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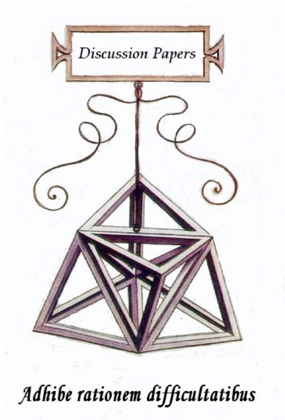

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Characterizing the pseudoconvexity of a wide class of generalized fractional functions

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Discussion Paper

n.

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Characterizing the pseudoconvexity of a wide class of generalized fractional functions

Abstract

We consider a wide class of generalized fractional functions, namely the sum between a linear one and a ratio which has an affine function as numerator and, as denominator, the p-th power of an affine one. For this class of functions we aim to derive necessary and/or sufficient conditions for pseudoconvexity on the nonnegative orthant. The obtained conditions are very easy to be checked and allow us to construct several subclasses of pseudoconvex generalized fractional functions.

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Contents

I. Introduction

Among different classes of generalized convex functions, pseudoconvexity plays a key role in Optimization theory and in many applied sciences such as Economics and Management Science. Pseudoconvexity owes its great relevance to the fact that it maintains some nice optimization properties of convex functions, such as critical and local minimum points are global minimum. Furthermore, if the objective function of a bicriteria problem is component-wise pseudoconvex, then the efficient frontier is connected.

Unlike their good properties, it is not easy to establish whether a function is pseudoconvex or not. Besides some theoretical characterizations (see for all 1 and references therein), there are only few results related to specific classes of functions. In this direction, the first contributions deal with quadratic functions (see for istance $2, 6, 8$) and, more recently, different approaches have been proposed for the study of pseudoconvexity of generalized fractional functions (see for all $3, 7, 9$).

In this paper, we aim to study the pseudoconvexity of the sum between a linear function and a ratio which has an affine one as numerator and, as denominator, the p-th power ($p > 0$) of a positive affine one. Since the case $p = 1$ has been recently analyzed $4, 5$, we focus our attention on $p > 0, p \neq 1$. Due to the fact that in Management Sciences and in Economics we often require the variable nonnegativity, we investigate the pseudoconvexity on the nonnegative orthant. The key tool of our analysis is a second order characterization of the pseudoconvexity related to an open and convex set. Unfortunately the pseudoconvexity of a function on an open and convex set does not guarantee its pseudoconvexity on the closure of the set. Therefore, a priori, we need to distinguish conditions which characterize pseudoconvexity on the interior on $\mathbb{R}^n_+$ and conditions related to the behavior of f on the boundary of $\mathbb{R}^n_+$. The performed analysis allows us to give necessary and sufficient conditions which are very easy to be checked and which can be used to construct several subclasses of pseudoconvex generalized fractional functions.

II. Statement of the problem and preliminaries

Consider the following class of generalized fractional functions

$$f(x) = a^T x + \frac{c^T x + c_0}{(d^T x + d_0)^p} \tag{1}$$

where: $d^T x + d_0 > 0$, $d \in \text{int}\mathbb{R}^n_+$, $d_0 > 0$, $p > 0$ and $p \neq 1$.
The gradient, the Hessian matrix $H(x)$ of f, and the quadratic form associated with $H(x)$ are the following:

$$\nabla f(x) = a + \frac{c\left(d^T x + d_0\right) - p\left(c^T x + c_0\right) d}{(d^T x + d_0)^{p+1}} \tag{2}$$

$$H(x) = \frac{p\left[\left(d^T x + d_0\right)\left(-dc^T - cd^T\right) + (p+1)\left(c^T x + c_0\right) dd^T\right]}{(d^T x + d_0)^{p+2}} \tag{3}$$

$$w^T H(x) w = \frac{p\left[-2\left(d^T x + d_0\right)\left(w^T d\right)\left(c^T w\right) + (p+1)\left(c^T x + c_0\right)(d^T w)^2\right]}{(d^T x + d_0)^{p+2}} \tag{4}$$

For the sake of completeness we recall the definition of a pseudoconvex function and the pseudoconvexity second order characterization we are going to use in our analysis (see for istance 3).

Definition 1 *Let f be a differentiable function defined on an open set $X \subseteq \mathbb{R}^n$ and let $S \subseteq X$ be a convex set.*
f is said to be pseudoconvex on S if the following implication holds:

$$x^1, x^2 \in S,\ \ f(x^1) > f(x^2) \quad \Rightarrow \quad \nabla f(x^1)^T (x^2 - x^1) < 0 \tag{5}$$

Theorem 2 *Let f be a twice continuously differentiable function defined on an open convex set $X \subseteq \mathbb{R}^n$.*
Then, f is pseudoconvex on X if and only if the following conditions hold:
i) $x \in X,\ w \in \mathbb{R}^n,\ w^T \nabla f(x) = 0\ \Rightarrow w^T H(x) w \geq 0$;
ii) If $x_0 \in X$ is a critical point, then x_0 is a local minimum point for f.

With respect to the introduced class of function (1), the following theorem gives a preliminary necessary pseudoconvexity condition related to the linear dependence of the vectors a, c and d.

Theorem 3 *Let $S \subseteq \mathbb{R}^n$ be a convex set with* $\mathrm{int}S \neq \emptyset$. *If f is pseudoconvex on S, then* $\mathrm{rank}[a, c, d] \leq 2$.

Proof Suppose on the contrary that $\mathrm{rank}[a, c, d] = 3$; this implies $\nabla f(x) \neq 0$ for every $x \in S$. Applying Theorem 2, let us consider $x \in \mathrm{int}S$ and a direction w such that $\nabla f(x)^T w = 0$.
We have $a^T w + \dfrac{\left(d^T x + d_0\right) c^T w - p\left(c^T x + c_0\right) d^T w}{\left(d^T x + d_0\right)^{p+1}} = 0$ so that

$$\left(d^T x + d_0\right) c^T w = p\left(c^T x + c_0\right) d^T w - \left(d^T x + d_0\right)^{p+1} a^T w$$

Substituting the value $c^T w \left(d^T x + d_0\right)$ in (4), we get

$$w^T H(x) w = \frac{p}{(d^T x + d_0)^{p+2}} d^T w \left[(1-p)\left(c^T x + c_0\right) d^T w + 2\left(d^T x + d_0\right)^{p+1} a^T w\right]$$

For every $x \in \mathrm{int}S$, consider the linear map $A : \mathbb{R}^n \to \mathbb{R}^3$, where $A = \begin{bmatrix} \nabla f(x)^T \\ a^T \\ d^T \end{bmatrix}$. Since $\mathrm{rank}[a, c, d] = 3$, the map A is surjective and hence we can choose $w \in \mathbb{R}^n$ such that $\nabla f(x)^T w = 0$, $d^T w < 0$ and $a^T w > \dfrac{(p-1)\left(c^T x + c_0\right)}{2\left(d^T x + d_0\right)^{p+1}} d^T w$, so that $w^T H(x) w < 0$. Consequently, f is not pseudoconvex on $\mathrm{int}S$ and this is a contradiction. □

According to Theorem 3, we must study the pseudoconvexity of f in the following exhaustive cases:

- $\mathrm{rank}[a, c, d] = 1$; $\mathrm{rank}[a, d] = 2$, $c = \beta d$; $\mathrm{rank}[c, d] = 2$.

In what follows we will prove that in the first case (see Section III.), the study of pseudoconvexity reduces to the study of pseudoconvexity of a suitable one variable function; in the second case (see Section IV.), f is pseudoconvex if and only if it is convex. As regard the third case, we must study the behavior of f on the interior of $\mathbb{R}^n_+$ and the behavior of f on the boundary of $\mathbb{R}^n_+$.

III. *Case rank*$[\mathbf{a}, \mathbf{c}, \mathbf{d}] = \mathbf{1}$

Setting $a = \alpha d$, $c = \gamma d$ in (1) we obtain

$$f(x) = \alpha d^T x + \frac{\gamma d^T x + c_0}{(d^T x + d_0)^p}$$

Substituting $a = \alpha d$ and $c = \gamma d$, the gradient and the Hessian matrix of f are specified as follows

$$\nabla f(x) = d \left[\alpha + \frac{\gamma \left(d^T x + d_0 \right) (1 - p) + p \left(\gamma d_0 - c_0 \right)}{\left(d^T x + d_0 \right)^{p+1}} \right]$$

$$H(x) = \frac{p d d^T}{\left(d^T x + d_0 \right)^{p+2}} \left[\gamma (p - 1) \left(d^T x + d_0 \right) + (p + 1) \left(c_0 - \gamma d_0 \right) \right] \tag{6}$$

Setting $d^T x = z$, function f becomes

$$\eta(z) = \alpha z + \frac{\gamma z + c_0}{(z + d_0)^p} \tag{7}$$

and we have

$$\eta'(z) = \alpha + \frac{\gamma (1 - p) z + \gamma d_0 - p c_0}{(z + d_0)^{p+1}}, \ z \geq 0 \tag{8}$$

$$\eta''(z) = \frac{p}{(z + d_0)^{p+2}} \left[\gamma (p - 1) z - 2 \gamma d_0 + (p + 1) c_0 \right], \ z \geq 0 \tag{9}$$

The following theorem proves that the pseudoconvexity of function f on $\mathbb{R}^n_+$ is equivalent to the pseudoconvexity of function η on the closed set $[0, +\infty)$.

Theorem 4 *Assume* $\operatorname{rank}[a, c, d] = 1$. *Then f is pseudoconvex on $\mathbb{R}^n_+$ if and only if η is pseudoconvex on $[0, +\infty)$.*

Proof Let $x^1, x^2 \in \mathbb{R}^n_+$ and set $z_1 = d^T x^1$ and $z_2 = d^T x^2$.
Note that $f(x^1) > f(x^2)$ if and only if $\eta(z_1) > \eta(z_2)$; furthermore, $\nabla f(x) = \eta'(z) d$. Consequently $\nabla f(x^1)^T (x^2 - x^1) < 0$ if and only if $\eta'(z_1) d^T (x^2 - x^1) = \eta'(z_1)(z_2 - z_1) < 0$ and the thesis follows. □

Remark 5 *Note that if $\bar{z}$ is a critical point for η, then every point of the hyperplane $d^T x = \bar{z}$ is a critical point for f.*

Since η is a function of one variable, it is known that η is pseudoconvex on $[0, +\infty)$ if and only if every critical point is a minimum point. We will use this result in proving the following theorems.

Theorem 6 *Assume* $\operatorname{rank}[a, c, d] = 1$ *and* $\gamma(p-1) > 0$.
Then f is pseudoconvex on $\mathbb{R}^n_+$ if and only if one of the following conditions holds:
i) $\gamma \leq \frac{p+1}{2}\frac{c_0}{d_0}$;
ii) $\gamma > \frac{p+1}{2}\frac{c_0}{d_0}$ *and either* $\alpha d_0^{p+1} + \gamma d_0 - p c_0 < 0$, *or*

$$\alpha + (c_0 - \gamma d_0)\left[\frac{\gamma(p-1)}{(p+1)(\gamma d_0 - c_0)}\right]^{p+1} > 0 \tag{10}$$

Proof According to Theorem 4 we prove that η is pseudoconvex in $[0, +\infty)$.
From (9), we have that $\eta(z)$ is convex (hence pseudoconvex) in $[0, +\infty)$ if and only if i) holds.
If $\gamma > \frac{p+1}{2}\frac{c_0}{d_0}$, then $\eta'(z)$ has a unique minimum point at

$$\bar{z} = \frac{2\gamma d_0 - (p+1)c_0}{\gamma(p-1)} > 0.$$

If $\eta'(\bar{z}) > 0$, that is (10) holds, η is increasing on $[0, +\infty]$.
If $\eta'(\bar{z}) = 0$, necessarily we have $\eta'(0) = \alpha d_0^{p+1} + \gamma d_0 - p c_0 > 0$, so that $\bar{z}$ turns out to be an inflection point for η.
If $\eta'(\bar{z}) < 0$ and $\eta'(0) \geq 0$, then $\eta'(z)$ has a zero corresponding to a maximum point for η.
If $\eta'(\bar{z}) < 0$ and $\eta'(0) < 0$, then either $\eta'(z) < 0$, $\forall z \in [0, +\infty)$ or η' has a zero corresponding to a minimum point for η.
Consequently, η is pseudoconvex if and only if ii) holds. The proof is complete. $\square$

Theorem 7 *Assume* $\operatorname{rank}[a,c,d]=1$ *and* $\gamma(p-1)<0$.
Then f *is pseudoconvex on* $\mathbb{R}^n_+$ *if and only if one of the following conditions holds:*
i) $\alpha \geq 0$;
ii) $\gamma \geq \frac{p+1}{2}\frac{c_0}{d_0}$ *and* $\alpha d_0^{p+1}+\gamma d_0 - pc_0 < 0$;
iii) $\gamma < \frac{p+1}{2}\frac{c_0}{d_0}$ *and*

$$\alpha + (c_0-\gamma d_0)\left[\frac{\gamma(p-1)}{(p+1)(\gamma d_0 - c_0)}\right]^{p+1} < 0 \tag{11}$$

Proof According to Theorem 4 we prove that η is pseudoconvex on $[0,+\infty)$.
Firstly note that, $\lim\limits_{z\to+\infty}\eta'(z)=\alpha$ and, from (9), η' has a maximum point (feasible or not) at $\bar{z}=\dfrac{2\gamma d_0-(p+1)c_0}{\gamma(p-1)}$.
Consider the case $\alpha\geq 0$; if $\eta'(0)\geq 0$, then $\eta'(z)\geq 0$, $\forall z\in[0,+\infty)$. Otherwise if $\eta'(0)<0$, then $\eta'(z)$ has a unique zero corresponding to a minimum point for η.
Consequently, when $\alpha\geq 0$, η is pseudoconvex on $[0,+\infty)$.
Consider now the case $\alpha<0$. If $\gamma\geq\frac{p+1}{2}\frac{c_0}{d_0}$, then η' is decreasing towards α. If $\eta'(0)<0$, i.e., $\alpha d_0^{p+1}+\gamma d_0-pc_0<0$, then $\eta'(z)<0$, $\forall z\in[0,+\infty)$, otherwise η' has a zero corresponding to a maximum point for η.
If $\gamma<\frac{p+1}{2}\frac{c_0}{d_0}$, η' has a feasible maximum point at $\bar{z}$. If $\eta'(\bar{z})<0$, i.e., (11) holds, then η does not have critical points, otherwise, once again, η' has a zero corresponding to a maximum point for η and this completes the proof. □

When $\gamma=0$, the following theorem holds.

Theorem 8 *Assume* $\operatorname{rank}[a,c,d]=1$ *and* $\gamma=0$.
Then f *is pseudoconvex on* $\mathbb{R}^n_+$ *if and only if one of the following conditions holds:*

i) $c_0 \geq 0$;
ii) $c_0 < 0$ *and either* $\alpha \geq 0$ *or* $\alpha < \dfrac{pc_0}{d_0^{p+1}}$.

Proof It is easy to verify that η is convex when $c_0 \geq 0$, while it does not have critical points when $c_0 < 0$ and $\alpha \geq 0$. In the case $c_0 < 0$ and $\alpha < 0$, η is pseudocovex if and only if $\eta'(0) < 0$, i.e., $\alpha < \dfrac{pc_0}{d_0^{p+1}}$. The proof is complete. □

As we will see in the following sections, the previous theorems will be relevant even in the case $\text{rank}[c, d] = 2$. More precisely we will apply them to study the pseudoconvexity of f on the boundary of $\mathbb{R}^n_+$ (see Section V.B.).

IV. *Case* $\mathbf{rank}[\boldsymbol{a}, \boldsymbol{d}] = \mathbf{2}$, $\boldsymbol{c} = \boldsymbol{\beta d}$

Theorem 9 *Assume* $\text{rank}[a, d] = 2$ *and* $c = \beta d$.
Then f *is pseudoconvex on* $\mathbb{R}^n_+$ *if and only if the following conditions hold:*
i) $\beta(p-1) \geq 0$;
ii) $\beta \leq \dfrac{c_0(p+1)}{2d_0}$.

Proof The gradient and the Hessian matrix of f become

$$\nabla f(x) = a + \frac{d}{(d^T x + d_0)^{p+1}} \left[\beta(1-p)\left(d^T x + d_0\right) + p(\beta d_0 - c_0)\right] \tag{12}$$

$$H(x) = \frac{pdd^T}{(d^T x + d_0)^{p+2}} \left[\left(d^T x + d_0\right)(p-1)\beta + (c_0 - \beta d_0)(p+1)\right] \tag{13}$$

Note that $\nabla f(x) \neq 0$, $\forall x$; furthermore, the Hessian matrix $H(x)$ is positive semidefinite for every $x \in \mathbb{R}^n_+$ if and only if $\beta(p-1) \geq 0$ and $\beta \leq \dfrac{c_0(p+1)}{2d_0}$. It follows that f is convex and hence pseudoconvex on $\mathbb{R}^n_+$ if and only if *i*) and *ii*) hold. In any other case f is not pseudoconvex. □

From Theorem 9 if follows that f is pseudoconvex on $\mathbb{R}^n_+$ if and only if f is convex on $\mathbb{R}^n_+$.

V. Case rank$[\mathbf{c}, \mathbf{d}] = \mathbf{2}$

The study of the pseudoconvexity on a closed convex set S is usually performed by studying the pseudoconvexity on the interior of S with the aim to extend the obtained results on the boundary of S. The particular structure of the function allows us to prove that the pseudoconvexity of f on $\mathrm{int}\mathbb{R}^n_+$ is equivalent to the pseudoconvexity of f on $\mathbb{R}^n_+$.

V.A. Pseudoconvexity on $\mathrm{int}\mathbb{R}^n_+$

Let us preliminary observe that, due to the linear independence of c and d, the quadratic form (4) is indefinite for every fixed $x \in \mathbb{R}^n_+$, and hence any critical point of f in $\mathrm{int}\mathbb{R}^n_+$, is not a minimum point. Consequently, the following result holds.

Theorem 10 *Assume* $\mathrm{rank}[c, d] = 2$. *If f is pseudoconvex on* $\mathrm{int}\mathbb{R}^n_+$, *then $\nabla f(x) \neq 0$ for every $x \in \mathrm{int}\mathbb{R}^n_+$.*

According to Theorem 2, pseudoconvexity is studied by characterizing the non-existence of critical points and by analyzing the behavior of the Hessian matrix on the directions which are orthogonal to the gradient.
Substituting $a = \alpha_1 c + \alpha_2 d$ in (2), we get

$$\nabla f(x) = \frac{c\left(d^T x + d_0\right)\left(\alpha_1\left(d^T x + d_0\right)^p + 1\right) + d\left(\alpha_2\left(d^T x + d_0\right)^{p+1} - p\left(c^T x + c_0\right)\right)}{\left(d^T x + d_0\right)^{p+1}}$$

From $\mathrm{rank}[c, d] = 2$, it follows that f has critical points on $\mathbb{R}^n_+$ if and only if the following system has solutions on $\mathbb{R}^n_+$:

$$\begin{cases} \alpha_1\left(d^T x + d_0\right)^p + 1 & = 0 \\ \alpha_2\left(d^T x + d_0\right)^{p+1} - p\left(c^T x + c_0\right) & = 0 \end{cases} \tag{14}$$

Setting

$$h(x) = \frac{(1-p)\left(c^T x + c_0\right)}{(d^T x + d_0)^p} + \left[(\alpha_1(p+1)c + 2\alpha_2 d)^T x + \alpha_1(p+1)c_0 + 2\alpha_2 d_0\right] \quad (15)$$

we have the following result.

Theorem 11 *Assume* $\text{rank}[c, d] = 2$.
i) f *is pseudoconvex on* $D_1^+ = \{x \in \text{int}\mathbb{R}_+^n : \ \alpha_1 \left(d^T x + d_0\right)^p + 1 > 0\}$ *if and only if* $h(x) \geq 0, \ \forall x \in D_1^+$.
ii) f *is pseudoconvex on* $D_1^- = \{x \in \text{int}\mathbb{R}_+^n : \ \alpha_1 \left(d^T x + d_0\right)^p + 1 < 0\}$ *if and only if* $h(x) \leq 0, \ \forall x \in D_1^-$.

Proof Note that $\alpha_1 \left(d^T x + d_0\right)^p + 1 \neq 0$ implies the non-existence of critical points, so that we have to apply i) of Theorem 2. Condition $w^T \nabla f(x) = 0$ holds if and only if

$$\left(d^T x + d_0\right) c^T w = \frac{\left(p\left(c^T x + c_0\right) - \alpha_2 \left(d^T x + d_0\right)^{p+1}\right)}{\alpha_1 \left(d^T x + d_0\right)^p + 1} d^T w \quad (16)$$

Substituting (16) in (4) we get

$$w^T H(x) w = \frac{p(d^T w)^2 h(x)}{\left(d^T x + d_0\right)^2 \left(\alpha_1 \left(d^T x + d_0\right)^p + 1\right)} \quad (17)$$

Therefore $w^T H(x) w \geq 0$ if and only if $\dfrac{h(x)}{\alpha_1 \left(d^T x + d_0\right)^p + 1} \geq 0$. The proof is complete. □

Theorem 12 *Assume* $\text{rank}[c, d] = 2$.
If both D_1^+ *and* D_1^- *are not empty, then* f *is not pseudoconvex on* $\text{int}\mathbb{R}_+^n$.

Proof Assume, by contradiction, the pseudoconvexity of f on $\text{int}\mathbb{R}_+^n$. Then, we have $h(x) \geq 0, \ \forall x \in D_1^+$ and $h(x) \leq 0, \ \forall x \in D_1^-$, so that, by continuity, $h(x) = 0$ for every $x \in \text{int}\mathbb{R}_+^n$ such that $\alpha_1 \left(d^T x + d_0\right)^p + 1 = 0$. Substituting $\alpha_1 = -\dfrac{1}{\left(d^T x + d_0\right)^p}$ in $h(x) = 0$ we get $\alpha_2 \left(d^T x + d_0\right)^{p+1} - p\left(c^T x + c_0\right) = 0$, so that from (14) f has critical points and this is a contradiction. □

Remark 13 *From Theorem 11 and Theorem 12, the pseudoconvexity of f on* $\mathrm{int}\mathbb{R}^n_+$ *implies* $\alpha_1 \geq 0$, *or* $\alpha_1 \leq -\dfrac{1}{d_0^p}$. *Whenever* $\alpha_1 \geq 0$, *or* $\alpha_1 < -\dfrac{1}{d_0^p}$, *it is* $\nabla f(x) \neq 0$ *for every* $x \in \mathbb{R}^n_+$, *while in the case* $\alpha_1 = -\dfrac{1}{d_0^p}$, *only the origin may be a critical point and this happens if and only if* $\alpha_2 d_0^{p+1} - pc_0 = 0$.

Note that, by continuity, condition $h(x) \geq 0$ ($h(x) \leq 0$), $\forall x \in \mathrm{int}\mathbb{R}^n_+$, implies $h(x) \geq 0$ ($h(x) \leq 0$), $\forall x \in \mathbb{R}^n_+$, or, equivalently, $\inf\limits_{x \in \mathbb{R}^n_+} h(x) \geq 0$ $\left(\sup\limits_{x \in \mathbb{R}^n_+} h(x) \leq 0\right)$.

Regarding the infimum (supremum) of h, we have the following result.

Theorem 14 *There exists an index* $i \in I = \{1, ..., n\}$ *such that*

$$\inf_{x \in \mathbb{R}^n_+} h(x) = \inf_{x_i \geq 0} h_i(x_i) \tag{18}$$

$$\sup_{x \in \mathbb{R}^n_+} h(x) = \sup_{x_i \geq 0} h_i(x_i) \tag{19}$$

where $h_i(x_i)$ *denotes the restriction of function* $h(x)$ *on the* i*-th edge of* $\mathbb{R}^n_+$.

Proof Let $\{x_n\} \subset \mathbb{R}^n_+$ be a sequence such that

$$h(x_n) \to \ell = \inf_{x \in \mathbb{R}^n_+} h(x).$$

For every fixed x_n, consider the linear problem

$$P_n : \inf_{x \in S_n} h(x), \quad S_n = \{x \in \mathbb{R}^n_+ : \ d^T x + d_0 = d^T x_n + d_0\}$$

Since S_n is a compact set, the infimum is attained as a minimum at a vertex $\hat{x}_n$, which belongs to an edge of $\mathbb{R}^n_+$, and obviously it is $h(\hat{x}_n) \leq h(x_n)$, $\forall n$. Consequently, $h(\hat{x}_n) \to \ell$. The finite number of edges implies the existence of a subsequence $\{y_n\}$ of $\{x_n\}$, contained in an edge, such that $h(y_n) \to \ell$, so that (18) holds.

The proof of (19) follows in a similar way. □

Remark 15 *Theorem 11 implies that the study of the pseudoconvexity of f on $\mathrm{int}\mathbb{R}^n_+$ reduces to the study of the sign of function h on $\mathbb{R}^n_+$. When $a = 0$, that is $\alpha_1 = \alpha_2 = 0$, the study is very simple; in fact $h(x)$ reduces to $h(x) = \dfrac{(1-p)(c^Tx + c_0)}{(d^Tx + d_0)^p}$, so that f is pseudoconvex on $\mathrm{int}\mathbb{R}^n_+$ if and only if $0 < p < 1$, $c \in \mathbb{R}^n_+$, $c_0 \geq 0$ or $p > 1$, $c \in \mathbb{R}^n_-$, $c_0 \leq 0$.*

From now on, taking into account Remark 15, we will consider the case $a \neq 0$. Furthermore, for sake of simplicity, we will assume $c \in \mathrm{int}\mathbb{R}^n_+$. Conditions for the case $c_i \leq 0$ can be obtained following the same strategies used in the results that we are going to present. The following theorem characterizes the pseudoconvexity of f on $\mathrm{int}\mathbb{R}^n_+$ in the case $\alpha_1 \geq 0$.

Theorem 16 *Assume $\mathrm{rank}[c, d] = 2$, $c \in \mathrm{int}\mathbb{R}^n_+$ and $\alpha_1 \geq 0$.*
Then f is pseudoconvex on $\mathrm{int}\mathbb{R}^n_+$ if and only if:
i) $\alpha_1(p+1)c + 2\alpha_2 d \in \mathbb{R}^n_+ \setminus \{0\}$;
ii) $(1-p)c_0 + d_0^p\,(\alpha_1(p+1)c_0 + 2\alpha_2 d_0) \geq 0$
and one of the following conditions holds:
iii) $0 < p < 1$;
iv) $p > 1$ and, either

$$\nabla h(0) = (1-p)\frac{d_0 c - pc_0 d}{d_0^{p+1}} + \alpha_1(p+1)c + 2\alpha_2 d \in \mathbb{R}^n_+,$$

or $\min\limits_{i \in J} h_i(\bar{x}_i) \geq 0$, where $J = \{i : \dfrac{\partial h}{\partial x_i}(0) < 0\}$, and $\bar{x}_i$ is such that $h_i'(\bar{x}_i) = 0$.

Proof The assumption $\alpha_1 \geq 0$ guarantees the non-existence of critical points, so that f is pseudoconvex on $\mathrm{int}\mathbb{R}^n_+$ if and only if $h(x) \geq 0,\ \forall x \in \mathrm{int}\mathbb{R}^n_+$.
Firstly we prove that $i)$ and $ii)$ are necessary conditions for pseudoconvexity.
Note that $h(x) \geq 0,\ \forall x \in \mathrm{int}\mathbb{R}^n_+$, implies $h(0) \geq 0$, i.e., $ii)$. On the other hand, if $\alpha_1(p+1)c + 2\alpha_2 d \notin \mathbb{R}^n_+ \setminus \{0\}$, then there exists a restriction of h on an edge of $\mathbb{R}^n_+$ for which $\inf\limits_{x_i \geq 0} h_i(x_i) = -\infty$ and

this is a contradiction.
Let us now consider the case $0 < p < 1$. Condition i) and ii) implies $h_i(x_i)$, $i = 1, ..., n$, is a sum of increasing functions with $h_i(0) \geq 0$. From Theorem 14 we get $h(x) \geq 0$, for every $x \in \text{int}\mathbb{R}^n$. We are left to prove that condition i) ii) and iv) imply $h(x) \geq 0$, $\forall x \in \text{int}\mathbb{R}^n_+$, or, equivalently (see Theorem 14), $h_i(x_i) \geq 0$, $\forall x_i \geq 0$, $\forall i \in \{1, ..., n\}$. We have

$$h_i'(x_i) = (1-p)\frac{(1-p)c_i d_i x_i + c_i d_0 - p d_i c_0}{(d_i x_i + d_0)^{p+1}} + \alpha_1(p+1)c_i + 2\alpha_2 d_i$$

and

$$h_i''(x_i) = \frac{(1-p)p}{(d_i x_i + d_0)^{p+2}} d_i\left[(p-1)c_i d_i x_i - 2c_i d_0 + (p+1)d_i c_0\right]$$

Let us preliminary observe that $h_i'(x_i)$ has a unique critical point which is a maximum point and $\lim\limits_{x_i \to +\infty} h_i'(x_i) = \alpha_1(p+1)c_i + 2\alpha_2 d_i \geq 0$. Therefore if $\nabla h(0) \in \mathbb{R}^n_+$, that is $h_i'(0) \geq 0$ for every i, then $h_i'(x_i) \geq 0$, $\forall x_i \geq 0$. Since $h_i(0) \geq 0$, it results $h_i(x_i) \geq 0$, $\forall x_i \geq 0$. On the other hand if $i \in J$, then there exists $\bar{x}_i$ such that $h_i'(\bar{x}_i) = 0$ which is a minimum point for h_i, with $h_i(\bar{x}_i) \geq 0$. Consequently $h_i(x_i) \geq 0$, $\forall i \in J$, and the thesis follows. □

Remark 17 *When $0 < p < 1$, condition i) of Theorem 16 implies $a_i \geq \frac{(1-p)c_i}{2}$. Consequently, if there exists i such that $a_i < 0$, then f is not pseudoconvex.*

The following example points out that in the case $J \neq \emptyset$, condition $\min\limits_{i \in J} h_i(\bar{x}_i) \geq 0$ can not be relaxed.

Example 18 *Consider the function $f(x, y) = \frac{1}{32}x + \frac{4x + y - 1}{(3x + y + 1)^2}$. The restriction of on $y = 1$ has a critical point at $x = 0,98142397$ which is a maximum point. Therefore f is not pseudoconvex. Referring to Theorem 16 it is easy to verify that $a = \frac{1}{32}c - \frac{1}{32}d$*

and $\text{rank}[c,d] = 2$. *Since* $\alpha_1 = \dfrac{1}{32} > 0$, f *has no critical points. Moreover, necessary conditions i) and ii) of Theorem 16 are verified. On the other hand,* $h_1(x) = h(x,0) = -\dfrac{(4x-1)}{(3x+1)^2} + \dfrac{3}{16}x - \dfrac{5}{32}$ *with* $h_1'(0) = -\dfrac{157}{16}$, *hence* $J \neq \emptyset$. *Function* $h_1(x)$ *has a minimum point at* $\bar{x} = 0,546247734$ *with* $h_1(\bar{x}) = -0,224$, *so that condition iv) of Theorem 16 is not verified.*

According to Remark 13, we are going to characterize the pseudoconvexity in the case $\alpha_1 \leq -\dfrac{1}{d_0^p}$.

Theorem 19 *Assume* $\text{rank}[c,d] = 2$, $c \in \text{int}\mathbb{R}^n_+$ *and* $\alpha_1 \leq -\dfrac{1}{d_0^p}$.
Then f *is pseudoconvex on* $\text{int}\mathbb{R}^n_+$ *if and only if:*
i) $\alpha_1(p+1)c + 2\alpha_2 d \in \mathbb{R}^n_- \setminus \{0\}$;
ii) $(1-p)c_0 + d_0^p\,(\alpha_1(p+1)c_0 + 2\alpha_2 d_0) \leq 0$;
and one of the following conditions holds
iii) $p > 1$;
iv) $0 < p < 1$, *and* $\max\limits_{i \notin J_1} h_i(\bar{x}_i) \leq 0$, *where*

$$J_1 = \left\{ i : \ \frac{\partial h}{\partial x_i}(0) \leq 0, \ \frac{c_i}{d_i} \geq \frac{p+1}{2}\frac{c_0}{d_0} \right\}$$

and $\bar{x}_i$ *is such that* $h_i'(\bar{x}_i) = 0$

Proof Note that the assumption $\alpha_1 \leq -\dfrac{1}{d_0^p}$ guarantees the nonexistence of critical points, consequently f is pseudoconvex on $\text{int}\mathbb{R}^n_+$ if and only if $h(x) \leq 0, \ \forall x \in \mathbb{R}^n_+$.
Firstly we prove that $i)$ and $ii)$ are necessary conditions for pseudoconvexity.
Infact, $h(x) \leq 0, \ \forall x \in \mathbb{R}^n_+$, implies $h(0) \leq 0$, i.e., condition $ii)$. On the other hand, if $\alpha_1(p+1)c + 2\alpha_2 d \notin \mathbb{R}^n_- \setminus \{0\}$, there exists a restriction on an edge of $\mathbb{R}^n_+$ for which $h(x) \to +\infty$ and this is a contradiction.

Let us now consider the case $p > 1$. Condition i) and ii) implies $h_i(x_i)$, $i = 1, ..., n$, is a sum of decreasing functions with $h_i(0) \leq 0$. From Theorem 14 we get $h(x) \leq 0$, for every $x \in \text{int}\mathbb{R}^n$.
We are left to prove that when $0 < p < 1$, condition i) ii) and iv) imply $h(x) \leq 0$, $\forall x \in \text{int}\mathbb{R}^n_+$, or, equivalently (see Theorem 14), $h_i(x_i) \geq 0$, $\forall x_i \leq 0$, $\forall i \in \{1, ..., n\}$.
Let us note that $\lim\limits_{x_i \to +\infty} h'_i(x_i) = \alpha_1(p+1)c_i + 2\alpha_2 d_i \leq 0$.
If $i \in J_1$, then $h'_i(x_i) \leq 0$, $\forall x_i \geq 0$; therefore, $h_i(x_i) \leq 0$, $\forall x_i \geq 0$.
We consider now the case $i \notin J_1$.
If $h'_i(0) > 0$, then h has a maximum point $\bar{x}_i$, so that condition $\max\limits_{i \notin J_1} h_i(\bar{x}_i) \leq 0$ implies $h_i(x_i) \leq 0$, $\forall x_i \geq 0$.
If $h'_i(0) < 0$, and $\dfrac{c_i}{d_i} < \dfrac{p+1}{2}\dfrac{c_0}{d_0}$, h' has a maximum point at $\tilde{x} = \dfrac{2c_i d_0 - (p+1)d_i c_0}{c_i d_i (p-1)}$. If $h'_i(\tilde{x}) \leq 0$, then h is decreasing so that $h_i(x_i) \leq 0$, $\forall x_i \geq 0$. If $h'_i(\tilde{x}) > 0$, then there exists $\bar{x}_i$ such that $h'_i(\bar{x}_i) = 0$ which is a maximum point for h_i. Condition $\max\limits_{i \notin J_1} h_i(\bar{x}_i) \leq 0$ implies $h_i(x_i) \leq 0$, $\forall x_i \geq 0$. The proof is complete. □

Remark 20 *When $0 < p < 1$, condition i) of Theorem 19 implies $a_i \leq \dfrac{(1-p)c_i}{2}$. Consequently, if there exists i such that $a_i > 0$, then f is not pseudoconvex.*

The following example points out that condition $\max\limits_{i \notin J_1} h_i(\bar{x}_i) \leq 0$, can not be relaxed.

Example 21 *Consider the function*

$$f(x, y) = -1,001x - 3,001y + \frac{2x + 3y + 8}{\sqrt{x + y + 1}},$$

the point $P = (898, 1)$ and the direction $w = (w_1, w_2)$ with $w_1 = -1212,843229$ and $w_2 = 400$; it can be verified that $\nabla f(P)^T w = 0$, and $w^T H(P) w < 0$. Therefore f is not pseudoconvex.
Referring to Theorem 19 , it is easy to verify that $a = -2c + 2,999d$

and $\operatorname{rank}[c, d] = 2$. *Since* $\alpha_1 = -2 < -1$, f *has no critical points and, by simple computations, it follows that necessary conditions i) and ii) of Theorem 19 are verified. Moreover* $h_1'(0) < 0$ *and* $\dfrac{c_1}{d_1} = 2 < \dfrac{p+1}{2}\dfrac{c_0}{d_0} = 6$, *so that* $1 \notin J_1$. *On the other hand, the maximum value of* h_1 *is positive and so* f *is not pseudoconvex.*

V.B. Pseudoconvexity on $\mathbb{R}^n_+$

In this section we prove that the pseudoconvexity of f on $\mathbb{R}^n_+$ is equivalent to the pseudoconvexity on $\operatorname{int}\mathbb{R}^n_+$. This result is obtained through several steps. We first state that the pseudoconvexity on $\operatorname{int}\mathbb{R}^n_+$ and on every face of $\mathbb{R}^n_+$ guarantees the pseudoconvexity of f on the whole set $\mathbb{R}^n_+$ (see Theorem 24). Then, thanks to the particular structure of the function, the characterization of the pseudoconvexity on every face can be substituted by the characterization of the pseudoconvexity on every edge. At last, the main result is obtained by proving that the pseudoconvexity on $\operatorname{int}\mathbb{R}^n_+$ implies the pseudoconvexity on every edge.

In order to study the behavior of f on the faces of $\mathbb{R}^n_+$, let us introduce the following notations.

Set $I = \{1, ..., n\}$, and let J be a subset of I with cardinality $\mid J \mid = k$, $1 \leq k < n$.

A face $\mathcal{F}_k$ of $\mathbb{R}^n_+$ with dimension k is defined as

$$\mathcal{F}_k = \{x \in \mathbb{R}^n_+ : x_i = 0, i \notin J\}.$$

An edge of $\mathbb{R}^n_+$ is a face of dimension 1.

Let f_k be the restriction of f on the face $\mathcal{F}_k$, that is

$$f_k(x^k) = a^{k^T} x^k + \frac{c^{k^T} x^k + c_0}{\left(d^{k^T} x^k + d_0\right)^p}$$

where a^k, c^k, d^k, x^k are obtained from a, c, d, x respectively, by deleting all the i-th components such that $i \notin J$. Observe that the quadratic form associated with the Hessian matrix of f_k is of the following form

$$w^T H_k(x^k)w = \frac{p\left[-2\left(d^{k^T}x^k + d_0\right)\left(w^T d^k\right)\left(c^{k^T}w\right) + (p+1)\left(c^{k^T}x^k + c_0\right)(d^{k^T}w)^2\right]}{\left(d^{k^T}x^k + d_0\right)^{p+2}} \tag{20}$$

with $w \in \mathbb{R}^k$. Moreover the critical points of f_k are the solutions of the system obtained from (14), by setting $x_i = 0, i \notin J$. As a consequence, the following theorem holds.

Theorem 22 *Let $\mathcal{F}_k$ be a face of $\mathbb{R}^n$ such that* $\operatorname{rank}[c^k, d^k] = 2$. *$\bar{x}^k$ is a critical point for f_k if and only if $\bar{x}$ is a critical point for f, where $\bar{x}_i = \bar{x}_i^k$ for every $i \in J$ and $\bar{x}_i = 0$ for every $i \notin J$.*

The following theorem points out that if f has a critical point $x \neq 0$, $x \in \mathcal{F}_k$, then it is not pseudoconvex.

Theorem 23 *Assume* $\operatorname{rank}[c, d] = 2$ *and let $\bar{x} \in \mathcal{F}_k$, $1 \leq k < n$, $\bar{x} \neq 0$. If $\nabla f(\bar{x}) = 0$, then f is not pseudoconvex on $\mathbb{R}^n_+$.*

Proof Let us preliminary observe that, from Theorem 22, $\bar{x}$ is a critical point for f if and only if $\bar{x}^k$ is a critical point for the restriction f_k.
Consider the case $\bar{x}$ belongs to the relative interior of a face $\mathcal{F}_k$ with $\operatorname{rank}[c^k, d^k] = 2$. The linear independence of c^k, d^k implies that the quadratic form (20) is indefinite for every fixed $x^k \in \mathcal{F}_k$. Therefore $\bar{x}^k$ is not a minumum point for f_k, so that f_k and consequently f, are not pseudoconvex.
Whenever $\bar{x}$ belongs to an edge e^i, take an edge e^j, such that $\operatorname{rank}[(c_i, c_j), (d_i, d_j)] = 2$. The existence of such an edge follows from the assumption $\operatorname{rank}[c, d] = 2$. Let $\mathcal{F}_2$ be the face containing the edges e^i and e^j and let f_2 be the restriction of f on this face. Since the corresponding quadratic form H_2 is indefinite, there exists an eigenvector $v = (v_i, v_j) \in \mathbb{R}^2$ such that $v^T H^2(\bar{x}^2)v < 0$. Let $y \in \mathbb{R}^n$ be such that $y_i = \epsilon w_i + \bar{x}_i$, $y_j = \epsilon w_j$ and $y_s = 0$ for $s = 1, ..., n, s \neq i, j$. Taking $w = v$ or $w = -v$, there exists a suitable $\epsilon > 0$ such that $y \in \mathcal{F}_2$, and $y^T H(\bar{x})y < 0$. Consequently f is not pseudoconvex.
It remains to consider the case $\bar{x}$ belongs to the relative interior of

any face $\mathcal{F}_k$ with $\mathrm{rank}[c^k, d^k] = 1$. In this case, every point of the hyperplane of equation $d^{k^T} x^k + d_0 = d^{k^T} \bar{x}^k + d_0$ is a critical point for f; in particular, there exists a critical point $\tilde{x}$ belonging to an edge of $\mathbb{R}^n$ and then, as in the previous case, f is not pseudoconvex. $\square$

The particular form of f allows to characterize the pseudoconvexity of f on $\mathbb{R}^n_+$ by means of the pseudoconvexity on $\mathrm{int}\mathbb{R}^n_+$ and of the pseudoconvexity on every face. With this regards, the following theorem holds.

Theorem 24 *Assume* $\mathrm{rank}[c, d] = 2$, $\nabla f(0) \neq 0$.
Then, f is pseudoconvex on $\mathbb{R}^n_+$ if and only if
i) f is pseudoconvex on $\mathrm{int}\mathbb{R}^n_+$*;*
ii) f is pseudoconvex on every face $\mathcal{F}_k$ of $\mathbb{R}^n_+$.

Proof If f is pseudoconvex, its restriction on a convex subset is still pseudoconvex, so that $i)$ and $ii)$ hold.
Assume now $i)$ and $ii)$ hold; we will prove that f is pseudoconvex on $\mathbb{R}^n_+$, by applying Definition 1.
Let $x^1, x^2 \in \mathbb{R}^n_+$ such that $f(x^1) > f(x^2)$.
Let us consider the line segment $[x^1, x^2]$; if $[x^1, x^2]$ is contained either in $\mathrm{int}\mathbb{R}^n_+$ or in a face, the result follows from i) and ii). In the other cases, $(x^1, x^2) \subset \mathrm{int}\mathbb{R}^n_+$. By continuity, there exists $\lambda \in (0, 1)$ such that $y = x^1 + \lambda(x^2 - x^1)$ with $f(x^1) > f(y)$. We must prove that $\nabla f(x^1)^T(x^2 - x^1) < 0$, or, equivalently, $\nabla f(x^1)^T(y - x^1) < 0$.
From Theorem 10 and Theorem 23, it follows that $\nabla f(x^1) \neq 0$. By contradiction, assume $\nabla f(x^1)^T(y - x^1) \geq 0$.
If $\nabla f(x^1)^T(y - x^1) > 0$, then $d = y - x^1$ is an increasing direction, so that the restriction $f(x^1 + td), t \in [0, 1]$ has a maximum point belonging to $\mathrm{int}\mathbb{R}^n_+$, and this contradicts $i)$. If $\nabla f(x^1)^T(y - x^1) = 0$, take $z = y + \epsilon\nabla f(x^1)$. For a suitable $\epsilon > 0$, we have $z \in \mathrm{int}\mathbb{R}^n_+$, $f(x^1) > f(z)$, and $\nabla f(x^1)^T(z - x^1) = \epsilon\|\nabla f(x^1)\|^2 > 0$. Once again, $f(x^1 + td^1)$, $d^1 = z - x^1, t \in [0, 1]$, has a maximum point which belongs to $\mathrm{int}\mathbb{R}^n_+$, and this contradicts $i)$. $\square$

Taking into account Theorem 23, the pseudoconvexity of f on the relative interior of a face $\mathcal{F}_k$, with $\text{rank}[c^k, d^k] = 2$, is completely characterized by the the behavior of f_k along the directions which are orthogonal to the gradient. Going back to the Theorems about the pseudoconvexity of f on $\text{int}\mathbb{R}^n$ (Theorem 16 and Theorem 19), we can easily seen that the stated conditions either involve only the parameters $\alpha_1, \alpha_2, p, c_0, d_0$ or they are componentwise conditions. Therefore, the pseudoconvexity of f on the relative interior of a face $\mathcal{F}_k$, with $\text{rank}[c^k, d^k] = 2$ follows directly by applying Theorem 16 or Theorem 19 and hence it remains to analyze the pseudoconvexity of f on the edges of $\mathbb{R}^n_+$. On the other hand, the results given in Section *III.* imply that the pseudoconvexity on the edges of a face $\mathcal{F}_k$ with $\text{rank}[c^k, d^k] = 1$ is equivalent to the pseudoconvexity on the whole face. Consequently, we get the following corollary

Corollary 25 *Assume* $\text{rank}[c, d] = 2$, $\nabla f(0) \neq 0$.
Then, f is pseudoconvex on $\mathbb{R}^n_+$ if and only if
i) f is pseudoconvex on $\text{int}\mathbb{R}^n_+$*;*
ii) f is pseudoconvex on every edge.

Consider the restriction $f_i(x_i)$ of f on the i-th edge of $\mathbb{R}^n_+$. By setting $c_i = \gamma d_i$ and $\alpha = \alpha_1\gamma + \alpha_2$,we rewrite f_i and h_i as follows

$$f_i(x_i) = \alpha d_i x_i + \frac{\gamma d_i x_i + c_0}{(d_i x_i + d_0)^p} \tag{21}$$

$$h_i(x_i) = \frac{(1-p)\gamma d_i x_i + c_0}{(d_i x_i + d_0)^p} + (\alpha_1(p+1)\gamma + 2\alpha_2)\, d_i x_i + \alpha_1(p+1)c_0 + 2\alpha_2 d_0 \tag{22}$$

The following lemma points out some relationships between $f_i(x_i)$ and $h_i(x_i)$.

Lemma 26 *i) $h_i''(x_i) = (1-p) f_i''(x_i)$;*
ii) there exists $\mu \in \mathbb{R}$ such that $h_i'(x_i) = (1-p) f_i'(x_i) + \mu$;
iii) $f_i'(\tilde{x}_i) = 2\,(d_i\tilde{x}_i + d_0)\, h_i(\tilde{x}_i)$ where $\tilde{x}_i = \dfrac{2\gamma d_0 - (p+1)\, c_0}{\gamma(p-1) d_i}$

Proof i) and ii) are obvious.
iii) Substituting $d_i\tilde{x}_i = \dfrac{2\gamma d_0 - (p+1)\, c_0}{\gamma(p-1)}$ in $h_i(\tilde{x}_i)$ and in $f_i'(\tilde{x}_i)$, we

obtain

$$h_i(\tilde{x}_i) = \frac{2\,(c_0 - \gamma d_0)}{(d_i\tilde{x}_i + d_0)^p} + 2\alpha(d_i\tilde{x}_i + d_0) = 2(d_i\tilde{x}_i + d_0)\left(\alpha + \frac{c_0 - \gamma d_0}{(d_i\tilde{x}_i + d_0)^{p+1}}\right);$$

$$f_i'(\tilde{x}_i) = d_i\left(\alpha + \frac{c_0 - \gamma d_0}{(d_i\tilde{x}_i + d_0)^{p+1}}\right).$$

The proof is complete. □

Finally, we are ready to prove our main theorem.

Theorem 27 *Assume* $\operatorname{rank}[c, d] = 2$, $\nabla f(0) \neq 0$ *and* $c \in \operatorname{int}\mathbb{R}^n_+$. *Then, f is pseudoconvex on $\mathbb{R}^n_+$ if and only if f is pseudoconvex on* $\operatorname{int}\mathbb{R}^n_+$.

Proof Taking into account Corollary 25, it is sufficient to prove that the pseudoconvexity on $\operatorname{int}\mathbb{R}^n_+$ implies the pseduconvexity on every edge i. Note that $\frac{c_i}{d_i} = \gamma > 0$.

Case $\alpha_1 \geq 0$ and $p > 1$. We refer to Theorem 6.

If $\frac{c_i}{d_i} \leq \frac{p+1}{2}\frac{c_0}{d_0}$, then i) of Theorem 6 is verified.

If $\frac{c_i}{d_i} > \frac{p+1}{2}\frac{c_0}{d_0}$, then $h_i'(x_i)$ has a maximum point at

$$\bar{x}_i = \frac{2c_i d_0 - (p+1)c_0 d_i}{(p-1)c_i d_i}.$$

Consequently f_i' has a minimum point at $\bar{x}_i$ and furthermore, from Lemma 26, $f_i'(\bar{x}_i) > 0$, so that ii) of Theorem 6 is verified.

Case $\alpha_1 \geq 0$ and $0 < p < 1$. We refer to Theorem 7.

The necessary condition $i)$ of Theorem 16 can be rewritten as follows $\alpha_1 c_i(p-1) + 2a_i \geq 0$, $\forall i$; therefore $a_i \geq 0$, $\forall i$, so that, for every one-dimensional face, we have $\alpha = \alpha_1\gamma + \alpha_2 \geq 0$. The thesis follows from $i)$ of Theorem 7.

Case $\alpha_1 \leq -\frac{1}{d_0^p}$ and $p > 1$. We refer to Theorem 6.

If $\frac{c_i}{d_i} \leq \frac{p+1}{2}\frac{c_0}{d_0}$, then i) of Theorem 6 holds.

If $\frac{c_i}{d_i} > \frac{p+1}{2}\frac{c_0}{d_0}$, then ii) of Theorem 19 becomes

$$2\left(\alpha d_0^{p+1} + d_0\frac{c_i}{d_i} - pc_0\right) + (1+\alpha_1 d_0^p)\left((p+1)c_0 - 2\frac{c_i}{d_i}d_0\right) \leq 0 \quad (23)$$

so that $\alpha d_0^{p+1} + d_0\frac{c_i}{d_i} - pc_0 < 0$ and hence ii) of Theorem 6 holds.
Case $\alpha_1 \leq -\frac{1}{d_0^p}$, $0 < p < 1$. We refer to Theorem 7.
If $\frac{c_i}{d_i} < \frac{p+1}{2}\frac{c_0}{d_0}$, then f_i' has a maximum point at $\bar{x}_i$. From Lemma 26 it is $f_i'(\bar{x}_i) < 0$, so that iii) of Theorem 7 holds.
If $\frac{c_i}{d_i} \geq \frac{p+1}{2}\frac{c_0}{d_0}$, then, from (23), we have $\alpha d_0^{p+1} + d_0\frac{c_i}{d_i} - pc_0 < 0$, so that ii) of Theorem 7 holds. The proof is complete. □

Corollary 28 *Assume* $\mathrm{rank}[c,d] = 2$, *and* $c \in \mathrm{int}\mathbb{R}^n_+$.
Then, f is pseudoconvex on $\mathbb{R}^n_+ \setminus \{0\}$ if and only if f is pseudoconvex on $\mathrm{int}\mathbb{R}^n_+$.

V.C. The particular case $\nabla f(0) = 0$

So far, all results are obtained in the case $\nabla f(0) \neq 0$. The following example shows that when the origin is a critical point f may be not pseudoconvex.

Example 29 *Consider the function*

$$f(x,y) = -\frac{1}{125}x - \frac{9}{500}y + \frac{x+2y+1}{(x+y+10)^2}.$$

It is to verify that conditions i), ii) and iii) of Theorem 19 hold with $\alpha_1 = -\frac{1}{100}$ and $\alpha_2 = \frac{1}{500}$, and hence f is pseudoconvex on $\mathbb{R}^n_+ \setminus \{0\}$. Moreover $(0,0)$ is a critical point and it is a maximum for the restriction of f on the half-line $y \geq 0$. Therefore f is not pseudoconvex $\mathbb{R}^n_+ \setminus \{0\}$.

The following theorem provides necessary and sufficient conditions for the pseudoconvexity of f when the origin is a critical point. Note that $\nabla f(0) = 0$ if and only if $\alpha_1 = -\frac{1}{d_0^p}$ and $\alpha_2 = \frac{pc_0}{d_0^{p+1}}$.

Theorem 30 *Assume* $\text{rank}[c, d] = 2$, $c \in \text{int}\mathbb{R}^n_+$, $\alpha_1 = -\frac{1}{d_0^p}$ *and* $\alpha_2 = \frac{pc_0}{d_0^{p+1}}$.
Then, f is pseudoconvex on $\mathbb{R}^n_+$ if and only if the following conditions holds:
i) $p > 1$;
ii) $\frac{2p}{p+1}\frac{c_0}{d_0} \leq \min_i \frac{c_i}{d_i}$;
iii) $\max_i \frac{c_i}{d_i} \leq \frac{p+1}{2}\frac{c_0}{d_0}$.

Proof Assume that f is pseudoconvex. Condition ii) follows immediately from i) of Theorem 19
Consider now the restriction $\varphi_u(t)$ of f on the half-line $x = tu$, $t \geq 0$, $u \in \mathbb{R}^n_+$. We have

$$\varphi_u'(t) = \frac{pc_0 d^T u - d_0 c^T u}{d_0^{p+1}} + \frac{(1-p)c^T u d^T u t + d_0 c^T u - pc_0 d^T u}{(td^T u + d_0)^{p+1}}$$

$$\varphi_u''(t) = \frac{pd^T u}{(td^T u + d_0)^{p+2}} \left[(p-1)c^T u d^T u \, t - 2d_0 c^T u + (p+1)c_0 d^T u\right] \tag{24}$$

Since $\varphi_u'(0) = 0$, necessarily we must have $\varphi_u''(0) \geq 0$, i.e.

$$\frac{c^T u}{d^T u} \leq \frac{p+1}{2}\frac{c_0}{d_0}, \forall u \in \Re^n_+$$

which is equivalent to *iii*). We are left to show that $p > 1$.
Suppose that $0 < p < 1$; since ii) holds, there exists $u \in \mathbb{R}^n_+$ such that $\frac{c^T u}{d^T u} > \frac{pc_0}{d_0}$ and hence $\lim_{t \to +\infty} \varphi_u'(t) < 0$. Moreover, from (24) and from condition iii), $\varphi_u'(t)$ has a maximum point $\tilde{t} > 0$. Since $\varphi_u'(\tilde{t}) > 0$, φ_u' has one zero corresponding to a maximum point for

$\varphi_u(t)$, contradicting the pseudoconvexity of f. Viceversa, assume that conditions i), ii) and iii) hold.
Taking into account $\alpha_1 = -\dfrac{1}{d_0^p}$ and $\alpha_2 = \dfrac{pc_0}{d_0^{p+1}}$, and conditions i) and ii), from Theorem 19, f is pseudoconvex on $\mathrm{int}\mathbb{R}^n_+$. From Corollary 28, f is pseudoconvex on $\mathbb{R}^n_+ \setminus \{0\}$. It remains to prove that any restriction of f on the half-line $x = tu$, $t \geq 0$, $u \in \mathbb{R}^n_+$ is pseudoconvex; observe that conditions $p > 1$ and $\dfrac{c^T u}{d^T u} \leq \dfrac{p+1}{2}\dfrac{c_0}{d_0}$, $\forall u \in \mathbb{R}^n_+$, guarantee the convexity of every restriction $\varphi_u(t)$ and this completes the proof. $\square$

VI. Conclusion

In this paper we have characterized the pseudoconvexity on $\mathbb{R}^n_+$ of a wide class of generalized fractional functions. The obtained conditions are very easy to be checked and according to them, several classes of pseudoconvex functions can be constructed.
The nice properties of pseudoconvexity suggest further developments. With respect to scalar optimization problems we aim to propose simplex-like sequential methods for solving problems having this kind of functions as objective and a polyhedral set as feasible region. Moving from the scalar to the bicriteria case, we aim also to derive the efficient frontier when one of the two objectives is linear and the other one belongs to the studied class.

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