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On maximizing a sum of ratios

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ON MAXIMIZING A SUM OF RATIOS *

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* The Paper has been discussed jointly by the Authors - CAMBINI has developed section 5; MARTEIN has developed sections 2-3-4, SCHAIBLE has developed section 1.

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INTRODUCTION. The problem of maximizing the sum of m concave-convex fractional functions defined on a convex set is considered; this problem is shown to be equivalent to the one whose objective function f is the sum of m linear fractional functions (of a very simple form) defined on a suitable convex set. Successively, using the Charnes-Cooper trasformation, f is transformed into the sum of one linear function and (m-1) linear fractional functions.
As a particular case, the problem P of maximizing the sum of two linear fractional functions subject to linear constraints is considered and some theoretical properties are established which allow us to propose two sequential methods to solve problem P. By means of these sequential methods, which are based on the concept of optimal level solution, a global maximum point for P is found (if one exists) generating a finite sequence of local maximum points. The algorithms are shown to be convergent in a finite number of iterations.

1. EQUIVALENCES OF FRACTIONAL PROGRAMS

Consider the following fractional program

$$(P_1) \qquad \alpha \overset{\Delta}{=} \sup_{x \in S} \sum_{i=1}^{m} \frac{f_i(x)}{g_i(x)}$$

where the feasible region S is a convex subset of $\mathbb{R}^n$, $f_i:\mathbb{R}^n \to \mathbb{R}$, $i=1,\dots,m$, is a nonnegative concave function and $g_i:\mathbb{R}^n \to \mathbb{R}$, $i=1,\dots,m$, is a positive convex function. Set $f(x) \overset{\Delta}{=} (f_1(x),\dots,f_m(x))$.

$g(x) \triangleq (g_1(x),\dots,g_m(x))$, $y=(y_1,\dots,y_m)$, $z = (z_1,\dots,z_m)$; problem P_1 turns out to be equivalent, in the sense given by theorem 1.1, to the following problem whose objective function is the sum of linear fractional functions and whose feasible region is convex:

$$(P_2) \qquad \beta \triangleq \sup \sum_{i=1}^{m} y_i/z_i \ , \ (x,y,z) \in S_1$$

where[(1)] $S_1 \triangleq \{(x,y,z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m : x \in S,\ y \leq f(x),\ z \geq g(x)\}$.

THEOREM 1.1. We have $\alpha = \beta$. Furthermore:

i) if $x^\circ \in S$ is an optimal solution for problem P_1 then $(x^\circ, y^\circ = f(x^\circ),\ z^\circ = g(x^\circ))$ is an optimal solution for problem P_2:

ii) if $(x^\circ, y^\circ, z^\circ) \in S_1$ is an optimal solution for problem P_2, then $y^\circ = f(x^\circ)$, $z^\circ = g(x^\circ)$ and x° is optimal for P_1.

Proof: - $\underline{\beta \geq \alpha}$. Let us note that $(x, y=f(x),\ z=g(x))$, $x \in S$, is a feasible point for P_2 and furthermore $\sum_{i=1}^{m} \frac{y_i}{z_i} =$ $= \sum_{i=1}^{m} \frac{f_i(x)}{g_i(x)}$, so that $\beta \geq \alpha$.

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(1) Let $a, b \in \mathbb{R}^n$, $a \leq b$ means $a_i \leq b_i$, $i=1,\dots,n$.

– $\underline{\alpha \geq \beta}$. Without loss of generality, we can suppose $y \geq 0$ in P_2; in fact, for any feasible point (x,y,z), $y_j < 0$, we can find $(x,\tilde{y},z) \in S_1$, $\tilde{y}_i = y_i$, $i \neq j$, $\tilde{y}_j = 0$, such that

$$\sum_{i=1}^{m} \frac{\tilde{y}_i}{z_i} > \sum_{i=1}^{m} \frac{y_i}{z_i} .$$

Let $(x,y,z) \in S_1$; we have $z_i \geq g_i(x)$, $i=1,\dots,m$, so that

$\sum_{i=1}^{m} \frac{y_i}{z_i} \leq \sum_{i=1}^{m} \frac{y_i}{g_i(x)} \leq \sum_{i=1}^{m} \frac{f_i(x)}{g_i(x)}$; this implies $\beta \leq \alpha$. At last

we will prove i) and ii); i) is obvious; ii) follows by noting

that $0 \leq \frac{y_i^\circ}{z_i^\circ} \leq \frac{f_i(x^\circ)}{g_i(x^\circ)}$ and $\sum_{i=1}^{m} \frac{y_i^\circ}{z_i^\circ} = \sum_{i=1}^{m} \frac{f_i(x^\circ)}{g_i(x^\circ)}$ since $\alpha = \beta$. □

Problem P_2 is a particular case of the following one

$$(P_3) \qquad \alpha' = \sup \phi(x) \triangleq \sum_{i=1}^{m} \frac{(c^i)^T x}{(d^i)^T x} , \quad x \in S$$

where S is a convex subset of $\mathbb{R}^p$, $c^i \in \mathbb{R}^p$, $d^i \in \mathbb{R}^p$, $i=1,\dots,m$; $(d^i)^T x > 0$, $\forall x \in S$, $i=1,\dots,m$.
By means of the following Charnes-Cooper transformation

$t = \frac{1}{(d^j)^T x}$, $y = tx$, problem P_3 can be transformed into the following one:

$$(P_4) \qquad \beta' = \sup \psi(t,y) \triangleq (c^j)y + \sum_{\substack{i=1 \\ i \neq j}}^{m} \frac{(c^i)^T y}{(d^i)^T y} , \quad (t,y) \in S'$$

where $S' = \{(t,y) \in \mathbb{R}^{p+1} : \frac{y}{t} \in S,\ (d^j)^\tau y = 1,\ t > 0\}$.

Problem P_3 and P_4 are equivalent in the sense given by the following theorem:

THEOREM 1.2. We have $\alpha' = \beta'$. Furthermore $x^\circ \in S$ is an optimal solution for problem P_3 iff $\left(t^\circ = \frac{1}{(d^j)^\tau x^\circ},\ y^\circ = t^\circ x^\circ\right)$ is optimal for P_4.

Proof: Let $\{x_n\} \subset S$ be a sequence such that $\lim_{n\to+\infty} \phi(x_n) = \alpha'$; set $t_n = \frac{1}{(d^j)^\tau x_n}$, $y_n = t_n x_n$. We have $\phi(x_n) = \psi(t_n, y_n)$,

$\lim_{n\to+\infty} \phi(x_n) = \lim_{n\to+\infty} \psi(t_n,y_n) = \alpha'$, so that $\alpha' \leqq \beta'$. Consider now a sequence $\{(t_n,y_n)\} \subset S'$ such that $\lim_{n\to+\infty} \psi(t_n,y_n) = \beta'$ and set $x_n = \frac{y_n}{t_n}$. Since $(d^j)^\tau y_n = 1$, $\forall n$, we have $t_n (d_j)^\tau x_n = 1$,

$t_n = \frac{1}{(d_j)^\tau x_n}$, $\psi(t_n,y_n) = \phi\left(\frac{y_n}{t_n}\right) = \phi(x_n)$. Then $\lim_{n\to+\infty} \psi(t_n,y_n) =$

$= \lim_{n\to+\infty} \phi(x_n) = \beta'$, so that $\beta' \leq \alpha'$.

To complete the proof it is sufficient to note that $\phi(x^\circ) = \psi(t^\circ, y^\circ)$.

□

2. THEORETICAL PROPERTIES OF A SPECIAL FRACTIONAL PROGRAM

Throughout the remainder of the article we will consiser problem P_3 where the objective function is the sum of two linear fractional functions and the feasible region is a polyhedral set (not necessarily bounded), that is the problem(2):

$$(P): \quad \sup \ \phi(x) \triangleq \frac{h^T x}{p^T x} + \frac{c^T x}{d^T x}, \ x \in S = \{x \in \mathbb{R}^n : Ax=b, \ x \geq 0\}$$

where h, p, c, d $\in \mathbb{R}^n$, $p^T x > 0$, $d^T x > 0$, $\forall x \in S$.
From Theorem 1.2, problem P is equivalent to the one whose objective function is the sum of a linear function and a linear fractional function, that is the problem:

$$(P'): \ \sup \ \psi(t,y) \triangleq h^T y + \frac{c^T y}{d^T y}, \ (t,y) \in S' =$$

$$= \{ (t,y) \in \mathbb{R}^{n+1} : Ay-bt=0 \ , \ p^T y=1, \ t > 0 \}$$

where $d^T y > 0$, $\forall (t,y) \in S'$.

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(2) Let us note that the quotient of two affine functions $\frac{a^t z+a_o}{b^t z+b_o}$ can be transformed into the quotient of two linear functions $\frac{\bar{a}^t z'}{\bar{b}^t z'}$ by setting $z' = (z,t)$, $\bar{a} = (a,a_o)$, $\bar{b} = (b,b_o)$ with the further constraint $t=1$.

Problem P' has been studied in [8,11,12] and some of these results allow us to establish some theoretical properties for problem P.

The following theorem holds:

THEOREM 2.1. Suppose that problem P has optimal solutions, then one of these solutions belongs to an edge of the feasible region S.

Proof: Let us note that the Charnes-Cooper transformation is a bijective function and furthermore the image of the intersection of k hyperplanes of $\mathbb{R}^n$ is the intersection of (k+1) hyperplanes of $\mathbb{R}^{n+1}$, taking $d^T y=1$ into account. As a consequence, a vertex (edge) of S is transformed into a vertex (edge) of S'.
The thesis follows by noting that the statement of the theorem holds for problem P' [8] . □

The objective function of problem P is not, in general, a strictly quasi-concave function, so that P can have several local maximum points; if x^k is a local maximum point, we will refer to $\phi(x^k)$ as a local maximum value.

The following theorem holds:

THEOREM 2.2. The set of local maximum values of problem P is finite.

Proof: As shown in [8] , the statement of the theorem holds for problem P'; the thesis follows by noting that the Charnes-Cooper transformation is locally an homeomorphism and $\psi(t,y) = \phi(\frac{y}{t})$. □

3. OPTIMAL LEVEL SOLUTIONS

Our aim is to propose an algorithm to solve problem P or, equivalently (see theorem 1.2) problem

$$(P^*):\quad \sup\left[z(x) \triangleq \left(h^T x + \frac{c^T x + c_0}{d^T x + d_0}\right)\right],\quad x \in S = \{x \in \mathbb{R}^n : Ax=b, x \geq 0\}$$

where $h, c, d \in \mathbb{R}^n$, $c_0, d_0 \in \mathbb{R}$, $d^T x + d_0 > 0$, $\forall x \in S$. (We will suppose that degeneracy does not occur in S).

Let x^o be a vertex of the feasible region S with corresponding basis B. We partition the vectors x, h, c and d as $x=(x_B,x_N)$, $h=(h_B,h_N)$, $c=(c_B,c_N)$, $d=(d_B,d_N)$ and the matrix A as $A=[B \,\vdots\, N]$.

Set $\bar{h}_N^T = h_N^T - h_B^T B^{-1}N$, $\bar{c}_N^T = c_N^T - c_B^T B^{-1}N$, $\bar{d}_N = d_N^T - \bar{d}_B^T B^{-1}N$, $\bar{h}_o = h^T x^o$, $\bar{c}_o = c^T x^o + c_o$, $\bar{d}_o = d^T x^o + d_o$, $x_B^o = B^{-1}b$.

The following theorem states a local condition for x^o to be optimal. Set $\Gamma = \bar{d}_o^2\, \bar{h}_N + \bar{d}_o \bar{c}_N - \bar{c}_o \bar{d}_N$ and $J = \{j : \Gamma_j > 0\}$.

THEOREM 3.1. Let $x^o \in S$ be a vertex. Then x^o is an optimal solution for problem P* iff $J = \emptyset$.

Proof: The thesis follows by calculating the partial derivatives of the function $z(x_N) = \bar{h}_N^T x_N + \dfrac{\bar{c}_N^T x_N + \bar{c}_o}{\bar{d}_N^T x_N + \bar{d}_o}$. □

The sequential method, which we will describe in section 5 is obtained combining two algorithms given in [4,8] and is based on the concept of an optimal level solution.

We will refer to $\bar{x} \in S$ as an optimal level solution (in short o.l.s.) iff $\bar{x}$ solves the problem

$$P^*(\bar{\xi}): \begin{cases} \dfrac{1}{\bar{\xi}} \max \; (\bar{\xi} h^T x + c^T x + c_o) \\ x \in S \\ d^T x + d_o = \bar{\xi} \end{cases}$$

where $\bar{\xi} = d^T \bar{x} + d_o$.

The idea of the algorithm is to find a local maximum point for problem P* by generating a finite sequence of optimal level solutions. To this aim we start to characterize a vertex $x^o \in S$ which is also an optimal level solution. Let us note that the feasible region S' of the problem $P^*(\xi^o)$, $\xi^o = \bar{d}_o$, is obtained adding to S the constraint $d^T x + d_o = \bar{d}_o$, so that the new basis associated with $x^o \in S'$ is of the form $[B \vdots N^{(k)}]$, where $N^{(k)}$ denotes a suitable column of N. Then, for x^o to be optimal for $P^*(\xi^o)$, the corresponding reduced costs must be nonpositive, namely:

there exists $\bar{d}_{N_k} \neq 0$ such that

$$\bar{d}_o \tilde{h}_N + \tilde{c}_N \leq 0 \tag{1}$$

where: $\tilde{h}_N = \bar{h}_N - \frac{\bar{h}_{N_k}}{\bar{d}_{N_k}} \bar{d}_N$, $\tilde{c}_N = \bar{c}_N - \frac{\bar{c}_{N_k}}{\bar{d}_{N_k}} \bar{d}_N$.

Consider now the edge $s_k \subset S$ whose equation, in the basic variables space, is:

$$x_B(x_{N_k}) = x_B^o - B^{-1}N^{(k)} x_{N_k} \ . \tag{2}$$

Let $\bar{x} = (x_B(\bar{x}_{N_k}),\ \bar{x}_{N_k}, 0) \in s_k$ and $\bar{\xi} = d^T\bar{x} + d_o$. Then x is an optimal solution for $P^*(\bar{\xi})$ iff (3) holds:

$$(\bar{d}_o + \bar{d}_{N_k} \bar{x}_{N_k}) \tilde{h}_N + \tilde{c}_N \leq 0 \tag{3}$$

As a consequence, a point of s_k is an optimal level solution for any x_{N_k} such that:

$$0 \leq x_{N_k} \leq \alpha_1 \text{ , where}$$

$$\alpha_1 = \begin{cases} +\infty & \text{if } \bar{d}_{N_k}\tilde{h}_N \leq 0 \\ \min\limits_{\bar{d}_{N_k}\tilde{h}_{N_i} > 0} \left(- \dfrac{\bar{d}_o \tilde{h}_{N_i} + \tilde{c}_{N_i}}{\bar{d}_{N_k}\tilde{h}_{N_i}} \right) & \text{otherwise .} \end{cases}$$

In order to find $\bar{d}_{N_k}$ satisfying (1), we will prove the following theorem:

THEOREM 3.2. Let x° be an optimal level solution and a vertex of S; suppose $J \neq \emptyset$. Then conditions i) and ii) hold.

i) <u>either</u> $\bar{d}_{N_j} > 0, \forall j \in J$ <u>or</u> $\bar{d}_{N_j} < 0, \forall j \in J$;

ii) $\bar{d}_{N_k}$ satisfies (1) iff (4a) or (4b) hold.

$$\frac{\bar{d}_o \bar{h}_{N_k} + \bar{c}_{N_k}}{\bar{d}_{N_k}} = \max_{\substack{j \in J \\ \bar{d}_{N_j} > 0}} \frac{\bar{d}_o \bar{h}_{N_j} + \bar{c}_{N_j}}{\bar{d}_{N_j}} \quad \text{if } \bar{d}_{N_j} > 0, \forall j \in J \qquad (4a)$$

$$\frac{\bar{d}_o\bar{h}_{N_k}+\bar{c}_{N_k}}{\bar{d}_{N_k}} = \min_{\substack{j\in J\\ \bar{d}_{N_j}<0}} \frac{\bar{d}_o\bar{h}_{N_j}+\bar{c}_{N_j}}{\bar{d}_{N_j}} \quad \text{if } \bar{d}_{N_j}<0\ ,\ \forall j\in J\ . \qquad (4b)$$

Proof: i) Since x° is an optimal solution for problem $P^*(\xi^\circ)$, $\xi^\circ=d^T x^\circ+d_o$, there exists an index $k\in J$ such that (1) holds; on the other hand we have

$$\bar{d}_o\bar{h}_{N_j}+\bar{c}_{N_j} > \frac{\bar{c}_o}{\bar{d}_o}\ \bar{d}_{N_j}\ ,\ \forall j\in J\ . \qquad (5)$$

It results $\bar{d}_{N_j}\neq 0$, $\forall j\in J$ since (1) and (5) are in contradiction for $\bar{d}_{N_j}=0$.

To prove that $\bar{d}_{N_k}>0$ implies $\bar{d}_{N_j}>0$, $\forall j\in J$ it is sufficient to note that if there exists $j\in J$ such that $\bar{d}_{N_j}<0$, we have, taking into account (1) and (5), that

$$\frac{\bar{c}_o}{\bar{d}_o} < \frac{\bar{d}_o\bar{h}_{N_k}+\bar{c}_{N_k}}{\bar{d}_{N_k}} \leq \frac{\bar{d}_o\bar{h}_{N_j}+\bar{c}_{N_j}}{\bar{d}_{N_j}} < \frac{\bar{c}_o}{\bar{d}_o}$$

which is impossible. The case $\bar{d}_{N_j}<0$, $\forall j\in J$ can be proven in a similar way.

ii): This follows taking into account (i) and relation (1).

This completes the proof. □

REMARK 3.1. In order to find an optimal level solution we suggest the following procedure: consider problem P_o: $\min_{x\in S} (d^T x+d_o)=\xi^o$; let us note that P_o has optimal solutions since its objective function is lower bounded on S. If P_o has the unique solution $\hat{x}$, then $\hat{x}$ is also an o.l.s. for P*; otherwise we solve the problem P^*_o: $\max\{(\xi^o h^T x+c^T x+c_o),\ x\in S,\ d^T x+d_o=\xi^o\}$. If P^*_o has no solution, then $\sup_{x\in S} z(x)=+\infty$; otherwise an optimal solution of P^*_o is also an o.l.s. for P*.

REMARK 3.2. The sequential method, which will be described in section 5, generates a finite sequence of o.l.s. corresponding to increasing levels of the linear function $d^T x+d_o$; this means that the value of d_{N_k} in theorem 3.2 is positive.

4. LOCAL OPTIMALITY CONDITIONS

Let x^o be an o.l.s. for problem P*, which is also a vertex of S and consider the restriction $z(x_{N_k})$ of the function

$$z(x_N)=\bar{h}^T x_N + \frac{\bar{c}_N^T x_N+\bar{c}_o}{\bar{d}_N^T x_N+\bar{d}_o}$$

on the edge $s_k\subset S$, defined by (2).

Let $\bar{\gamma}_k = \bar{d}_o \bar{c}_{N_k} - \bar{c}_o \bar{d}_{N_k}$. Then the following theorem holds:

THEOREM 4.1.

i) If $\bar{h}_{N_k} \leq 0$ and $\bar{\gamma}_k \leq 0$, then x^o is a local maximum point for problem P*;

ii) If $\bar{h}_{N_k} = 0$ and $\bar{\gamma}_k > 0$, or $\bar{h}_{N_k} > 0$, then $z(x_{N_k})$ is an increasing function in $[0, \alpha_2[$, with $\alpha_2 = +\infty$;

iii) If $\bar{h}_{N_k} < 0$ and $\bar{\gamma}_k > 0$, then $z(x_{N_k})$ is increasing in

$[0, \alpha_2]$, where $\alpha_2 = \dfrac{-\bar{d}_o}{\bar{d}_{N_k}} - \dfrac{1}{\bar{h}_{N_k} \bar{d}_{N_k}} \sqrt{-\bar{h}_{N_k} \bar{\gamma}_k}$.

Proof: The derivative of $z(x_{N_k})$ is

$$z'(x_{N_k}) = \frac{\bar{h}_{N_k} (\bar{d}_{N_k} x_{N_k} + \bar{d}_o)^2 + \bar{\gamma}_k}{(\bar{d}_{N_k} x_{N_k} + \bar{d}_o)^2} \quad ;$$

the thesis follows by analysing the sign of $z'(x_{N_k})$ taking into account Remark 3.2.

□

Let a_{i_k} be the i-th element of the column $B^{-1}N^{(k)}$; set

$I = \{i: a_{i_k} > 0\}$ and

$$\alpha_3 = \begin{cases} +\infty & \text{if } I = \emptyset \\ \min_{i \in I} \dfrac{x^\circ_{B_i}}{a_{i_k}} & \text{if } I \neq \emptyset \end{cases}.$$

The sequential method, which we will propose, is based on an analysis of feasibility, level optimality and the sign of the derivative $z'(x_{N_k})$. More precisely using $\alpha = \min(\alpha_1, \alpha_2, \alpha_3)$, the following cases arise:

1) $\alpha = \alpha_1 < +\infty$; in this case no point of the edge s_k with $x_{N_k} > \alpha_1$ is an o.l.s.; we will see later how to restore the level optimality.

2) $\alpha = \alpha_2 < +\infty$; in such a case the point $\bar{x}=(x_B(\alpha_2), \alpha_2, 0)$ is a local maximum point for the restriction $z(x_{N_k})$; on the other hand, since $\alpha_2 \leq \alpha_1$ any point of the edge s_k, $0 \leq x_{n_k} \leq \alpha_2$, is an o.l.s., so that $\bar{x}$ is a local mximum point for P*.

3) $\alpha = \alpha_3 < +\infty$; since $\alpha_3 \leq \alpha_2$, $z(x_{N_k})$ is increasing in $[0, \alpha_3]$; on the other hand any point of the edge s_k is an o.l.s., so that $\bar{x} = (x_B(\alpha_3), \alpha_3, 0)$ is a vertex of S which is also

an o.l.s. and we have $z(\bar{x}) > z(x^{o})$.

4) $\alpha = \alpha_1 = \alpha_2 = \alpha_3 = +\infty$; taking into account that

$$\lim_{x_{N_k} \to +\infty} (z_{N_k}) = \begin{cases} +\infty & \text{if } \bar{h}_{N_k} \neq 0 \\ \dfrac{\bar{c}_{N_k}}{\bar{d}_{N_k}} & \text{if } \bar{h}_{N_k} = 0 \end{cases}$$

and $\sup_{x \in S} z(x) = \sup_{\xi} \sup_{\substack{x \in S \\ d^T x + d_o = \xi}} z(x)$, we have

$$\sup_{x \in S} z(x) = \begin{cases} +\infty & \text{if } \bar{h}_{N_k} \neq 0 \\ \dfrac{\bar{c}_{N_k}}{\bar{d}_{N_k}} & \text{if } \bar{h}_{N_k} = 0 \end{cases} .$$

In order to be able to give an algorithm to solve problem P*, we have to study the loss of level optimality (case 1)). When $\alpha = \alpha_1 < +\infty$, the point $\bar{x} = (x_B(\alpha_1), \alpha_1, 0)$ is an o.l.s. for

problem P*, but it is not a vertex of S, so that, in order to restore the level optimality, we must generate a finite sequence of o.l.s. which are not verteces of S.
To this aim, we propose a parametric approach.
Let us note that $\bar{x}$ is an optimal basic solution for problem $P^*(\bar{\xi})$, $\bar{\xi}=d^T\bar{x}+d_o$.

Consider the parametric problem

$$P^*(\bar{\xi}+\theta):\ z(\theta) = \frac{1}{\bar{\xi}+\theta} \max((\xi+\theta)h+c)^T x+c_o$$

$$Ax = b$$

$$d^T x = -d_o + \bar{\xi} + \theta$$

$$x \geqq 0$$

Let $\hat{B}$ be the basis associated to $\bar{x}$ and set:
$\hat{b}^T = (b, \bar{\xi}-d_o)$, $w = \hat{B}^{-1}e^{(m+1)}$ where $e^{(m+1)}$ is the unit vector whose last component is 1, $\bar{c}_o = c^T_{\hat{B}}\hat{B}^{-1}\hat{b}$, $\bar{c}_1 = c^T_{\hat{B}}w$, $\bar{h}_o=\hat{h}^T_B\hat{B}^{-1}\hat{b}$, $\bar{h}_1 = h^T_{\hat{B}}w$.
We can study again feasibility, level optimality and the sign of $z'(\theta)$.
As regards to feasibility, $\bar{x}(\theta)$ is feasible iff $\bar{b} + \theta w \geqq 0$.
Set $I_1=\{i: w_i < 0\}$; if $I_1 \neq \emptyset$, then $\bar{x}(\theta)$ is feasible

$\forall \theta \in [0,\theta_1]$ with $\theta_1 = \min\limits_{i\in I_1} -\dfrac{\bar{b}_i}{w_i}$, otherwise $\bar{x}(\theta)$ is feasible

$\forall\, \theta \in [0, \theta_1[\,, \quad \theta_1 = +\infty$.

As regards to level optimality, taking into account the definition of o.l.s., we must solve the inequality

$$(\bar{d}_o + \theta)\bar{h}_{\hat{N}} + \bar{c}_{\hat{N}} \leq 0 \quad . \tag{6}$$

Set $I_2 = \{i : \bar{h}_{\hat{N}_i} > 0\}$; if $I_2 \neq \emptyset$, then we have level optimality $\forall\, \theta \in [0, \theta_2]$ with $\theta_2 = \min_{i \in I_2} - \left[\bar{d}_o + \frac{\bar{c}_{\hat{N}_i}}{\hat{h}_{N_i}} \right]$, otherwise we have level optimality $\forall\, \theta \in [0, \theta_2[\,, \quad \theta_2 = +\infty$.

Consider now the function

$z(\theta) = \bar{h}_o + \bar{h}_1 \theta + \dfrac{\bar{c}_o + \bar{c}_1 \theta}{\bar{\xi} + \theta}$. We have

$z'(\theta) = \dfrac{\bar{h}_1 \theta^2 + 2\bar{h}_1 \bar{\xi}\, \theta + \bar{h}_1 \bar{\xi}^2 + \bar{c}_1 \bar{\xi} - \bar{c}_o}{(\bar{\xi} + \theta)^2}$, so that

$z'(0) = \bar{h}_1 \bar{\xi}^2 + \bar{c}_1 \bar{\xi} - \bar{c}_o$.

It is easy to show that:

- if $z'(0) \leq 0$, then $\bar{x}$ is a local maximum point for problem P*;
- if $z'(0) > 0$ and $\bar{h}_1 < 0$, then $z(\theta)$ is an increasing function

in $[0,\theta_3]$, with $\theta_3 = -\bar{\xi} + \sqrt{\dfrac{\bar{c}_o - \bar{c}_1\bar{\xi}}{\bar{h}_1}}$;

- if $z'(0) > 0$ and $\bar{h}_1 \geq 0$, then $z(\theta)$ is an increasing function in $[0,\theta_3[$, $\theta_3 = +\infty$.

The obtained results allow us to describe in the next section two procedures for solving problem P.

5. ALGORITHMS

Now we are able to suggest two sequential methods for solving problem P. Any algorithm will find the optimal solution, if one exists, by generating a finite sequence of local maximum points. These methods are shown to be convergent in a finite number of iterations.
First of all, problem P is transformed into problem P* (see sections 1, 2) and an o.l.s. x^o is found as in Remark 3.1.

Algorithm 1 (Parametric procedure)

STEP 1. Set $\xi^o = d^T x^o + d_o$; solve problem $P(\xi^o)$ and let $\hat{B}$ be the basis corresponding to the optimal solution. Go to step 2.

STEP 2. Calculate θ_1, θ_2, θ_3 and $\bar{\theta} = \min(\theta_1, \theta_2, \theta_3)$. If $\bar{\theta} = 0$, $\bar{x} = (x_B(0), 0)$ is a local maximum for P*, go to step 7; otherwise: if $\bar{\theta} = \theta_1$, go to step 3; if

$\bar{\theta} = \theta_2$ go to step 4; if $\bar{\theta} = \theta_3$ go to step 5; if $\bar{\theta} = +\infty$ go to step 6.

STEP 3. The basic variable, which becomes zero for $\theta = \theta_1$, leaves the basic and a pivot operation is performed according to the dual simplex algorithm. Let x^k be the optimal solution, set k=0 and go to step 1.

STEP 4. The non basic variable, whose reduced cost becomes zero for $\theta = \theta_2$, enters the basis and a pivot operation is performed according to the simplex algorithm. Let x^k be the optimal solution, set k=0 and go to step 1.

STEP 5. The point $\bar{x} = (x_B(\bar{\theta}), \bar{\theta}, 0)$ is a local maximum point for P*. Go to step 7.

STEP 6. Problem P* has no optimal solution and the algorithm stops; more exactly if $\bar{h}_1 > 0$ then $\sup_{x \in S} z(x) = +\infty$, while if $\bar{h}_1 = 0$ then $\sup_{x \in S} z(x) = \bar{h}_o + \bar{c}_1$.

STEP 7. Set $\bar{\xi} = d^T\bar{x} + d_o$ and solve $P^*(\bar{\xi} + \tilde{\theta})$ where

$$\tilde{\theta} = \frac{\bar{c}_o - \bar{\xi} z(\bar{\theta}) + \bar{\xi}\bar{h}_o}{\bar{h}_1 \bar{\theta}} \quad \text{if} \quad \bar{\theta} = \theta_3, \text{ while}$$

$$\tilde{\theta} = \frac{\bar{c}_o - \bar{c}_1\bar{\xi} - \bar{h}_1\bar{\xi}^2}{\bar{h}_1 \bar{\xi}} \quad \text{if} \quad \bar{\theta} = 0.$$ Let x^k be the optimal

solution, set k=0 and go to step 1.

Algorithm 2

The sequential method, that we are going to describe, differs from the previous one, since it uses the parametric approach only if the level optimality is lost.

STEP 1. Let x^o be an o.l.s. with corresponding besis B; go to step 2.

STEP 2. Calculate α_1, α_2, α_3 and set $\alpha = \min(\alpha_1, \alpha_2, \alpha_3)$. If $\alpha = 0$ then $\bar{x} = (x_B(0), 0)$ is a local maximum point for P*, go to step 7; otherwise: if $\alpha = \alpha_1$ go to step 3; if $\alpha = \alpha_2$ go to step 4; if $\alpha = \alpha_3$ go to step 5; if $\alpha = +\infty$ go to step 6.

STEP 3. Set $\xi^o = d^T x^o + d_o$. Starting from $P^*(\xi^o)$, apply the parametric procedure until either a local maximum point for P* is found or a basic optimal solution x^k is generated. In the first case, go to step 7, otherwise set k=0 and go to step 1.

STEP 4. The point $\bar{x} = (x_B(\alpha_2), \alpha_2, 0)$ is a local maximum point for P*, go to step 7.

STEP 5. $x_{N_k} = \alpha_3$ becomes a basic variable; a pivot operation is performed according to the simplex algorithm. Let x^k be the optimal solution, set k=0 and go to step 1.

STEP 6. Problem P* has no optimal solution, and the algorithm stops; more exactly if $\bar{h}_{N_k} \neq 0$ then $\sup_{x \in S} z(x) = +\infty$,

while if $\bar{h}_{N_k} = 0$ then $\sup_{x \in S} z(x) = \dfrac{\bar{c}_{N_k}}{\bar{d}_{N_k}}$.

STEP 7. Set $\hat{x}_{N_k} = \dfrac{\bar{c}_o - \bar{d}_o z(\alpha)}{\bar{h}_{N_k} \bar{d}_{N_k} \alpha}$ if $\alpha = \alpha_3$ and

$$\hat{x}_{N_k} = \frac{\bar{c}_o \bar{d}_{N_k} - \bar{d}_o \bar{c}_{N_k} - \bar{d}_o^2 \bar{h}_{N_k}}{\bar{d}_o \bar{h}_{N_k} \bar{d}_{N_k}}$$ if $\alpha = 0$. Let x^o be the

optimal solution of $P^*(\hat{\xi})$, $\hat{\xi} = d^T \hat{x} + d_o$,.
$\hat{x} = (x_B(\hat{x}_{N_k}), \hat{x}_{N_k}, 0)$, go to step 1; if $P^*(\hat{\xi})$ has no optimal solution, stop; $\bar{x}$ is an optimal solution for P^*.

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