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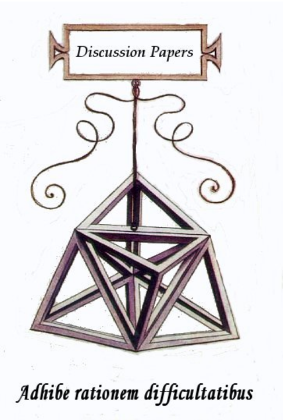

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Simplex-like sequential methods for a class of generalized fractional programs

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Discussion Paper

n. 168

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Simplex-like sequential methods for a class of generalized fractional programs

Abstract

We deal with a class of generalized fractional programming problems having a polyhedral feasible region and as objective the ratio of an affine function and the power $p > 0$ of an affine one. We aim to propose simplex-like sequential methods for finding the global maximum points. As the objective function may have local maximum points not global, we analyze the theoretical properties of the problem; in particular, we study the maximal domains of the pseudoconcavity of the function. Depending on whether or not the objective is pseudoconcave on the feasible set, we suggest different algorithms.

Classificazione JEL: C61
Classificazione AMS: 90C32, 90C30, 26B25
Keywords: Generalized fractional programming, Pseudoconcavity, Sequential methods

Contents

I. Introduction

Fractional programming (FP) problem is a special class of nonlinear programming which optimizes one or some ratio objective functions over a feasible region. FP problems usually arise for modelling real life problems such as production planning, financial, health care, and engineering (see for instance Frenk and Schaible (2005)).
Motivated by the interest on FP in the literature (see for instance the extensive bibliography in Stancu-Minasian (2006)), this paper considers a class of generalized fractional programming problems whose feasible region is a polyhedron and whose objective is the ratio of an affine function and the power p of an affine one. With the aim of solving the problem, we will give an algorithm based on the so called optimal level solutions method. This approach has been proposed, for the first time, for solving a linear fractional problem Cambini and Martein (1990), and then, it has been used for generalized class of FP (see for istance Cambini (1994), Carosi and Martein (2008), Cambini and Martein (2012), Ellero (1996)). Even if the analyzed problem may have several local optimum points which are not global, the theoretical results stated in the paper allow us to solve it by means of a simplex-like algorithm.
The paper is organized as follows: in Section II. the problem is stated, Section III. is devoted to determine the maximal domain of pseudoconcavity of the objective function; according with the different specification of problem parameters, different theoretical properties and sequential methods are proposed in Section IV. and in Section V. In order to clarify how the algorithm works, in Section 6 some easy examples are illustrated.

II. Statement of the problem

We consider the following optimization problem:

$$\begin{aligned} P: \quad \sup \quad & f(x) = \frac{c^T x + c_0}{(d^T x + d_0)^p} \\ s.t. \quad & x \in X = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\} \subset D, \end{aligned} \tag{1}$$

where $D = \{x \in \mathbb{R}^n : d^T x + d_0 > 0\}$, $p > 0$, A is a real $m \times n$ matrix, with rank$[A] = m < n$.
We aim to establish conditions under which Problem P has optimal solutions and to propose sequential methods for finding them. A key tool of our analysis is the pseudoconcavity of the objective function. It is well known that if the objective function is pseudoconcave, then local maximum points are global ones and therefore, Problem P can be solved more easily with respect to the general case. Since f is not in general pseudoconcave (see Example 4), in Section III. we will characterize the maximal domain of pseudoconcavity. The stated conditions are related to the values of the objective function parameters c, d, c_0, d_0, and p. In particular, we are going to distinguish the cases rank$[c, d] = 2$ and rank$[c, d] = 1$; we will derive different characterizations for the pseudoconcavity. The value of rank$[c, d]$ bears on the theoretical properties of Problem P and consequently, we will propose different sequential methods accordingly. Therefore, Section IV. deals with the case rank$[c, d] = 2$, while Section V. is devoted to the case rank$[c, d] = 1$.

Regardless the parameter specifications of function f, if an optimal solution exists, it belongs to a feasible edge. Furthermore, if the supremum is not attained as a maximum, then there exists an extreme direction along which the function converges to the supremum. With this regards, the following theorem holds. We omit its proof since it is similar to the one given for Theorem 3.1 in Carosi and Martein (2008).

Theorem 1 *Let L be the supremum of Problem P.*

i) *L is attained as a maximum if and only if there exists a feasible point x^0 belonging to an edge of X such that $f(x^0) = L$;*

ii) *If L is not attained as a maximum, then there exists an extreme direction u and a feasible point x^0 such that $L = \lim_{t \to +\infty} f(x^0 + tu)$.*

III. On the maximal domain of pseudoconcavity

We study the pseudoconcavity of the function

$$f(x) = \frac{c^T x + c_0}{(d^T x + d_0)^p}, \tag{2}$$

on the domain $D = \{x \in \mathbb{R}^n : d^T x + d_0 > 0\}$, $p > 0$. Throughout the paper, we will assume $p \neq 1$ and $c \in \mathbb{R}^n \setminus \{0\}$ since it is well known that, when $p = 1$ or $c = 0$, f is both pseudoconvex and pseudoconcave on D.
In what follows $\nabla f(x)$ and $H(x)$ will denote the gradient and the Hessian matrix of f evaluated at x, respectively.
For the sake of completeness, we recall the definition of a pseudoconcave function.

Definition 2 *Let f be a real-valued differentiable function defined on a convex set $C \subseteq \mathbb{R}^n$. f is said to be pseudoconcave on C if and only if*

$$\forall x^1, x^2 \in C, \quad f(x^1) < f(x^2) \Rightarrow (x^2 - x^1)^T \nabla f(x^1) > 0$$

In order to characterize the maximal domain of pseudoconcavity of f, we will use the following second order characterization (see for all Cambini and Martein (2009)).

Theorem 3 *Let f be a twice differentiable function defined on an open convex set $A \subseteq \mathbb{R}^n$. Then, f is pseudoconcave on A if and only if the following two conditions hold:*
i) $x \in A, v \in \mathbb{R}^n,\ v^T \nabla f(x) = 0 \Rightarrow v^T H(x) v \leq 0$;
ii) If $x^0 \in A$ is a critical point, then x^0 is a local maximum point for f on A.

According to Theorem 3, we have to analyze the behavior of the Hessian matrix on the directions which are orthogonal to the gradient and we have to establish whether the critical points are maximum points. The gradient and Hessian matrix are as follows:

$$\nabla f(x) = \frac{c(d^T x + d_0) - pd(c^T x + c_0)}{(d^T x + d_0)^{p+1}}, \tag{3}$$

$$H(x) = \frac{p[(d^T x + d_0)(-cd^T - dc^T) + (p+1)(c^T x + c_0)dd^T]}{(d^T x + d_0)^{p+2}}. \quad (4)$$

We will separately consider the case $\text{rank}[c, d] = 2$ and the case $\text{rank}[c, d] = 1$.
Assume first that $\text{rank}[c, d] = 2$; the following example shows that f is not in general pseudoconcave. The example takes a function where $p > 1$, but similar examples can be constructed for the case $0 < p < 1$.

Example 4 *Consider* $f(x) = \dfrac{-x_1 + x_2 - 2}{(x_1 + x_2 + 1)^p}$, $p > 1$ *and its restriction* φ *on the half-line* $x_2 = 2$ *and* $x_1 > -3$, *i.e.,* $\varphi(x_1) = \dfrac{-x_1}{(x_1 + 3)^p}$. *It can be easily verified that* $x_1 = \dfrac{3}{p-1} > 0$ *is a minimum point for* φ. *Hence,* φ *is not pseudoconcave on the half-line and, therefore,* f *is not pseudoconcave on* D, $\forall p > 1$.

Taking into account (3), condition $\text{rank}[c, d] = 2$ implies the non-existence of critical points. Thus, f is pseudoconcave on D if and only if condition i) in Theorem 3 holds.
By setting

$$D^1_+ = D \cap \{x \in \mathbb{R}^n : c^T x + c_0 \geq 0\}$$

and

$$D^1_- = D \cap \{x \in \mathbb{R}^n : c^T x + c_0 \leq 0\}$$

the following theorem characterizes the maximal domain of pseudoconcavity of f.

Theorem 5 *Assume* $\text{rank}[c, d] = 2$.
i) If $p > 1$, *then* f *is pseudoconcave on* D^1_+.
ii) If $0 < p < 1$, *then* f *is pseudoconcave on* D^1_-.

Proof Let us first observe that the linear independence of vectors c and d implies that both D^1_+ and D^1_- are non-empty sets. From (3), $\text{rank}[c, d] = 2$ implies $\nabla f(x) \neq 0, \forall x \in D$. Let $v \neq 0$ be a direction such that $\nabla f(x)^T v = 0$. It implies that $c^T v = \dfrac{p(c^T x + c_0)}{(d^T x + d_0)} d^T v$

and $v^T H(x)v = \dfrac{p}{(d^T x + d_0)^{p+2}}((1-p)(c^T x + c_0))(d^T v)^2$. Then, $v^T H(x)v \leq 0$ for every $x \in D$ if and only if $(1-p)(c^T x + c_0) \leq 0$. Consequently, from condition i) of Theorem 3, f is pseudoconcave on $\mathrm{Int}D^1_+$ if and only if $p > 1$ and f is pseudoconcave on $\mathrm{Int}D^1_-$ if and only if $0 < p < 1$.
Consider now, $x \in D^1_+ \cap D^1_-$.
For every $x^2 \in \mathrm{Int}D^1_+$, we have $f(x) < f(x^2)$, and $(x^2 - x)^T \nabla f(x) = \dfrac{c^T x^2 + c_0}{(d^T x + d_0)^p} > 0$. For every $x^2 \in \mathrm{Int}D^1_-$, we have $f(x) > f(x^2)$ and $(x - x^2)^T \nabla f(x^2) = -\dfrac{c^T x^2 + c_0}{(d^T x + d_0)^p} > 0$. Consequently, taking into account Definition 2, the proof is complete. □

The particular structure of the function allows us to easily characterize the maximal domain of the pseudoconvexity too.

Theorem 6 *Assume* $\mathrm{rank}[c, d] = 2$.
i) If $p > 1$*, then* f *is pseudoconvex on* D^1_-.
ii) If $0 < p < 1$*, then* f *is pseudoconvex on* D^1_+.

Proof Since f has no critical points, f is pseudconvex on an open convex set A if and only if for every $x \in A, v \in \mathbb{R}^n$, it is $v^T \nabla f(x) = 0 \Rightarrow v^T H(x)v \geq 0$.
Recalling that $v^T H(x)v = \dfrac{p}{(d^T x + d_0)^{p+2}}((1-p)(c^T x + c_0))(d^T v)^2$, the proof is obtained along the lines of the previous theorem. □

Consider now the case $\mathrm{rank}[c, d] = 1$, i.e, $c = \gamma d$, $\gamma \neq 0$. Then, we have

$$f(x) = \frac{\gamma d^T x + c_0}{(d^T x + d_0)^p},$$

and the gradient and Hessian matrix can be specified as follows:

$$\nabla f(x) = \frac{d}{(d^T x + d_0)^{p+1}}[\gamma(1-p)(d^T x + d_0) - p(c_0 - \gamma d_0)], \quad (5)$$

$$H(x) = \frac{dd^T}{(d^T x + d_0)^{p+2}} p[\gamma(p-1)(d^T x + d_0) + (p+1)(c_0 - \gamma d_0)]. \quad (6)$$

The following theorem characterizes the pseudoconcavity of f on D.

Theorem 7 *Assume* $\mathrm{rank}[c, d] = 1$ *and* $c = \gamma d$, $\gamma \neq 0$.
Then, f is pseudoconcave on D if and only if one of the following conditions holds:
i) $\gamma(1-p) < 0$*;*
ii) $\gamma(1-p) > 0$ *and* $c_0 - \gamma d_0 \leq 0$.

Proof Setting $d^T x + d_0 = z$, function f becomes $\eta(z) = \dfrac{\gamma(z-d_0)+c_0}{z^p}$ and we have $\eta'(z) = \dfrac{\gamma(1-p)z - p(c_0-\gamma d_0)}{z^{p+1}}$.
Let $x^1, x^2 \in D$ and set $z_1 = d^T x^1 + d_0$ and $z_2 = d^T x^2 + d_0$. Then, $f(x^1) < f(x^2)$ if and only if $\eta(z_1) < \eta(z_2)$. Furthermore, $\nabla f(x) = \eta'(z) d$ and, consequently, $\nabla f(x^1)^T(x^2 - x^1) > 0$ if and only if $\eta'(z_1) d^T (x^2 - x^1) = \eta'(z_1)(z_2 - z_1) > 0$. Therefore, f is pseudoconcave on D if and only if η is pseudoconcave on $(0, +\infty)$.
Since η is a single variable function, η is pseudoconcave if and only if its critical points are maximum points.
If $\gamma(1-p) < 0$, then either η is decreasing on $(0, +\infty)$ or it has a critical point which is a maximum point. If $\gamma(1-p) > 0$ and $c_0 - \gamma d_0 > 0$, then η has a feasible critical point which is a minimum point. If $\gamma(1-p) > 0$ and $c_0 - \gamma d_0 \leq 0$, then η is increasing on $(0, +\infty)$.
Consequently, η is pseudoconcave in $(0, +\infty)$ if and only if i) or ii) holds. The proof is complete. □

Remark 8 *Taking into account (6), it is easy to verify that if ii) of Theorem 7 holds, then f is concave on D.*

As a direct conseguence of Theorem 7, we are able to characterize the pseudoconvexity of the function.

Theorem 9 *Assume* $\mathrm{rank}[c, d] = 1$ *and* $c = \gamma d$, $\gamma \neq 0$.
Then, f is pseudoconvex on D if and only if $\gamma(1-p) > 0$ *and* $c_0 - \gamma d_0 > 0$.

IV. Case rank*[c, d] = 2. *Theoretical properties and sequential method

We consider the case rank$[c, d] = 2$. First some theoretical properties of problem P are established and, successively, a simplex-like sequential method is suggested for solving the problem.

Theorem 10 *Assume* rank$[c, d] = 2$, $p > 1$.
i) If $X \cap \{x \in \mathbb{R}^n : c^T x + c_0 \geq 0\} \neq \emptyset$, then, the supremum L of problem P is $L = +\infty$ if and only if there exists an extreme direction u such that $c^T u > 0$ and $d^T u = 0$. In any other case, the supremum is attained as a maximum.
ii) If $X \subset \{x \in \mathbb{R}^n : c^T x + c_0 < 0\}$, then, the supremum L of problem P is $L = 0$ if and only if there exists an extreme direction u such that $d^T u > 0$. In any other case the supremum is attained as a maximum.

Proof From ii) of Theorem 1, we have to study the behaviour of f along every extreme direction u of X, i.e., $d^T u \geq 0$.
Let φ be the restriction of f on the half-line $x = x^0 + tu,\ t \geq 0$, $x^0 \in X$, i.e.,

$$\varphi(t) = \frac{c^T x^0 + tc^T u + c_0}{(d^T x^0 + td^T u + d_0)^p}. \tag{7}$$

We are going to prove i) and ii) separately.
i) Without loss of generality, we can assume that $c^T x + c_0 \geq 0$, $\forall x \in X$. Hence, $c^T u \geq 0$ for any extreme direction u.
If $d^T u = 0$ and $c^T u = 0$, then $\varphi(t) = f(x^0),\ \ \forall t \geq 0$; if $d^T u > 0$, then $\lim\limits_{t \to +\infty} \varphi(t) = 0$.
Consequently, $L = +\infty$ if and only if there exists an extreme direction u such that $d^T u = 0$ and $c^T u > 0$.
ii) Since $c^T x + c_0 < 0,\ \forall x \in X$, it results $c^T u \leq 0$ for any extreme direction u. If $d^T u = 0$ and $c^T u = 0$, then $\varphi(t) = f(x^0),\ \forall t \geq 0$; if $d^T u = 0$ and $c^T u < 0$, then $\lim\limits_{t \to +\infty} \varphi(t) = -\infty$; if $d^T u > 0$, then $\lim\limits_{t \to +\infty} \varphi(t) = 0$. Since $f(x) < 0\ \forall x \in X$, $L = 0$ is the supremum if and only if $d^T u > 0$.

□

Theorem 11 *Assume* $\operatorname{rank}[c,d] = 2$, $0 < p < 1$.
i) If $X \subset \{x \in \mathbb{R}^n : c^T x + c_0 < 0\}$*, then, the supremum* L *of problem* P *is* $L = 0$ *if and only if there exists an extreme direction* u *such that* $d^T u > 0$ *and* $c^T u = 0$*. In any other case the supremum is attained as a maximum.*
ii) If $X \cap \{x : c^T x + c_0 = 0\} \neq \emptyset$*, then, the supremum of problem* P *is* $L = +\infty$ *if and only if there exists an extreme direction* u *such that* $c^T u > 0$*. In any other case the supremum is attained as a maximum.*

Proof Take an extreme direction u, i.e., $d^T u \geq 0$ and the restriction φ of f on the half-line $x = x^0 + tu,\ t \geq 0,\ x^0 \in X$. We are going to prove i) and ii) separately.
i) We have $c^T u \leq 0$. If $c^T u = 0$ and $d^T u = 0$, then $\varphi(t) = f(x^0)$, $\forall t \geq 0$; if $c^T u < 0$, then $\lim_{t \to +\infty} \varphi(t) = -\infty$; if $c^T u = 0$ and $d^T u > 0$ then $\lim_{t \to +\infty} \varphi(t) = 0$. Consequently, $L = 0$ if and only if $c^T u = 0$ and $d^T u > 0$.
ii) If $c^T u < 0$, then $\lim_{t \to +\infty} \varphi(t) = -\infty$; if $c^T u = 0$ and $d^T u = 0$, then $\varphi(t) = f(x^0),\ \forall t \geq 0$; if $c^T u = 0$ and $d^T u > 0$, then $\lim_{t \to +\infty} \varphi(t) = 0$. Consequently, $L = +\infty$ is if and only if there exists an extreme direction u such that $c^T u > 0$. □

When $\operatorname{rank}[c,d] = 2$ and f is not pseudoconcave, the following theorem shows that if Problem P takes the maximum, then it is attained at a vertex.

Theorem 12 *Assume* $\operatorname{rank}[c,d] = 2$*, and one of the following conditions holds:*
i) $p > 1$ *and* $X \subset \{x \in \mathbb{R}^n : c^T x + c_0 \leq 0\}$.
ii) $0 < p < 1$ *and* $X \subset \{x \in \mathbb{R}^n : c^T x + c_0 \geq 0\}$.
If the supremum of problem P *is attained as a maximum, then there exists a vertex of* X *which is a maximum point.*

Proof From Theorem 6, f is pseudoconvex. The result follows from the properties of pseudoconvex functions (see for all Cambini and

Martein (2009)). Nevertheless, for sake of completeness we provide an independent proof for the case i).
Let x^0 be an optimal solution. Since $\nabla f(x) \neq 0, \forall x \in X$, x^0 belongs to the boundary of X. If x^0 is a vertex, then there is nothing to prove. Otherwise consider the restriction $\varphi(t)$ of f on an edge containing x^0, i.e., $x = x^0 + tu$, $t \in (-\epsilon, \epsilon)$. We have

$$\varphi'(t) = \frac{t(1-p)c^T u d^T u + c^T u(d^T x^0 + d_0) - p d^T u(c^T x^0 + c_0)}{(d^T x^0 + t d^T u + d_0)^{p+1}}.$$

Since x^0 is an optimal solution, then $t = 0$ is a local maximum point for φ so that $\varphi'(0) = 0$. If $d^T u = 0$, then, necessarily, we have $c^T u = 0$, otherwise $\dfrac{c^T u}{d^T u} = p\dfrac{c^T x^0 + c_0}{d^T x^0 + d_0} \leq 0$ and $(1-p)c^T u d^T u \leq 0$. Since $p > 1$, necessarily we have $(1-p)c^T u d^T u = 0$. Consequently, f is constant on the edge so that there exists a vertex which is optimal.
$\square$

- **Sequential method**

We are going to propose a sequential method which allows to solve the problem either the objective function is pseudoconcave on the feasible set X or not. Referring to Theorems 5 and 6, we have to analyze four different cases.
Note that, when function f is not pseudoconcave on X, solving problem P on X means to look for a maximum of a pseudoconvex function or, eqivalently, to look for a minimum of a pseudoconcave function. This last problem, like finding a minimum for a concave function, is a hard problem. Nevertheless, taking into account the particular structure of f, we can suggest a simple sequential method for solving P also in this case.
The proposed solution method extends the so called “optimal level solutions” approach (see Cambini and Martein (1990)).
First observe that the denominator function $d^T x + d_0$ is lower bounded

on X. Thus, the linear problem

$$P_d : \min_{x\in X} (d^T x + d_0),$$

has optimal solutions. Let $\theta_{\min}$ be the minimum value of P_d. Consider the linear program

$$P_c : \max\ (c^T x + c_0),\ x \in X \cap \{x : d^T x + d_0 = \theta_{\min}\}.$$

Assume that the supremum of P_c is finite and let x_0 be a vertex of X which is an optimal solution of P_c (note that if P_c does not have solutions, then the supremum of problem P is $+\infty$). Starting from x_0, we suggest an algorithm for determining a local maximum point (if one exists) of problem P.
Consider the linear parametric problem

$$P(\theta) :\ \psi(\theta) = \max\ (c^T x + c_0),\ x \in X(\theta) = X \cap \{x : d^T x = d^T x_0 + \theta\}$$

and set $\Theta = \{\theta :\ X(\theta) \neq \emptyset\} = [0, \theta_{\max}]$, where $\theta_{\max}$ may be $+\infty$. We have

$$\max_{x\in X} f(x) = \max_{\theta\in\Theta} \max_{x\in X(\theta)} f(x).$$

Setting $h(\theta) = \max_{x\in X(\theta)} f(x)$, it results

$$\max_{x\in X} f(x) = \max_{\theta\in\Theta} h(\theta),\ h(\theta) = \frac{\psi(\theta)}{(\theta_{\min} + \theta)^p}.$$

If $h(\theta)$ increases (decreases), then the function $f(x)$ increases (decreases) so that a local maximum of $h(\theta)$ corresponds to a local maximum of $f(x)$.
The following sequential method determines (if one exists) a local maximum point for $h(\theta)$.
The idea is the following: corresponding to the vertex x_0, which is an optimal solution of $P(\theta_0)$, $\theta_0 = 0$, denote by B_0 the set of indices associated with the basic variables and set $x_0 = (x_{B_0}, 0)$. Applying sensitivity analysis, we find $(x_{B_0}(\theta), 0) = (x_{B_0} + \theta u_{B_0}, 0)$ which is optimal for $P(\theta)$ for every θ belonging to the stability interval $[\theta_0, \theta_1] = \{\theta :\ x_{B_0}(\theta) \geq 0\}$. If $h'(\theta_0) \leq 0$, then $(x_{B_0}, 0)$ corresponds

to a local maximum point of P. If there exists $\widetilde{\theta} \in [\theta_0, \theta_1]$ such that $h'(\widetilde{\theta}) = 0$, then $(x_{B_0}(\widetilde{\theta}), 0)$ corresponds to a local maximum point of P which belongs to an edge of X. In any other case, for $\theta > \theta_1$ the feasibility is lost and it is restored by applying an iteration of the dual simplex algorithm. We find a new stability interval and we repeat the analysis. Proceeding in this way, we develop a finite sequence of basis B_k, $k = 0, 1, ...$ and a finite number of stability intervals $[\theta_k, \theta_{k+1}]$, $k = 0, 1, ...$

With the usual notations, corresponding to the basis B_k, we have:
$x(\theta) = (x_{B_k}(\theta), 0) = (x_{B_k} + \theta u_{B_k}, 0)$,
$\psi(\theta) = c^T_{B_k} x_{B_k} + \theta c^T_{B_k} u_{B_k} + c_0$, $\theta \in [\theta_k, \theta_{k+1}]$ so that

$$h(\theta) = \frac{c^T_{B_k} x_{B_k} + \theta c^T_{B_k} u_{B_k} + c_0}{(\theta_{\min} + \theta)^p}, \ \theta \in [\theta_k, \theta_{k+1}], \tag{8}$$

$$h'(\theta) = \frac{(1-p) c^T_{B_k} u_{B_k} \theta + \xi_{B_k}}{(\theta_{\min} + \theta)^{p+1}}, \theta \in [\theta_k, \theta_{k+1}]. \tag{9}$$

where $\xi_{B_k} = \theta_{\min} c^T_{B_k} u_{B_k} - p(c^T_{B_k} x_{B_k} + c_0)$.

A local maximum point θ_k of $h(\theta)$ corresponds to a global maximum for P if f is pseudoconcave, or when f is pseudoconvex and $h'(\theta) < 0$ for every $\theta \geq \theta_k$. In the other cases, we have to look for another value $\widetilde{\theta}_k$ of θ such that $h(\widetilde{\theta}_k) = h(\theta_k)$. The uniqueness of $\widetilde{\theta}_k > 0$ is guaranteed by the pseudoconvexity of $h(\theta)$, together with $\lim\limits_{\theta \to +\infty} h(\theta) = 0$ (if $p > 1$) or $\lim\limits_{\theta \to +\infty} h(\theta) = +\infty$ (if $0 < p < 1$).

We make a jump setting $d^T x + d_0 = \theta_{\min} + \widetilde{\theta}_k + \theta$ and we find a new optimal level solution by solving the problem

$$\widetilde{P}(\theta): \ \psi(\theta) = \max_{x \in X(\theta)} (c^T x + c_0)$$

where $X(\theta) = X \cap \{x : d^T x + d_0 = \theta_{\min} + \widetilde{\theta}_k + \theta\}$.

Observe that if $\widetilde{P}(\theta)$ does not have solutions, then $x(\theta_k)$ is a global maximum point for P.

As a final remark, let us observe that, with respect to the original problem P, the parametric problem $P(\theta)$ has the additional constraint $d^T x = d^T x_0 + \theta$. This leads to the introduction of an

additional slack variable x_{n+1}. According to the idea of the optimal level solution method, for any value of θ, every optimal solution of $P(\theta)$ is binding to the parametric constraint, so that there exists a basic optimal solution $(x_{B_0}, 0)$ such that x_{n+1} is a non-basic variable. Therefore, with a little abuse of notation, we will refer to $(x_{B_0}, 0)$ as a basic solution of the original Problem P.
As it has been outlined before, we have to deal with four different cases, i.e., we have four different procedures inside the main algorithm. With respect to the case $p > 1$, Procedure A is related to the pseudoconcave case, while Procedure B to the pseudocovex case; with respect to the case $0 < p < 1$, Procedure C is related to the pseudoconcave case, while Procedure D to the pseudocovex case.

The main algorithm

Step 0. Compute $C_{\max} = \sup\limits_{x \in X}(c^T x + c_0)$. If $p > 1$, then go to Step 1, else go to Step 2.

Step 1. If $C_{\max} \geq 0$, then Procedure A, else Procedure B.

Step 2. If $C_{\max} = +\infty$, then STOP: the supremum of Problem P is $+\infty$ too, otherwise go to Step 3.

Step 3. If $C_{\max} \leq 0$, then Procedure C, else Procedure D.

Procedure A: pseudoconcave case, $p > 1$

Step 0. Solve problem P_d and let $\theta_{\min}$ be its optimal value. Solve Problem P_c; if P_c does not have solutions, STOP: the supremum of P is $+\infty$. Otherwise let x_0 be an optimal solution of P_c which is also an optimal solution of Problem $P(\theta_0)$ with $\theta_0 = 0$. Set $k = 0$ and go to Step 1.

Step 1. Determine the stability interval $[\theta_k, \theta_{k+1}]$ associated with the optimal solution $x(\theta_k) = (x_{B_k} + \theta_k u_{B_k}, 0)$ of $P(\theta_k)$.
Compute $h'(\theta_k)$. If $h'(\theta_k) \leq 0$, STOP: $x(\theta_k)$ is the optimal solution of P otherwise go to Step 2.

Step 2. Compute $\widetilde{\theta} = -\dfrac{\xi_{B_k}}{(1-p)c_{B_k}^T u_{B_k}}$. If $\widetilde{\theta} \in (\theta_k, \theta_{k+1}]$, STOP: $x(\tilde{\theta})$ is the optimal solution of P; otherwise let i be such that $x_{B_{k_i}} + \theta_{k+1} u_{B_{k_i}} = 0$. Perform a pivot operation by means of the dual simplex algorithm, set $k = k+1$ and go to Step 1.

Procedure B: pseudoconvex case, $p > 1$

Step 0. Determine $\theta_{\max} = \sup_{x \in X} d^T x$. If $\theta_{\max} = +\infty$, then STOP: the supremum of Problem P is 0. Otherwise go to Step 1

Step 1. Solve problem P_d and let $\theta_{\min}$ be its optimal value. If $\theta_{\min} - d_0 > 0$, then set $\theta_{\max} = \theta_{\max} - (\theta_{\min} - d_0)$. Solve problem P_c and let x_0 be an optimal solution of Problem P_c which is also an optimal solution of Problem $P(\theta_0)$ with $\theta_0 = 0$. Set $k = 0$, $X^M = \emptyset$, and $\mathrm{Val}_f = -\infty$. Go to Step 2.

Step 2. Determine the stability interval $[\theta_k, \theta_{k+1}]$ associated with the optimal solution $x(\theta_k) = (x_{B_k}(\theta_k), 0) = (x_{B_k} + \theta_k u_{B_k}, 0)$ of $P(\theta_k)$. Compute $h'(\theta_k)$. If $h'(\theta_k) \geq 0$, go to Step 4. Otherwise $x(\theta_k)$ is a local maximum point for f. If $f(x(\theta_k)) > \mathrm{Val}_f$ then $X^M = \{x(\theta_k)\}$; if $f(x(\theta_k)) = \mathrm{Val}_f$, then $X^M = X^M \cup \{x(\theta_k)\}$. Go Step 3.

Step 3. Solve $h(\theta) = h(\theta_k)$, and let $\tilde{\theta}_k > \theta_k$ be the solution. If $\tilde{\theta}_k > \theta_{\max}$, then STOP: X^M is the set of optimal solutions. Otherwise if $\tilde{\theta}_k \leq \theta_{k+1}$, go to Step 4, else if $\theta_{k+1} < \tilde{\theta}_k \leq \theta_{\max}$, set $\theta = \theta + \tilde{\theta}_k$, $\theta_{\max} = \theta_{\max} - \widetilde{\theta}_k$; by means of an iteration of the dual simplex algorithm find a new feasible solution for Problem $P(\theta)$, compute the associated stability interval $[\theta_k, \theta_{k+1}]$ and go to Step 4.

Step 4. If $\theta_{k+1} = \theta_{\max}$ then $k = k+1$ and go to Step 5. Otherwise, let i be such that $x_{B_i} + \theta_{k+1} u_{B_{k_i}} = 0$. Perform a pivot operation by means of dual simplex algorithm. Set $k = k+1$ and go to Step 2.

Step 5. If $f(x(\theta_k)) > \mathrm{Val}_f$ then $X^M = \{x(\theta_k)\}$; if $f(x(\theta_k)) = \mathrm{Val}_f$, then $X^M = X^M \cup \{x(\theta_k)\}$. STOP: X^M is the set of optimal solutions

Procedure C, pseudoconcave case, $0 < p < 1$

Step 0. Solve problem P_d and let $\theta_{\min}$ be its optimal value. Solve problem P_c and let x_0 be an optimal solution of Problem P_c which is also an optimal solution of Problem $P(\theta_0)$ with $\theta_0 = 0$. Set $k = 0$ and go to Step 1.

Step 1. Determine the stability interval $[\theta_k, \theta_{k+1}]$ associated with the optimal solution $x(\theta_k) = (x_{B_k} + \theta_k u_{B_k}, 0)$ of $P(\theta_k)$.
Compute $h'(\theta_k)$. If $h'(\theta_k) \leq 0$, STOP: $x(\theta_k)$ is the optimal solution of P, otherwise go to Step 2.

Step 2. If $h'(\theta) > 0$, $\forall \theta \in [\theta_k, \theta_{k+1}]$ and $\theta_{k+1} = +\infty$, then STOP: the supremum of problem P is 0.
If $h'(\theta) > 0$, $\forall \theta \in [\theta_k, \theta_{k+1}]$ and $\theta_{k+1} < +\infty$, then let i be such that $x_{B_{k_i}} + \theta_{k+1} u_{B_{k_i}} = 0$. Perform a pivot operation by means of the dual simplex algorithm, set $k = k + 1$ and go to Step 1.
If there exists $\tilde{\theta} \in (\theta_k, \theta_{k+1}]$ such that $h'(\tilde{\theta}) = 0$, then STOP: $x(\tilde{\theta})$ is an optimal solution of P.

Procedure D: pseudoconvex case, $0 < p < 1$

Step 0. Determine optimal value $\theta_{\max}$ of problem $\max_{x \in X} d^T x$. Solve problem P_d and let $\theta_{\min}$ be its optimal value. If $\theta_{\min} - d_0 > 0$, then set $\theta_{\max} = \theta_{\max} - (\theta_{\min} - d_0)$. Suppose x_0 be an optimal solution of Problem P_c which is also an optimal solution of Problem $P(\theta_0)$ with $\theta_0 = 0$.
Set $k = 0$ and $X^M = \emptyset$, $\mathrm{Val}_f = -\infty$ and go to Step 1.

Step 1. Determine the stability interval $[\theta_k, \theta_{k+1}]$ associated with the optimal solution $x(\theta_k) = (x_{B_k}(\theta_k), 0) = (x_{B_k} + \theta_k u_{B_k}, 0)$ of $P(\theta_k)$. Compute $h'(\theta_k)$. If $h'(\theta_k) \geq 0$, go to Step 3. Otherwise $x(\theta_k)$ is a local maximum point for f. If $f(x(\theta_k)) > \mathrm{Val}_f$ then $X^M = \{x(\theta_k)\}$; if $f(x(\theta_k)) = \mathrm{Val}_f$, then $X^M = X^M \cup \{x(\theta_k)\}$.

If $h'(\theta) < 0$ for $\theta > \theta_k$, then STOP: X^M is the set of optimal solutions, otherwise go Step 2.

Step 2. Solve $h(\theta) = h(\theta_k)$, and let $\tilde{\theta}_k > \theta_k$ be the solution.
If $\tilde{\theta}_k > \theta_{\max}$, then STOP: X^M is the set of optimal solutions.
Otherwise if $\tilde{\theta}_k \le \theta_{k+1}$, go to Step 3, else if $\theta_{k+1} < \tilde{\theta}_k \le \theta_{\max}$, set $\theta = \theta + \tilde{\theta}_k$ and $\theta_{\max} = \theta_{\max} - \tilde{\theta}_k$; by means of dual simplex algorithm find a new feasible solution for Problem $P(\theta)$. Determine the new stability interval $[\theta_k, \theta_{k+1}]$ associated with the feasible solution and go to Step 3.

Step 3. If $\theta_{k+1} = \theta_{\max}$ then $k = k+1$ and go to Step 4. Otherwise, set i be such that $x_{B_i} + \theta_{k+1} u_{B_{k_i}} = 0$. Perform a pivot operation by means of dual simplex algorithm. Set $k = k+1$ and go to Step 1.

Step 4. If $f(x(\theta_k)) > \mathrm{Val}_f$, then $X^M = \{x(\theta_k)\}$; otherwise if $f(x(\theta_k)) = \mathrm{Val}_f$, then $X^M = X^M \cup \{x(\theta_k)\}$ is the set of optimal solutions. STOP: X^M is the set of optimal solutions.

Remark 13 *When $p > 1$, $X \cap \{x \in \mathbb{R}^n : c^T x + c_0 \ge 0\} \ne \emptyset$ and there exists $x \in X$ such that $c^T x + c_0 < 0$, then f is not pseudoconcave on X (see Theorem 5). Nevertheless, Problem P is equivalent to the following one:*

$$\begin{array}{ll} \sup & f(x) = \dfrac{c^T x + c_0}{(d^T x + d_0)^p} \\ s.t. & x \in X_1 = \{x \in X : c^T x + c_0 \ge 0\} \end{array} \tag{10}$$

Since f is pseudoconcave on X_1, we can apply, also in this case, Procedure A for solving the original problem. Similarly, for the case $0 < p < 1$, $X \cap \{x \in \mathbb{R}^n : c^T x + c_0 \le 0\} \ne \emptyset$, and there exists $x \in X$ such that $f(x) > 0$, Problem P is equivalent to Problem 10. In this case f is pseudoconvex on X_1, so that it is possible to apply Procedure B for solving the problem.

Remark 14 *We would like to point out that in Cambini (1994) it is considered a more general formulation of Problem P, namely*

$p \in \mathbb{R}$. Nevertheless the theoretical properties stated before allow us to obtain a sequential method easier to be handled. The main differences are related to Procedures B and D when a jump is needed. Furthermore our algorithm proposes faster stop criteria when the Problem may have no solutions.

V. Case rank$[\mathbf{c}, \mathbf{d}] = 1$. Theoretical properties and sequential method

We consider function f when the vectors c and d are linearly dependent, i.e., $c = \gamma d$. Setting $z = d^T x + d_0$, the behavior of function f on the domain D can be studied by means of the behavior of the one variable function $\eta(z)$ on $(0, +\infty)$:

$$\eta(z) = \frac{\gamma(z - d_0) + c_0}{z^p}, \tag{11}$$

whose derivative is

$$\eta'(z) = \frac{\gamma(1-p)z - p(c_0 - \gamma d_0)}{z^{p+1}}. \tag{12}$$

The particular structure of the function allows us to establish when the supremum of problem P is not attained as a maximum and, when this is not the case, it allows us to completely characterize the set of the optimal solutions. The results are provided regardless the objective function is pseudoconcave or not.

Set $z^* = \dfrac{p(c_0 - \gamma d_0)}{\gamma(1-p)}$, $z_{\min} = \min\limits_{x \in X} (d^T x + d_0)$, $z_{\max} = \max\limits_{x \in X} (d^T x + d_0)$, where $z_{\max}$ may be also equal to $+\infty$.

The following theorems hold.

Theorem 15 *Assume* $\operatorname{rank}[c, d] = 1$ *and f is pseudoconcave on D.*
i) If $z_{\max} = +\infty$, $p > 1$, then the supremum of problem P is $L = 0$ and it is not attained as a maximum.
ii) If $z_{\max} = +\infty$, $0 < p < 1$, $\gamma > 0$ then, the supremum of Problem P is $L = +\infty$.
In any other case, the supremum is attained as a maximum.

Proof Consider first the case $\gamma(1-p) > 0$ and $c_0 - \gamma d_0 < 0$.
From (12), the pseudoconcavity of f implies that $\eta(z)$ is increasing in $(0, +\infty)$.
If $z_{\max} = +\infty$, then $\lim\limits_{z \to +\infty} \eta(z)$ is equal to $+\infty$ when $0 < p < 1$ or it is equal to zero when $p > 1$, consequently i) and ii) hold.
If $z_{\max} < +\infty$, then the optimal solutions of P correspond to the optimal solutions of the problem $\max\limits_{x \in X} (d^T x + d_0)$.
In the case $\gamma(1-p) < 0$, function $\eta(z)$ has a maximum point at z^* and the supremum of problem P is attained as a maximum. □

The proof of Theorem 15 allows us to specify the set of optimal solutions when the supremum of Problem P is attained as a maximum.

Theorem 16 *Assume* $\operatorname{rank}[c, d] = 1$*, f is pseudoconcave on D.*
i) If $\gamma(1-p) > 0$ and $z_{\max} < +\infty$, then the optimal solutions of P correspond to the optimal solutions of the problem $\max\limits_{x \in X} (d^T x + d_0)$.
ii) If $\gamma(1-p) < 0$ and $z^* \in [z_{\min}, z_{\max}]$ then optimal solutions of P are the intersection between X and the set $\{x \in \mathbb{R}^n : d^T x + d_0 = z^*\}$.
iii) If $\gamma(1-p) < 0$ and $z^* < z_{\min}$, then the optimal solutions of P correspond to the optimal solutions of the problem $\min\limits_{x \in X} (d^T x + d_0)$.
iv) If $\gamma(1-p) < 0$ and $z^* > z_{\max}$, then the optimal solutions of P correspond to the optimal solutions of the problem $\max\limits_{x \in X} (d^T x + d_0)$.

Theorem 17 *Assume* $\operatorname{rank}[c, d] = 1$ *and f is not pseudoconcave on D.*
i) If $z_{\max} = +\infty$, $p > 1$, and $z_{\min} > -\dfrac{c_0}{\gamma} + d_0$, then the supremum of problem P is $L = 0$ and it is not attained as a maximum.
ii) If $z_{\max} = +\infty$, $0 < p < 1$, then, the supremum of Problem P is $L = +\infty$.
In any other case, the maximum value of f is

$$\max\left\{\frac{\gamma z_{\min} - \gamma d_0 + c_0}{z_{min}^p}, \frac{\gamma z_{\max} - \gamma d_0 + c_0}{z_{\max}^p}\right\}.$$

Proof Taking into account Theorem 7, f is not pseudoconcave if and only if $\gamma(1-p) > 0$ and $c_0 - \gamma d_0 > 0$.
If $z_{\max} = +\infty$ and $0 < p < 1$, then $\lim_{z\to+\infty} \eta(z) = +\infty$ and ii) holds.
If $z_{\max} = +\infty$ and $p > 1$, then $\lim_{z\to+\infty} \eta(z) = 0$ and the supremum of P is zero if and only if $\eta(z) < 0$, $\forall z \in (0, +\infty)$, i.e., if and only if $z_{\min} > -\frac{c_0}{\gamma} + d_0$, thus i) holds. Otherwise the optimal solutions of P correspond to the optimal solutions of the problem $\min_{x\in X} (d^T x + d_0)$.
If $z_{\max} < +\infty$, then the optimal solutions of the two problems: $\min_{x\in X} (d^T x + d_0)$, $\max_{x\in X} (d^T x + d_0)$ are local maximum for problem P.
Therefore, the maximum value of f is

$$\max\left\{\frac{\gamma z_{\min} - \gamma d_0 + c_0}{z_{\min}^p}, \frac{\gamma z_{\max} - \gamma d_0 + c_0}{z_{\max}^p}\right\}.$$

The proof is complete. □

Theorems 15, 16 and 17 allow us to suggest the following sequential method to solve Problem P.

Main Algorithm

Step 0. Compute the optimal values $z_{\min}$ and $z_{\max}$. If $0 < p < 1$, then go to Step 1 else go to Step 3.

Step 1. If $\gamma < 0$, then go to Step 7, else go to Step 2.

Step 2. If $z_{\max} = +\infty$, then STOP: the supremum of the problem is $+\infty$, else go to Step 8.

Step 3. If $z_{\max} < +\infty$ go Step 4, else go to Step 5.

Step 4. If $\gamma > 0$, then go to Step 7, else go to Step 8.

Step 5. If $\gamma > 0$, then STOP: the supremum is 0, otherwise go to Step 6.

Step 6. If $z_{\min} > -\dfrac{c_0}{\gamma} + d_0$, then, STOP: the supremum of P is 0, else STOP: the optimal solutions of P correspond to the optimal solutions of the problem $\min\limits_{x \in X} (d^T x + d_0)$.

Step 7. Compute $z^* = \dfrac{p(c_0 - \gamma d_0)}{\gamma(1-p)}$.
If $z^* \in [z_{\min}, z_{\max}]$, then STOP: $X \cap \{x \in \mathbb{R}^n : d^T x + d_0 = z^*\}$ is the set of all optimal solutions of Problem P, otherwise go to Step 8.

Step 8. If $\dfrac{\gamma z_{\min} - \gamma d_0 + c_0}{z_{\min}^p} > \dfrac{\gamma z_{\max} - \gamma d_0 + c_0}{z_{\max}^p}$, then STOP: the optimal solutions of P correspond to the optimal solutions of the problem $\min\limits_{x \in X} (d^T x + d_0)$, else STOP: the optimal solutions of P correspond to the optimal solutions of the problem $\max\limits_{x \in X} (d^T x + d_0)$.

VI. Examples

In order to clarify how the proposed procedures work, we present some easy examples concerning the most representative case, that is rank$[c, d] = 2$. Example 18 considers pseudoconcave objective functions, while the pseudoconvex case is handled in Examples 19 and 20.

Example 18 *Consider the following problem P*

$$\begin{array}{ll} \sup & f(x) = \dfrac{3x_1 + 4x_2 + 1}{(x_1 + x_2 + 4)^3} \\ s.t. & -x_1 + x_2 + x_3 = 1/2 \\ & x_1 + x_2 + x_4 = 7 \\ & x_1 - x_2 + x_5 = 3 \\ & x = (x_1, ..., x_5) : x_i \geq 0, i = 1, ..., 5 \end{array} \tag{13}$$

Note that $C_{\max} > 0$ and $p > 1$, then f is pseudoconcave on the feasible set. According to the algorithm we apply Procedure A. We have

$\theta_{\min} = 4$; starting from $x^0 = \left(0, 0, \frac{1}{2}, 7, 3, 0\right)$ which is the optimal solution of problem P_c, we consider the parametric problem $P(\theta)$. The simplex table associated with the feasible basic solution x^0 is the following:

$-1-4\,\theta$	-1	0	0	0	0	-4
$1/2-\theta$	-2	0	1	0	0	-1
$7-\theta$	0	0	0	1	0	-1
$3+\theta$	2	0	0	0	1	1
θ	1	1	0	0	0	1

Remember that we force the slack variable associated with the parametric constraint to be a non-basic variable. The stability interval is $\left[0, \frac{1}{2}\right]$, $h(\theta) = \dfrac{1+4\,\theta}{(\theta+4)^3}$ and $h'(\theta) = \dfrac{13-8\,\theta}{(\theta+4)^4}$. Since $h'(0) > 0$, and $\widetilde{\theta} = \dfrac{13}{8} > \dfrac{1}{2}$, we perform an iteration of the dual simplex algorithm; we get the following simplex table:

$-5/4-7/2\,\theta$	0	0	$-1/2$	0	0	$-7/2$
$-1/4+1/2\,\theta$	1	0	$-1/2$	0	0	$1/2$
$7-\theta$	0	0	0	1	0	-1
$7/2$	0	0	1	0	1	0
$1/4+1/2\,\theta$	0	1	$1/2$	0	0	$1/2$

The new stability interval is $\left[\frac{1}{2}, 7\right]$ and we have $h(\theta) = \dfrac{\frac{5}{4}+\frac{7}{2}\,\theta}{(\theta+4)^3}$, $h'(\theta) = \dfrac{41-28\theta}{(\theta+4)^4}$. Since $h'\left(\frac{1}{2}\right) > 0$ and $\widetilde{\theta} = \frac{41}{28} \in \left(\frac{1}{2}, 7\right)$, we get $x\left(\frac{41}{28}\right) = \left(\frac{27}{56}, \frac{55}{56}, 0, \frac{155}{28}, \frac{7}{2}, 0\right)$ which corresponds to the global optimal solution of problem P, that is $x^1 = \left(\frac{27}{56}, \frac{55}{56}, 0, \frac{155}{28}, \frac{7}{2}\right)$. Observe that x^1 belongs to an edge of the feasible region.

Consider now function f on the following feasible region S_2:

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 2 \\ x_1 - x_2 + x_4 &= 3 \\ -x_1 + x_2 + x_5 &= 2 \\ x_i \geq 0 \qquad & i = 1, ..., 5 \end{aligned}$$

In this case $C_{\max} = \theta_{\max} = +\infty$. *By applying Procedure A, we obtain that the optimal value is attained at the vertex* $(0,2,0,0,5)$.

Example 19 *Consider the following problem P*

$$\begin{array}{ll} \sup & f(x) = \dfrac{-18x_1 - 3x_2 - \frac{1}{2}}{(x_1+x_2+1)^4} \\ s.t. & -x_1 + x_2 + x_3 = 3 \\ & \frac{1}{2}x_1 + x_2 + x_4 = 6 \\ & 2x_1 - x_2 + x_5 = 3 \\ & x = (x_1,...,x_5) : x_i \geq 0, i = 1,...,5 \end{array} \tag{14}$$

Since $C_{\max} = -\dfrac{1}{2} < 0$ *and* $p > 1$, *function* f *is pseudoconvex on the feasible set and, according to the algorithm, we apply procedure B. We get* $\theta_{\max} = \dfrac{39}{5}$ *and* $\theta_{\min} = 1$. *Starting from* $x^0 = (0,0,3,6,3,0)$ *which is the optimal solution of problem* P_c, *we consider the parametric problem* $P(\theta)$. *The simplex table associated with the feasible basic solution* x^0 *is the following:*

$1/2 + 3\,\theta$	-15	0	0	0	0	3
$3-\theta$	-2	0	1	0	0	-1
$6-\theta$	$-1/2$	0	0	1	0	-1
$3+\theta$	3	0	0	0	1	1
θ	1	1	0	0	0	1

Remember that we force the slack variable associated with the parametric constraint to be a non-basic variable. The stability interval is $[0,3]$, $h(\theta) = -\dfrac{\frac{1}{2}+3\theta}{(\theta+1)^4}$ *and* $h'(\theta) = -\dfrac{9\theta-1}{(\theta+1)^5}$. *Since* $h'(0) < 0$, x^0 *corresponds to the local maximum point* $\widehat{x}^0 = (0,0,3,6,3)$ *with* $f(\widehat{x}^0) = -\frac{1}{2}$. *We get* $X^M = \{\widehat{x}^0\}$.*We find* $\widetilde{\theta} = 0,278 \in [0,3]$ *such that* $h(\widetilde{\theta}) = h(0)$; *by means of an iteration of the dual simplex algo-*

rithm we get the following simplex table:

$-22+21/2\,\theta$	0	0	$-15/2$	0	0	$21/2$
$-3/2+1/2\,\theta$	1	0	$-1/2$	0	0	$1/2$
$21/4-3/4\,\theta$	0	0	$-1/4$	1	0	$-3/4$
$15/2-1/2\,\theta$	0	0	$3/2$	0	1	$-1/2$
$3/2+1/2\,\theta$	0	1	$1/2$	0	0	$1/2$

The new stability interval is $[3,7]$ *and we have* $h(\theta)=\dfrac{22-\frac{21}{2}\theta}{(\theta+1)^4}$, $h'(\theta)=\dfrac{1}{2}\dfrac{63\theta-197}{(\theta+1)^5}$. *Since* $h'(3)<0$, $x^1=(0,3,0,3,6,0)$ *corresponds to* $\widehat{x}^1=(0,3,0,3,6)$ *which is a local maximum point for* f *with* $f(\widehat{x}^1)>f(\widehat{x}^0)$. *Therefore* $X^M=\{\widehat{x}^1\}$. *We find* $\widetilde{\theta}=3,268\in[3,7]$ *such that* $h(\widetilde{\theta})=h(3)$*; by means of an iteration of the dual simplex algorithm we get the following simplex table:*

$-359/2+33\,\theta$	0	0	0	-30	0	33
$-12+2\theta$	1	0	0	-2	0	2
$-21+3\theta$	0	0	1	-4	0	3
$39-5\theta$	0	0	0	6	1	-5
$12-\theta$	0	1	0	2	0	-1

The new stability interval is $[7,\frac{39}{5}]$ *and we have* $h(\theta)=\dfrac{\frac{359}{2}-33\theta}{(\theta+1)^4}$, $h'(\theta)=\dfrac{99\theta-751}{(\theta+1)^5}$. *Since* $h'(7)<0$, $x^2=(2,5,0,0,4,0)$ *corresponds to the local maximum point* $\widehat{x}^2=(2,5,0,0,4)$ *with* $f(\widehat{x}^2)>f(\widehat{x}^1)$, *then* $X^M=\{\widehat{x}^2\}$. *We find* $\widetilde{\theta}=8.344$ *such that* $h(\widetilde{\theta})=h(7)$. *Since* $\widetilde{\theta}=8.344\notin[7,\frac{39}{5}]$, *then STOP,* $\widehat{x}^2$ *is the global maximum point.*

Example 20 *Consider the following problem P:*

$$
\begin{aligned}
\sup \quad & f(x)=\frac{2x_1+3x_2+8}{\sqrt{\frac{3}{2}x_1+\frac{3}{2}x_2+1}} \\
s.t. \quad & x_1+6x_2+x_3=30 \\
& 3x_1+4x_2+x_4=48 \\
& x=(x_1,...,x_4): x_i\geq 0, i=1,...,4
\end{aligned}
\tag{15}
$$

Since $C_{\max} \geq 0$ and $0 < p < 1$, function f is pseudoconvex on the feasible set and, according to the algorithm, we apply procedure D. We get $\theta_{\max} = 24$ and $\theta_{\min} = 1$. Starting from $x^0 = (0, 0, 30, 48, 0)$ which is the optimal solution of problem P_c, we consider the parametric problem $P(\theta)$. The simplex table associated with the feasible basic solution x^0 is the following:

$-8-2\theta$	-1	0	0	0	-2
$30-4\theta$	-5	0	1	0	-4
$48-8/3\,\theta$	-1	0	0	1	$-8/3$
$2/3\,\theta$	1	1	0	0	$2/3$

The stability interval is $[0, \frac{15}{2}]$, $h(\theta) = \dfrac{8+2\theta}{\sqrt{\theta+1}}$ and $h'(\theta) = \dfrac{\theta-2}{(\theta+1)^{\frac{3}{2}}}$. Since $h'(0) < 0$, x^0 corresponds to $\widehat{x}^0 = (0, 0, 30, 48)$ which is a local maximum point for f with $f(\widehat{x}^0) = 8$. $X^M = \{\widehat{x}^0\}$ We find $\widetilde{\theta} = 8$ such that $h(\widetilde{\theta}) = h(0)$. Since $\frac{15}{2} < \widetilde{\theta} < \theta_{\max} = 24$, we update the previous simplex table by setting $\theta = \theta + \widetilde{\theta}$, and we determine the new value $\theta_{\max} = \theta_{\max} - \widetilde{\theta} = 16$. We perform an iteration of the dual simplex algorithm for finding a new feasible solution, getting

$-118/5-6/5\,\theta$	0	0	$-1/5$	0	$-6/5$
$2/5+4/5\,\theta$	1	0	$-1/5$	0	$4/5$
$406/15-28/15\theta$	0	0	$-1/5$	1	$-28/15$
$74/15-2/15\,\theta$	0	1	$1/5$	0	$-2/15$

The new stability interval is $[0, \frac{29}{2}]$ and we go to the adjacent vertex, by performing an iteration of the dual simplex algorithm.

$-152/3+2/3\theta$	0	0	0	-1	$2/3$
$-80/3+8/3\theta$	1	0	0	-1	$8/3$
$-406/3+28/3\theta$	0	0	1	-5	$28/3$
$32-2\theta$	0	1	0	1	-2

The stability interval is $[\frac{29}{2}, 16]$, $h(\theta) = \dfrac{\frac{152}{3}+\frac{2}{3}\theta}{\sqrt{\theta+9}}$, $h'(\theta) = -\dfrac{1}{3}\dfrac{\theta+94}{(\theta+9)^{\frac{3}{2}}}$. Since $h'\left(\frac{29}{2}\right) < 0$, $x^1 = (12, 3, 0, 0, 0)$ corresponds to the local maximum point $\widehat{x}^1 = (12, 3, 0, 0)$ with $f(\widehat{x}^1) = 8,458 > f(\widehat{x}^0)$.

$X^M = \{\widehat{x}^1\}$. *Furthermore, since* $h'(\theta) < 0$ *for every* $\theta > \frac{29}{2}$, *then* $\widehat{x}^1$ *is the global maximum point of the problem.*

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