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Walrasian dynamics and the Phillips curve

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Abstract

The paper aims to contribute to the debate on the theoretical foundation as well as the policy implications of the Phillips' curve. First, we show that Walrasian labour markets with persistent excess demand provide predictions on the relation between the growth rate of wages and the unemployment rate. Second we provide a careful theoretical characterisation of such a relation. This allows to show that what has been traditionally called the Phillips curve would in such a case only be a statistical artifact. Such findings may have far-reaching implications. First, the simplest dynamic neoclassical labour market "resurrects" the Phillips curve in the long run, showing that such a curve is correctly predicted by the law of demand and supply, as in the neoclassical view, though as a long-run, rather than transitory, phenomenon. Second, this theoretically funded Phillips, though persisting in the long run as in the Keynesian view, should be considered with a larger caution as a macroeconomic policy tool.

JEL codes: J20, G32.

Keywords: Walrasian labour market, persistent excess demand, unemployment, Phillips curve.

1. Introduction

The Walrasian dynamics is a key economic concept defining the dynamic law of demand and supply according to which the price of a commodity or service (including labour) changes over time at a speed proportional to the excess demand for it. The Walrasian dynamics of the labour market generates, under plausible conditions, steady oscillations of wage, demand and supply of labour (Fanti and Manfredi 2007). This means that a situation of excess demand may persist in the neoclassical labour market.

Another pervasive economic concept is the empirical (but "in search for a theory" according to Tobin (1972)) relation between the growth rate of the wage rate and the unemployment rate, universally known as the Phillips' curve. Few key economic concepts have a controversial interpretation as the Phillips' curve. The reason lies in its lack of micro-foundations that has been considered a major drawback. Neoclassical economists have attempted to found the Phillips' curve in terms of the law of demand and supply, i.e. on the basis of Walrasian arguments (e.g. Lipsey, 1960). The drawback of this explanation lies in its difficulty to ensure the persistence of the relation in the long-run. More modern theoretical explanations distinguished between short-run and long-run effects on unemployment: 1) the so-called "expectations-augmented Phillips curve" predicts that in the short run such effects exist but in the long run disappear; 2) the so-called New Keynesian Phillips curve, like the expectations-augmented Phillips curve, implies that increased inflation can lower unemployment temporarily, but cannot lower it permanently.

The above mentioned persistent excess demand emerging in Walrasian markets in principle bears no relation with the Phillips curve. In this paper we show that, instead, models of a standard neoclassical labour market endowed with the simplest Walrasian adjustment rule generate, under conditions of persistent excess demand, theoretical predictions also on the relationship between the growth rate of wages and the unemployment rate. We carefully investigate the theoretical properties of such a relation, suggesting that its shape is fully consistent with those empirically observed for the Phillips curve. Moreover we show that the "statistical form" of this relation is that of the Phillips curve, in the sense that statistical tests applied on time series of data generated by such a curve would identify a decreasing relationship between the growth rate of wages and the unemployment rate.

Two interesting implications arises. The first one is that, according to our findings, the Phillips curve is nothing else than an outcome of the Walrasian labour market under persistent excess demand. In other words, a purely neoclassical labour market under plausible conditions, “naturally” embeds the Phillips curve. The second is that since this “Walrasian” Phillips curve is a dynamic realization of the Walrasian labour market, it can no longer be interpreted, according to the long lasting policy interpretation popularised, e.g. by Samuelson-Solow (1960), as a locus of equilibrium points.

The plan of the paper is as follows: in section 2 we briefly describe our the Walrasian labour market model and summarise its main dynamical properties, as studied in Fanti and Manfredi 2007; in section 3 we introduce our new perspective on the Phillips curve, and investigate its properties. Concluding comments follow.

2. A model of a dynamic Walrasian economy and its properties

We describe the Walrasian model of a labour market used in Fanti and Manfredi (2007), and briefly summarises its main findings, as fundamental to understand our new perspective on the Phillips curve. The model considers a one-good economy with a single representative firm and a single representative worker-consumer. Output (*Y*) is produced using labour (*L*) as the only input by a Cobb-Douglas technology: $Y = DL^a \quad 0 < a < 1,\ D > 0$. The optimal demand for labour (setting to one the price of unit output) is:

$$L^D = f_1(w) = \left(\frac{w}{aD}\right)^{1/(a-1))} \qquad (1)$$

The worker-consumer maximises the CES utility function $U(C,L) = \left[C^b + (N-L)^b\right]^{1/b},\ b \in (-\infty,1).$ (Chichilnisky et al., 1995), where $N>0$ is the maximal labour supply and $C>0$ the worker’s consumption level.

We take *N=1*. This yields the optimal labor supply:

$$L^S = f_2(w) = \left(1 + w^{\frac{b}{b-1}}\right)^{-1} \qquad (2)$$

The optimal demand (1) has the traditional decreasing shape whereas the optimal supply (2) has the traditional increasing shape for $b>0$, and it bends backward for $b<0$. The latter case is an important one (Chichilnisky et al. 1995, Benhabib-Farmer 2000, Farmer-Guo 1995). Note moreover that if worker households receive profit or capital income, the labour supply curve could be positively sloped also in the case $b<0$.

In this model the Walrasian dynamics arises by assuming, as in simple labour market analyses (Hamermesh 1996, pp. 305), that i) the actual labour demand (L) adjusts to its optimal counterparts (L^D) following an adaptive adjustment rule (arising from quadratic adjustment costs), with speed of adjustment $g>0$; ii) the actual labour supply (S) adaptively adjusts, with speed $d>0$, to the optimal supply (L^S); and iii) the wage adaptively adjusts, with speed $\lambda>0$, to the actual excess demand for labour, according to the Walrasian law. The final form of the dynamic Walrasian model is therefore:

$$\begin{aligned} \dot{L} &= g\left(L^D - L\right) = g\left(f_1(w) - L\right) && g > 0 \\ \dot{S} &= d\left(L^S - S\right) = d\left(f_2(w) - S\right) && d > 0 \\ \dot{w} &= \lambda\left(L - S\right) && l > 0. \end{aligned} \qquad (3.a,b,c)$$

The main dynamic features of model (3) are investigated in detail in Fanti and Manfredi (2007). We summarise them to assist the subsequent analysis. Model (3) is well posed and always admits a unique and economically meaningful equilibrium point $E_1 = \left(L^*, S^*, w^*\right)$. E_1 is the *Walrasian equilibrium* of the economy. Under an upward sloping ($b>0$) or rigid ($b=0$) optimal labour supply, i.e. when consumption and leisure are strong substitute, the Walrasian equilibrium is always locally asymptotically stable (LAS). Simulations suggest that this holds globally. On the other hand, under a backward-bending labour supply ($b<0$) (i.e. when consumption and leisure are weak substitutes) the Walrasian equilibrium can be destabilised: for any given speed of adjustment of the labour supply d a value g_H of the speed of adjustment of demand can be found such that for any $g<g_H$ the Walrasian equilibrium is unstable. At $g=g_H$ persistent oscillations of the economy appear through a Hopf bifurcation of the Walrasian equilibrium. Fig. 1 reports possible shapes of the bifurcation locus $g_H(d)$). Simulations suggest that the limit cycle emerged though the bifurcation is unique and Globally Asymptotically Stable (GAS). A symmetrical result holds using d as a bifurcation parameter. Fig. 2 depicts the shape of one such limit cycles. Finally, further decreases of the adjustment speeds can favour the appearance of chaotic oscillations.

To sum up, the previous results state that, under fairly general conditions, an excess demand may persist in the neoclassical labour market. This is favoured by flexibility in the realisation of the employment decisions of the households as well as stickiness in those of the firms.

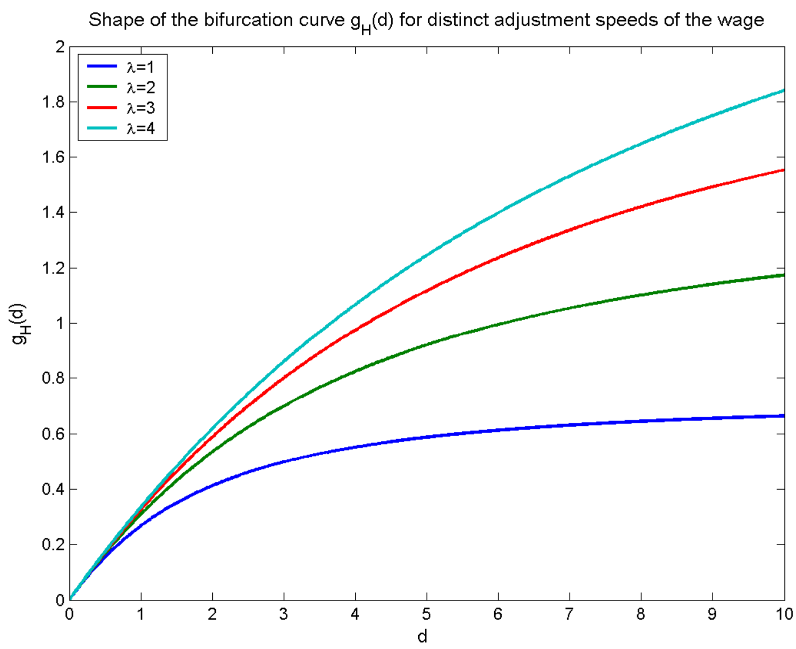

Fig. 1. *Bifurcation curves of the Walrasian equilibrium in the (g,d) plane for various λ's (parameter values: D=1,a=0.15,b=-10).*

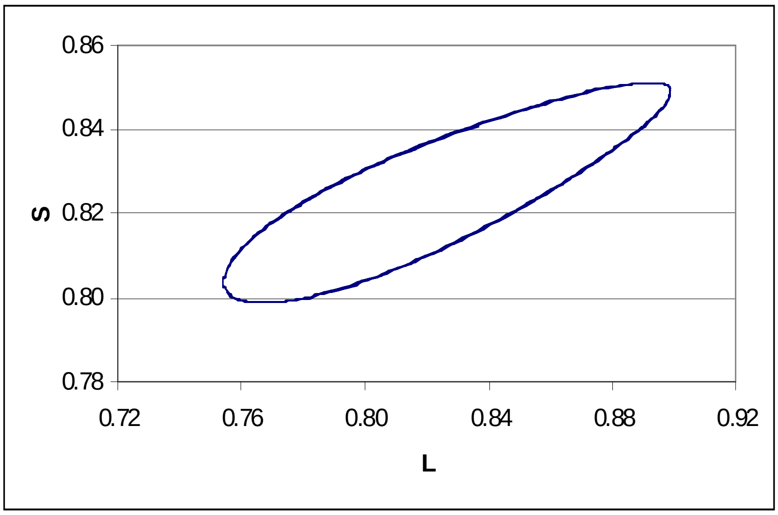

Fig. 2. Phase-plane view in the L,S plane of a limit cycle of model (3) . Parameter values as in fig. 1 and additionally d=4, g=1.07, λ=4.

3. Is the Phillips curve just a long-term representation of Walrasian dynamics ? A new perspective.

According to his proponents (Tinbergen 1959, Phillips 1958, Lipsey 1960, Klein-Goldberger, 1955) a Phillips' curve is an empirical relation of the form $\hat{w} = h(u)$ between the growth rate of wages $\hat{w} = \dot{w}/w$ and the unemployment rate *u*. Lipsey attempted to found the Phillips curve by resorting to the apparatus of the law of demand and supply. His formulation however is "ad hoc" exactly as the Phillips's original one, in that he postulates a Walrasian adjustment related to the proportional and not the absolute excess demand. Indeed, the right hand-side of his relation has the required Phillips form, whereas his left hand-side is still in the absolute rather than in the percentage growth rate. Subsequently when he uses his equation empirically he transforms the left-hand side in percentage terms. This appears a rather ad hoc approach. In any case his interpretation of the curve is not capable to justify a long-run Phillips curve, unless resorting to inter-markets frictions.

In this work we rather intend to investigate the existence and properties of the Phillips curve as it appears within the framework of the purest formulation of the Walrasian adjustment scheme. We show that models of dynamic Walrasian economies as (3) with persistent excess demand necessarily generate non trivial long-term predictions on such a relationship. We further show, as our main result, that the shape of this relationship might be “complex” but fully compatible with what is traditionally called the Phillips’ curve. We finally show that the traditional “policy-oriented” interpretation seems to be rather weakened.

Let us first proceed to the derivation of the “theoretical” Phillips curve (TP since now on). We term it “Theoretical” because it is an output of the fully funded economic model (3), rather than an “empirically based” relation. Let us first define the unemployment rate as follows:

$$u = \frac{S-L}{S} = 1 - \frac{L}{S} \qquad (4)$$

Remark 1. The expression (4) defines the unemployment rate as the proportionate excess demand Though the true unemployment rate is always positive by definition, contrary to (4), due to the existence of frictional unemployment, we keep formulation (4) for its simplicity. We note however that our main Theorem below is robust to different definitions of the unemployment rate, for instance taking $u = g((L-S)/S)$, where g is a non-linear positive decreasing function mapping the values of the excess demand onto [0,1].

The wage equation (3c) can then be manipulated to obtain the required relation between Phillips’ variables:

$$\frac{\dot{w}}{w} = \lambda \frac{L-S}{w} = (-\lambda)\frac{S}{w} \cdot u \qquad (5)$$

Relation (6) is the “Theoretical Phillips’ curve” predicted by the Walrasian dynamics. Let us investigate the properties of (5) by considering the locus *P(t)* defined in the parametric form:

$$P(t) = (u(t), Z(t)) \qquad (6)$$

where $u(t)$ and $Z(t)$ are the following functions defined on the solutions of system (3):

$$u(t)=\frac{S(t)-L(t)}{S(t)} \quad , \quad Z(t)=\frac{\dot{w}(t)}{w(t)}=(-\lambda)\frac{S(t)-L(t)}{w(t)} \qquad (7)$$

The shape of (6) in the (u,Z) plane might be complicate if one considers the full transient dynamics of (3). Given however that we are essentially interested in the Phillips' curve as a long term phenomenon, we disregard transient dynamics, and only focus on the long-term form $P_\infty=(u_\infty,Z_\infty)$ of (6)-(7). This is facilitated from the sharp behaviour of model (3), i.e. the fact that depending on the chosen parameter constellation a unique Globally Asymptotically Stable attractor (either an equilibrium point or a stable limit cycle, or a chaotic attractor) exists. Let us denote this attractor by *A*.

Lemma 1. The mapping $F: R_+^3 \to R^2$ defined as**:**

$$F(L,S,w)=\left(\frac{S-L}{S},\lambda\frac{L-S}{w}\right) \qquad (8)$$

is continuous but not one to one.

The proof of the first part is obvious; the second part trivially follows from the fact that the set Γ of points $\Gamma=\{(L,S,w)|S=L=\sigma\in R_+\}$ are all mapped into the origin.

The continuity of the mapping implies that $P_\infty=P(A)$ i.e. that the long-term shape of the locus $P(t)$ is simply the image of the attractor *A* through the mapping (8). Thus it trivially follows that if $A=E_1$ (the Walrasian equilibrium) then $P_\infty=P(E_1)$ is a trivial locus (a point) and therefore no long-term Phillips' curve "exists". Let us therefore consider the more interesting case in which $A=C$, i.e. *A* is the (unique) limit cycle $C=(S_\infty,L_\infty,w_\infty)$ described in the previous section. In this case $P_\infty=P(C)$ is the image of a cycle. We assume that the cycle *C* has period $T>0$. The cycle

C has the interesting economic property, following from the pure Walrasian adjustment rule for wages (3c), that the labour demand and supply have the same long-term averages (see Proposition 2 in the appendix):

$$\frac{1}{T}\int_0^T L_\infty(t)dt = \frac{1}{T}\int_0^T L_\infty(t)dt \qquad (9)$$

Next lemma, of which we omit the easy proof, ensures that this image is also a periodic function:

Lemma 2. If system (5) converges to the limit cycle C, then *u(t)* and *Z(t)* also converge to the periodic functions:

$$u_\infty(t) = 1 - \frac{L_\infty(t)}{S_\infty(t)} \quad , \qquad Z_\infty(t) = \lambda \frac{L_\infty(t) - S_\infty(t)}{w_\infty(t)} \qquad (10)$$

The shape of $P_\infty = P(C)$ is fully characterised by the following general result which is valid whatever be the actual form of the optimal labour demand and supply schedules f_1, f_2 (as the proof requires some preparatory results is postponed to the appendix).

Theorem 1. If the curve *P(t)* has asymptotically bounded periodic behavior $P_\infty = P(C)$, then in each period it crosses an even number of times *2n* the origin in a cross-like fashion. If in particular *2n=2* then the curve is a "ribbon".

The meaning of Theorem 1 is illustrated by fig. 3 reporting the graphical shape in the (u, $\hat{w}$) plane of the "theoretical Phillips" curve P_∞ predicted by the steady oscillating Walrasian dynamics of Fig. 2. Fig. 3 makes evident the cross-like shape. In particular the number of crossings per period in this case (and simulation suggests that this holds in most cases) is only 2, i.e. the curves *S(t), L(t)* intersect exactly twice. The intersections point *S(t)=L(t)* are mapped onto the origin. Points such that *S(t)>L(t)* or *S(t)<L(t)* are mapped onto two closed disjoint curves $P_{\infty,1}$ and $P_{\infty,2}$ living respectively only in the second and fourth quadrant. During the long-term cyclical dynamics of (3) there is a dynamics in the space $(u,\hat{w})$ which takes place on P_∞; in particular the dynamics on the $P_{\infty,1}$ branch occurs in a counter-clock wise fashion up to the origin, where the dynamics moves on the $P_{\infty,2}$ branch and becomes clock-wise as a consequence of the cross-like shape.

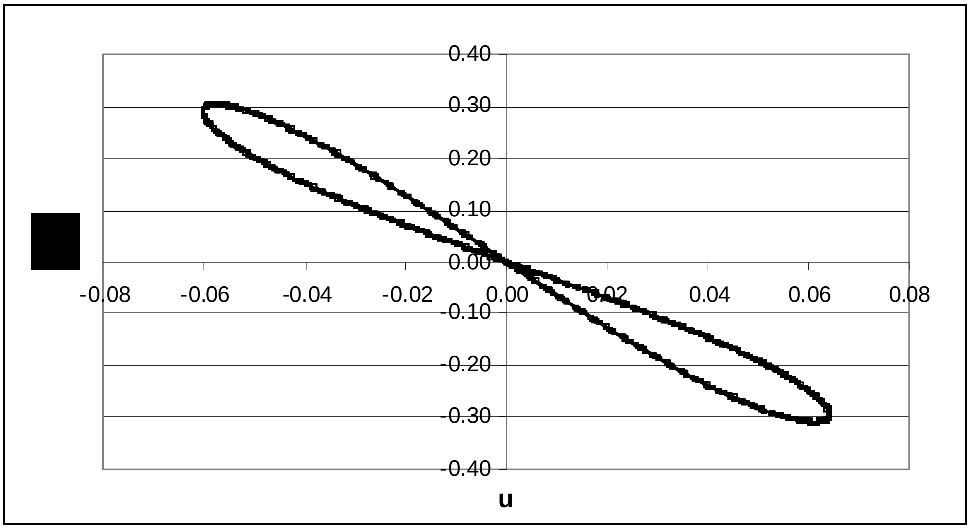

Fig. 3. Shape of the "Theoretical Phillips" curve predicted by the steady oscillating Walrasian dynamics of Fig. 2.

Remark 2. The fact that the left branch of the $P_\infty(t) = P(A)$ curve associates positive values of the unemployment rate with wage deflation is due to the simplicity of the adjustment rule (3c) which postulates that wages increase only if there is a positive excess demand. In the spirit of Remark 1, one could allow for a NAIRU, i.e. a "natural rate of inflation", which is the rate at which prices and wages increase when markets clear. In the latter case wage inflation may well coexist with positive unemployment.

Our result leads to fully new interpretations of the Phillips' curve. When the Walrasian equilibrium is destabilised our economy shows persistent excess demand in the form of stable oscillations. In this case the model predicts a persistent long term relation between the growth rate of wages and the unemployment rate. In other words the relation is not only a transitory disequilibrium phenomenon, but a steady one. The shape of this relation is “cross-like”, i.e. the ribbon in fig. 3. The points of the ribbon represent the time realizations of the Walrasian economy in the plane between the “Phillips’ variables” $\hat{w}$ and u. Data generated from this economy would likely lead to identify a decreasing statistical relationship between the “Phillips’ variables. An example is showed in fig. 4 where a linear regression curve is fitted to the scatter-plot obtained from a discrete equi-spaced time sampling of the “Theoretical Phillips’ curve” of fig. 3. We call this line the “Statistical Phillips’ curve” (SP).

Obviously the SP curve would be meaningless as a policy tool. Since the points belonging to the true underlying TP curve are just a dynamic realisation of the Walrasian economy, they cannot represent, as in the traditional view popularised in Samuelson and Solow (1960), a locus connecting different equilibrium points among which the policy maker could choose. Let us explain the point with an example. According to the traditional interpretation policy makers dispose of an estimated SP line and believe to face a real “wage inflation - unemployment” equilibrium trade-off. Thus when they judge unsatisfactory the “today” pair ($\hat{w}$,u) they choose a more desirable pair along the curve and set some economic policy measures in order to achieve the target. This measure could be direct (i.e. by directly affecting unemployment, by hiring or firing people) or indirect (i.e. moving some economic parameters which are known to affect unemployment). According to the interpretation of this paper this is a naïve approach. The direct strategy would only change the initial conditions of the Walrasian market that would return after some time to the long term TP curve. In this case the policy maker would observe a movement of the target variable $\hat{w}$ in the desired direction only by chance. The consequences of the indirect strategy would be even more unpredictable because it would imply a change in some structural parameters of the Walrasian system (3), that could imply a change in the shape of the attractor (or even its disappearance), and thus in the shape of the TP curve. The interpretation offered here therefore suggests a different perspective from previous ones, i.e. that no relationships between wage inflation and unemployment exists: what is called the Phillips curve might just be a statistical artifact. In this case the traditional view might be seriously weakened and policy makers should be cautious while providing policy recipes.

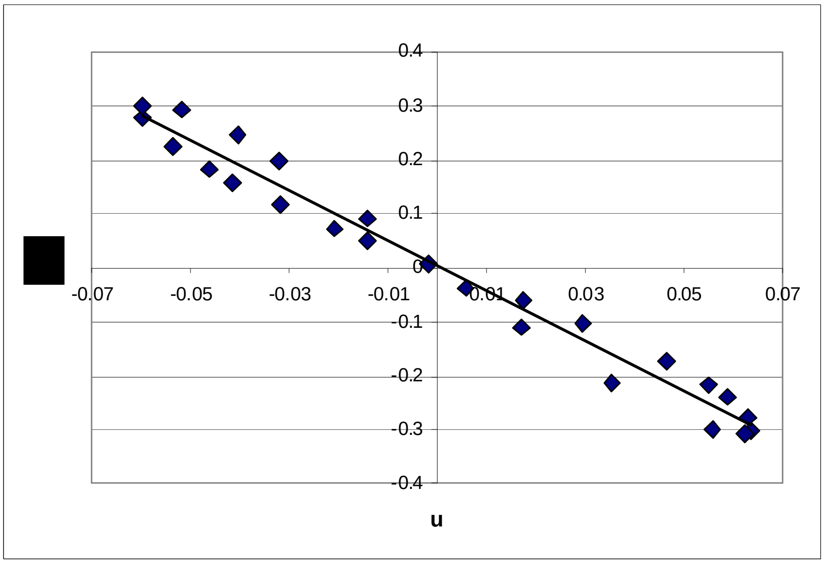

Fig. 4. Scatter-plot obtained from a discrete equi-spaced time sampling of the "Theoretical Phillips' curve" depicted in fig. 3, and related linear regression curve.

4. Discussion

The paper has shown that Walrasian labour markets with persistent excess demand provide predictions on the shape on the relation between economic time series of the growth rate of wages and unemployment. The shape of this theoretically founded relation suggests that what has been traditionally called the Phillips curve could just be a statistical artifact. Our findings have a twofold flavour for the debate on the interpretations of the Phillips curve. First, the simplest neoclassical dynamic labour market "resurrects" the Phillips curve in the long run. This reconciles, although for reasons at all different from what believed by previous interpretations, two different views: (a) the fact that the Phillips curve is correctly predicted from the law of demand and supply in the labour market but, in contrast with the neo-classical interpretations, as a long run rather than a transitory phenomenon; and (b) the fact that the Phillips curve actually is a long run phenomenon, as argued by the Keynesian view, who rather interpret it as a locus of macroeconomic equilibrium points with involuntary unemployment. Second, this long run theoretically funded Phillips curve is not anymore a policy tool as in the Keynesian view. These results are of course tentative, since for instance pertaining to specific utility and production functions; however, it is of value the provision of another interpretation of the historical inverse relationship between the rate of unemployment and the rate of inflation in an economy. Therefore these results complement those obtained by the vast literature aiming to provide a theoretical explanation of the Phillips curve and provide new light on the theoretical foundations as well policy implications of the Phillips' curve.

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Appendix: proof of theorem 1

The proof requires some preparatory result and lemmas. We assume that the trajectories *(S,L,w)* of the system are globally attracted by a limit cycle *C* of period *T*: $C=\left(S_\infty(t),L_\infty(t),w_\infty(t)\right)$. Defining the average values:

$$S_m = \int_0^T S_\infty(t)dt \quad , \quad L_m = \int_0^T L_\infty(t)dt \quad , \quad w_m = \int_0^T w_\infty(t)dt$$

(A.1)

we can write:

$$S_\infty(t) = S_m + \int_0^T \Omega_S(t)dt \quad , \quad L_\infty(t) = L_m + \int_0^T \Omega_L(t)dt \quad , \quad w_\infty(t) = w_m + \int_0^T \Omega_w(t)dt$$

(A.2)

where the functions $\Omega_i \ (i = L,S,w)$ are periodic of period *T* with zero average.

Proposition 2. If the orbit (*S(t),L(t),W(t)*) is periodic with period *T*, then *S(t),L(t)* have the same average, i.e.:

$$S_m = L_m \qquad \text{(A.3)}$$

Proof. Integrating the equation for wage adjustment over *(0,T)*:

$$0 = w(T) - w(0) = \int_0^T (L(t) - S(t))dt = L_m - S_m + \int_0^T (\Omega_L(t) - \Omega_S(t))dt = L_m - S_m$$

€ (A. 4)

Next, we need some results regarding the structure of solutions of equations of the form:

$$f_1(t) = f_2(t) \qquad t \in [0,T] \quad i = 1,2,... \qquad \text{(A.5)}$$

where $f_1(t)$ and $f_2(t)$ are zero-average T periodic continuous functions. The following lemmas hold. We only prove lemmas 4 and 5 (other proofs available on request).

Lemma 3. Equation (A.5) has at least two solutions in each period.

Lemma 4. Assume additionally that $f_1(t)$ and $f_2(t)$ are analytic functions.

If $t_i \le t_{i+1}$ are two consecutive solutions of (A.5) then:

$$f_1'(t_i) \le f_2'(t_i) \quad \Rightarrow \quad f_1'(t_{i+1}) \ge f_2'(t_{i+1}) \tag{A.6}$$

Proof. It follows immediately from the non existence of zeroes in (t_i, t_{i+1}).

Lemma 5. Assume $f_1(t)$ and $f_2(t)$ are analytic. Then (A.5) cannot have an odd number of solutions in each period.

Proof. Assume (A.5) has an odd number $2n+1$ of solutions. Let $\hat{t}$ be the last one. Let $f_1(0) > f_2(0)$. Analyticity implies we may find an ε such that:

$$f_1(\hat{t}-\varepsilon) \ge f_2(\hat{t}-\varepsilon) \quad , \quad f_1(\hat{t}+\varepsilon) \le f_2(\hat{t}+\varepsilon)$$

However, since $f_1(0) > f_2(0)$ the theorem of existence of zeros implies that there must be a further solution in (0,T) which is a contradiction with the hypothesis that $\hat{t}$ was the last. In other words, there cannot be an odd number of solutions.

We are now in the position to prove the main Theorem:

Proof. Note first that the function $P_\infty(t)$ passes through the origin only at those times at which the *L* and *S* curve intersect:

$$P_\infty(t) = (u_\infty(t), Z_\infty(t)) = (0,0) \quad \Leftrightarrow \quad L_\infty(t) = S_\infty(t) \tag{A.7}$$

Let us consider the solutions of the equation

$$L_\infty(t) = S_\infty(t) \tag{A.8}$$

Thanks to Lemma 5 the number of zeros of (A.8) in each period is even. Let t_x be one such zeros. It holds (we omit the subfix ∞ for simplicity)

$$u'(t_x)=\frac{S'(t_x)-L'(t_x)}{S(t_x)}-(L(t_x)-S(t_x))\frac{S'(t_x)}{S^2(t_x)}=\frac{S'(t_x)-L'(t_x)}{S(t_x)}$$

$$Z'(t_x)=(-1)\left(\frac{S'(t_x)-L'(t_x)}{w(t_x)}-(L(t_x)-S(t_x))\frac{w'(t_x)}{w^2(t_x)}\right)=(-1)\frac{S'(t_x)-L'(t_x)}{S(t_x)}$$

(A.9)

Thus, either it is: $sign\left(P'_\infty(t)\right)=sign\left(u'_\infty(t),Z'_\infty(t)\right)=(+,-)$ or $sign\left(P'_\infty(t)\right)=sign\left(u'_\infty(t),Z'_\infty(t)\right)=(-,+)$, unless $L'_\infty(t)=S'_\infty(t)=0$. Let now t_{2h-1} , t_{2h} , $h\geq 1$ be two consecutive zeroes of (A.8). Thanks to Lemma 4 we have either:

$$sign\left(P'_\infty(t_{2h-1})\right)=(+,-)\ \ ;\ \ sign\left(P'_\infty(t_{2h})\right)=(-,+) \qquad \text{(A.10)}$$

or:

$$sign\left(P'_\infty(t_{2h-1})\right)=(-,+)\ \ ;\ \ sign\left(P'_\infty(t_{2h})\right)=(+,-) \qquad \text{(A.11)}$$

Thus unless both the derivatives are null, the curve $P_\infty(t)$ has a cross-like shape.€

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