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On the Charnes-Cooper Transformation in Linear Fractional Programming

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Abstract : A parametric version of the Charnes-Cooper algorithm, which is able to solve a linear fractional problem for any feasible region (bounded or unbounded), is given. The equivalence between this new algorithm and the one proposed in [2,3] is shown.

Keywords : Fractional programming ; Linear fractional problem.

Introduction

In the linear fractional program (shortly L.F.P.) a function is to be maximized (minimized) which is a ratio of linear functions subject to linear constraints.

Such programs arise under various circumstances in management science and in other economic areas. Among its applications we may quote stock cutting problems, maintenance problems as well as allocation of resources [7,8,13].

For this reason many algorithms have been proposed to solve L.F.P..

Experiments performed by Birtran [1] have shown that the Martos [6] algorithm is one of the most efficient among those proposed in the literature and, even before, Wagner and Yuan [14] have demonstrated that the Charnes-Cooper [5] algorithm and the preceding one, which are two of the most important algorithms for the resolution of linear fractional programming problems, are equivalent. In other words, starting from the same initial solution, both algorithms determine the problem's optimal solution, if one exists, throughout the same sequence of basic feasible solutions.

A common peculiarity of all these algorithms is to work only in the case of a bounded region.

In the line of the studies of Wagner and Yuan, Cambini and Sodini [4] give a theoretical justification of the computational results of Birtran and extend the comparison , still by a theoretical point of view, over an algorithm of a parametric nature, proposed by Cambini [3] , which has the advantage of solving problems for any region (boundedlunbounded) and is not equivalent to the preceding two algorithms in the case of a compact region.

The aim of this work is to show that, interpreting the linear problem obtained from L.F.P. by means of the Charnes-Cooper transformation as a linear parametric problem, the parametric algorithm, used to resolve it and which we shall describe, becomes equivalent to Cambini's algorithm in the sense that will be stated below. We will also emphasize how the new parametric algorithm makes it possible to solve the L.F.P. even if the region is unbounded.

1. The Linear Fractional Problem and the Charnes-Cooper Transformation.

Let us consider the linear fractional problem

$$P_F : z^*=\sup[\, z(x) = (c^tx+c_0)\, (d^tx+d_0)^{-1}] \quad , \; x \in S_F \equiv \{ x\in R^n : Ax=b \, , \; x\geq 0 \}$$

where A is a mxn matrix and $b\in R^m$; $c_0,d_0\in R$, $c,d\in R^n$ with the additional constraint $d^tx+d_0>0 \quad \forall x\in S_F$; we will suppose that degeneracy does not occur.

By means of the Charnes-Cooper transformation

$$(1.1) \qquad y = t\,x \quad , \quad t= 1/(d^tx+d_0)$$

the problem P_F may be rewritten as the linear problem

$$P_L: w^*=\sup[\, w(y,t) = c^ty+c_0t \,] \, , (y,t) \in S_L\equiv\{y\in R^n, t\in R : Ay -bt=0, \, d^ty+d_0t=1, \, y \geq 0, \, t > 0 \}$$

where A is a mxn matrix and $b\in R^m$; $c_0,d_0\in R$, $c,d\in R^n$.

Let us note that S_L is a non closed set, because of the constraint $t > 0$, and thus the usual algorithms of the linear programming cannot be applied. For this reason Charnes-Cooper have proposed the transformation (1.1) only when the feasible region S_F is bounded, i.e. $d^tx+d_0 \leq D^*$, $\forall x\in S_F$, to ensure a value $t > 0$ in any optimal solution for problem P_L.

In order to investigate in S_L the critical point $t = 0$, according to Schaible's studies [9,12] , we establish the following theorem and corollary

Theorem 1.1 P_F and P_L are equivalent in the following sense : we have $z^*=w^*$; furthermore if $x^*\in S_F$ is an optimal solution for P_F then $(y^*,t^*)\in S_L$, with $t^*=1/(d^tx^*+d_0)$ and $y^*=t^*x^*$, is an optimal solution for P_L; on the contrary if $(y^*,t^*)\in S_L$, $t^*>0$ is an optimal solution for P_L then $x^*=y^*/t^*$ is an optimal solution for P_F.

proof. : Let $\{x_n\}\in S_F$ be a sequence such that $\lim_n z(x_n)=z^*$ and set $t_n=1/(d^tx_n+d_0)$, $y_n=t_nx_n$; since $z(x_n)=w(y_n,t_n)$ we have $\lim_n z(x_n)=\lim_n w(y_n,t_n)=z^*$, so that $z^*\leq w^*$.
Consider now a sequence $\{(y_n,t_n)\}\in S_L$, $t_n>0$ such that $\lim_n w(y_n,t_n)=w^*$ and set as before $t_n=1/(d^tx_n+d_0)$ $y_n=t_nx_n$; we have $w(y_n,t_n)=w(t_nx_n,t_n)=z(x_n)$ and we obtain $\lim_n w(y_n,t_n)=\lim_n z(x_n)=w^*$, so that $w^*\leq z^*$.
To complete the proof it is sufficient to note that $z(x^*)=w(y^*,t^*)$.

Consider problem P^*_L obtained by P_L substituting the constraint $t > 0$ with $t \geq 0$.

Corollary 1.1 Problem P_F does not have optimal solution and the supremum is finite iff any optimal solution of problem P^*_L is of the kind $(y^*,0)$.

proof : The proof follows immediately from Theorem 1.1 .

In the case $t = 0$ we shall see how, in the corresponding problem P_F, it is possible to find a half-line r, contained in the feasible region S_F , moving along which the supremum is reached, i.e. $\sup_{x\in r} z(x) = z^*$.

2. A Parametric Method for the Problem P_F .

Consider the following linear problem $P(\xi)$ obtained from P_F by adding the constraint $d^tx+d_0=\xi$, $\xi\in R^+$

$$P(\xi) : \qquad \psi(\xi) = \frac{1}{\xi} \sup(c^tx+c_0) \quad \begin{cases} Ax=b \\ d^tx+d_0=\xi \\ x\geq 0 \end{cases}$$

Clearly $\sup_{\xi} \psi(\xi) = \sup_{x\in S_F} z(x)$ and the problem P_F is equivalent to the linear problem $P(\xi')$ when ξ' is the level corresponding to an optimal solution x' of P_F (i.e. $\xi' = d^tx'+d_0$) .

Definition 2.1 (Optimal Level Solution) : Any optimal solution $x(\xi)$ of the problem $P(\xi)$ is referred to as an optimal level solution for the linear fractional problem P_F .

The following procedure is suggested in [3,4] , in order to find the optimal solution x^* or to decide that the problem P_F has no solution and, in this case, whether the supremum is a finite value or not (i.e. $z^* = +\infty$) .

This procedure starts by finding an optimal solution x_0 of the linear problem $\min(d^tx+d_0)$, $x\in S_F$ which certainly exists, since the function d^tx+d_0 is lower bounded on the feasible region S_F . If x_0 is unique, then $x^1=x_0$ is an initial optimal level solution for P_F ; otherwise we consider the problem $P(\xi_0)$ with $\xi_0 = d^tx_0+d_0$: if $P(\xi_0)$ has no solution, neither does the fractional problem P_F , $z^* = +\infty$ and the procedure stops; otherwise any solution x_0 of $P(\xi_0)$ gives an initial $x^1= x_0$ optimal level solution for P_F. From this initial optimal level solution x^1, the parametric method given in [3] solves the linear fractional problem P_F by generating a finite sequence of optimal level solutions, which are vertices of S_F .

More exactly, let x^j be an optimal level solution for P_F and consider the parametric problem $P(\theta) = P(\xi_j+\theta)$ with $\xi_j=d^tx^j+d_0$.

At the generic j iteration, the procedure tests if the feasible region of $P(\theta)$ is empty for $\theta > 0$: in such a case, x^j is the optimal solution of the problem P_F, $z^*= z(x^j)$ and the procedure stops; otherwise the procedure tests a suitable optimality condition on the optimal level solution x^j.

If x^j is the optimal solution of the fractional problem P_F, then $x^* = x^j$, $z^* = z(x^j)$ and the procedure stops; otherwise it tries to find a new adjacent vertex x^{j+1}, which is also an optimal level solution, by applying a sensitivity analysis to the parametric problem $P(\theta)$.

If x^{j+1} does not exist, then the fractional problem P_F has no solution, the supremum is a finite value and the procedure stops; otherwise the procedure starts again, by means of a new $j = j+1$ iteration.

More exactly, starting from the basic optimal level solution x^j, $j \geq 1$, for the linear problem $P(\xi_j)$ with $\xi_j=d^tx^j+d_0$, consider the parametric problem

$$P(\theta): \qquad \psi(\theta) = \frac{1}{\xi_j+\theta} \sup(c^tx+c_0) \quad \begin{cases} Ax=b \\ d^tx=(\xi_j-d_0)+\theta \\ x\geq 0 \end{cases} \quad \text{with} \quad \theta \geq 0$$

Set in the problem $P(\theta)$

$$\hat{A} = \begin{pmatrix} A \\ \\ d^t \end{pmatrix} \qquad\qquad \hat{u} = \begin{pmatrix} b \\ \\ (\xi_j-d_0)+\theta \end{pmatrix}$$

and

B_j the set of indexes of a basis at iteration j,

$B^{-1} = \hat{A}_j^{-1}$ the inverse of the basic matrix associated to the set B_j,

we obtain

$$(2.1) \quad x^j(\theta) = B^{-1} \begin{pmatrix} b \\ \\ \xi_j-d_0 \end{pmatrix} + \theta B^{-1}e^{m+1} = x^j + \theta \mu^j$$

where e^{m+1} is the unit vector whose last component is 1 and $\mu^j = B^{-1} e^{m+1}$.

$$(2.2)\quad \psi^j(\theta) = \frac{c_B{}^t x^j(\theta) + c_0}{\xi_j+\theta} = \frac{c_B{}^t x^j + \theta\, c_B{}^t \mu^j + c_0}{\xi_j+\theta} = \frac{\xi_j\, \psi^j + \theta\, \lambda^j}{\xi_j+\theta} \quad , \quad \forall\, \theta \in K^j$$

where $\psi^j = \psi^j(0) = \dfrac{c_B{}^t x^j + c_0}{\xi_j}$, $\lambda^j = c_B{}^t \mu^j$ and $K^j = \{\ \theta : x^j(\theta) \geq 0\ , \theta \geq 0\ \}$

Remark 2.1

Let us note that $\psi(\xi_0) \leq \psi(\xi_j) \leq \psi(\xi_{j+1})$ for $\xi_0 \leq \xi_j \leq \xi_{j+1}$ and consequently $\psi(\theta) \leq \psi(0)$ for $\theta < 0$ and $\psi(0) \leq \psi(\theta)$ for $0 < \theta$.

Thus we are interested to study problem $P(\theta)$ for $\theta \geq 0$.

The following Lemma and Theorem hold

Lemma 2.1 If x^* is a local maximum point for P_F , then x^* is a global maximum point for P_F.

Theorem 2.1 (Optimality Condition) x^j is a maximum point for problem P_F if the following condition holds :

(2.3) $\lambda^j \leq \psi^j$

proof: we have $\psi'(\theta) = (\lambda^j - \psi^j)\, \xi_j / (\xi_j+\theta)^2$; then condition (2.3) implies $\psi'(0) \leq 0$ and $\psi(\theta)$ is non increasing for $\theta \geq 0$. Since $\psi(\theta) \leq \psi(0)$ for $\theta < 0$, $\theta = 0$ is a local maximum point for problem $P(\theta)$ and, for Lemma 2.1 , x^j is an optimal solution for the problem P_F .

Using the above notations and condition (2.3) , the algorithm steps are the following:

Step 0 set j=0 ; solve $\min_{x \in S_F} d^t x + d_0$ and set $\xi_0 = d^t x_0 + d_0$ where x_0 is any optimal solution ;

if x_0 is unique, set $x^1 = x_0$ in $P(\theta)$ ***goto*** Step1

otherwise solve $P(\xi_0)$

if $P(\xi_0)$ has an optimal solution x^1 ***goto*** Step 1

otherwise $P(\xi_0)$ does not have optimal solution, neither does P_F : $z^* = +\infty$ ***stop*** .

Step 1 set $j=1$, $\xi_1 = \xi_0$; compute $\theta^* = \sup\ \{\ \theta : x^j(\theta) \geq 0\ , \theta \geq 0\ \}$ in $P(\theta)$

if $\theta^*=0$ ***goto*** Step 2

otherwise ***goto*** Step 3

Step 2 ***if*** the basis changes by dual simplex algorithm, compute θ^* ***goto*** Step 3

otherwise $x^j(0)$ is the optimal solution for P_F , $z^* = z(x^j)$ ***stop***.

Step 3 *if* $\lambda^j \le \psi^j$ then $x^j(0)$ is the optimal solution for P_F, $z^* = z(x^j)$ ***stop.***
otherwise

if $\theta^*=+\infty$ then P_F does not have optimal solution : $z^* = \lambda^j$ ***stop.***
otherwise set $\xi_{j+1}=\xi_j+\theta^*$, $j=j+1$ ***goto*** Step 2

3. A Parametric Version of the Charnes-Cooper Algorithm

Consider once more the linear problem P_L obtained from the fractional problem P_F by applying the Charnes-Cooper transformation. Wagner-Yuan [14] have studied some algorithm equivalence in linear fractional programming and, in particular, they have shown that the Martos and the Charnes-Cooper methods are equivalent in the sense that, starting from the same basic solution, both algorithms find the optimal solution by generating the same sequence of vertices.

On the contrary, the parametric method given in section 2 is not equivalent to the two above mentioned methods [4]. Our aim is now to make an extension to the Wagner-Yuan studies.

For this reason we suggest, in this section, a parametric version of the Charnes-Cooper algorithm which is able to solve the problem P_F for any feasible region; such a new algorithm will be proved, in the next section, to be equivalent to the parametric one studied in the previous section.

Let us now consider the problem

P^*_L: $w^*=\sup[\ w(y,t) = c^ty+c_0t\ ]$, $(y,t) \in S^*_L \equiv \{y\in R^n, t\in R : Ay - bt=0, d^ty+d_0t=1, y \ge 0, t \ge 0\}$

as a parametric problem, where t plays the role of a parameter

$$P(t): \qquad \varphi(t) = \sup\,(c^ty+c_0t) \quad \begin{cases} Ay = bt \\ d^t y = 1 - d_0t \\ y \ge 0 \qquad \text{with} \qquad t \ge 0 \end{cases}$$

Clearly $\sup_t \varphi(t) = \sup_{(y,t)\in S^*_L} w(y,t)$ and the problem P^*_L is equivalent to the problem $P(t')$ when t' is the level corresponding to an optimal solution $(y'(t'),t')$ of P^*_L.

Definition 3.1 <u>(Optimal Level Solution) : Let y(t) be an optimal solution for the linear problem P(t). (y(t),t) is referred to as an optimal level solution for the linear problem P^*_L</u> .

We suggest the following procedure in order to find the optimal solution (y^*,t^*) and the finite value of the supremum $w^*= w(y^*,t^*)$, or to decide that the problem P^*_L has no solution and supremum $w^* = +\infty$.

This procedure starts by finding an optimal solution x_0 of the linear problem $\min(d^tx+d_0)$,

$x \in S_F$ which certainly exists, as already said, since the function $d^t x + d_0$ is lower bounded on the feasible set S_F and by setting $t_0 = 1/(d^t x_0 + d_0)$.
If x_0 is unique, then (y^1, t_1) with $t_1 = t_0$ and $y^1 = t_0 x_0$ is an initial optimal level solution for P^*_L; otherwise we consider the problem $P(t_0)$: if $P(t_0)$ has no solution, neither does the problem P^*_L, $w^* = +\infty$ and the procedure stops; otherwise any solution y^1 of $P(t_0)$, with $t_1 = t_0$, gives an initial (y^1, t_1) optimal level solution for P^*_L. From this initial optimal level solution (y^1, t_1), the parametric method we give, solves the linear problem P^*_L by generating a finite sequence of optimal level solutions, which are vertices of S^*_L.

More exactly, let (y^j, t_j) be an optimal level solution and a vertex for P^*_L too, and consider the parametric problem $P(t)$ with $0 \le t \le t_j$.
At the generic j iteration, the procedure tests if the feasible region of $P(t)$ is empty for $0 \le t \le t_j$; in such a case (y^j, t_j) is the optimal solution of the problem P^*_L, $w^* = w(y^j, t_j)$ and the procedure stops; otherwise the procedure tests a suitable optimality condition on the optimal level solution (y^j, t_j).
If (y^j, t_j) is the optimal solution of the problem P^*_L, then $(y^*, t^*) = (y^j, t_j)$, $w^* = w(y^j, t_j)$ and the procedure stops; otherwise it finds a new adjacent vertex y^{j+1}, which is also an optimal level solution, by applying a sensitivity analysis to the parametric problem $P(t)$ and the procedure starts again, by means of a new $j = j+1$ iteration.

More exactly set in the problem $P(t)$

$$\hat{A} = \begin{pmatrix} A \\ d^t \end{pmatrix} \qquad \hat{u} = \begin{pmatrix} b\,t \\ 1 - d_0 t \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} b \\ -d_0 \end{pmatrix}$$

and

B_j the set of indexes of a basis at the iteration j,

$B^{-1} = \hat{A}_j^{-1}$ the inverse of the basic matrix associated to the set B_j,

we obtain

$$(3.1) \quad y^j(t) = B^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t B^{-1} \begin{pmatrix} b \\ -d_0 \end{pmatrix} = y^j + t\,\eta^j$$

where $y^j = B^{-1} e^{m+1}$ and $\eta^j = B^{-1} \begin{pmatrix} b \\ -d_0 \end{pmatrix}$

$$(3.2) \quad \varphi^j(t) = c_B{}^t y^j(t) + c_0 t = c_B{}^t y^j + t\, c_B{}^t \eta^j + c_0 t = \varphi^j + t \rho^j, \quad \forall t \in H^j$$

where $\varphi^j = c_B{}^t y^j$, $\rho^j = c_B{}^t \eta^j + c_0$ and $H^j = \{ t : y^j(t) \geq 0 , t \in [0,t_j] \}$

The following Theorem holds

Theorem 3.1 <u>(Optimality Condition) (y^j,t_j) is a maximum point for the problem $P*_L$ if the following condition holds</u> :

(3.3) $\rho^j \geq 0$

proof : it is sufficient to note that the derivative of the function $\varphi^j(t)$ for $t \in [0,t_j]$ is ρ^j , so that $\varphi^j(t)$ decreases for t decreasing in $[0,t_j]$, starting from the value t_j, if $\rho^j \geq 0$.

Using the above notations and conditions (3.3) , the algorithm steps are the following:

Step 0 set j=0 ; solve $\min_{x \in S_F} d^t x + d_0$ and set $t_0 = 1/(d^t x_0 + d_0)$ where x_0 is any optimal solution ;

if x_0 is unique, set $y^1 = t_0 x_0$ in P(t) ***goto*** Step1

otherwise solve $P(t_0)$

if $P(t_0)$ has an optimal solution y^1 ***goto*** Step 1

otherwise $P(t_0)$ does not have optimal solution, neither does $P*_L$: $w* = +\infty$ ***stop*** .

Step 1 set j=1, $t_1 = t_0$; compute $t^* = \inf \{ t : y^j(t) \geq 0 , 0 \leq t \leq t_j \}$ in P(t)

if $t^* = t_j$ ***goto*** Step 2

otherwise ***goto*** Step 3

Step 2 ***if*** the basis changes by dual simplex algorithm, compute t^* ***goto*** Step 3

otherwise (y^j,t_j) is the optimal solution for $P*_L$, $w* = w(y^j,t_j)$ ***stop.***

Step 3 ***if*** $\rho^j \geq 0$, then (y^j,t_j) is the optimal solution for $P*_L$, $w* = w(y^j,t_j)$ ***stop.***

otherwise

if $t^* = 0$, then $(y^j(0),0)$ is the optimal solution for $P*_L$: $w* = w(y^j,0)$ ***stop.***

otherwise set $t_{j+1} = t^*$, j=j+1 ***goto*** Step 2

4. Equivalence Between the Two Parametric Methods

In this section we shall prove the equivalence between the parametric version of the Charnes-Cooper algorithm and the parametric method given in section 2.

First of all, let us note that if x' is a vertex of S_F then (y',t') is a vertex of $S*_L$, where $t' = 1 / (d^t x' + d_0)$, $y' = t' x'$; viceversa if (y',t') is a vertex of $S*_L$ then y'/ t' is a vertex of S_F.

Consider now the problems P(θ) and P(t) with $\theta = 0$ and $t = t_j = 1/\xi_j$; because of the constraints $d^t x = \xi_j - d_0$ and $d^t y = 1 - d_0 t_j$, by (2.1) and (3.1) we have $x^j = y^j / t_j + \eta^j$ and one of the basic variables is zero , i.e. there exists an index h such that $x_h{}^j = y_h{}^j / t_j + \eta_h{}^j = 0$: for this reason we refer to $x^j(0)$, $y^j(t_j)$ as the same optimal level basic solution.

Furthermore

$$\eta^j = B^{-1}\begin{pmatrix} b \\ -d_0 \end{pmatrix} = \frac{1}{t_j} B^{-1}\begin{pmatrix} b\,t_j \\ -d_0 t_j \end{pmatrix} = \frac{1}{t_j} B^{-1}\left(\begin{pmatrix} b\,t_j \\ 1-d_0 t_j \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right),$$

so that

$$(4.1) \qquad \eta^j = x^j - \frac{\mu^j}{t_j}$$

taking into account the value of x^j and μ^j in (2.1) .

Now we explicit the relation between t and θ in such a way that $y^j(t) = t\,x^j(\theta)$.

By (3.1) and (2.1) , using (4.1), and noting that $y^j = \mu^j$, we obtain

$$(4.2)\ \ y^j(t) = y^j + t\,\eta^j = \mu^j + t\,(x^j - \frac{\mu^j}{t_j}) = t\,x^j + \frac{t_j - t}{t_j}\mu^j = t\,(x^j + \frac{t_j - t}{t_j\,t}\mu^j) =$$

$$= t\,(x^j + \theta\,\mu^j) = t\,x^j(\theta) \ ,$$

so we state the relation

$$(4.3) \qquad \theta = \frac{t_j - t}{t_j\,t} \quad \text{or equivalently} \quad t = \frac{t_j}{1 + \theta\,t_j}$$

Now we are able to establish the following

Theorem 4.1 :

a) <u>(2.3) holds iff (3.3) holds (equivalence of the optimality conditions).</u>

b) <u>$K^j = \{0\}$ iff $H^j = \{0\}$ (equivalence of the stability conditions).</u>

c) <u>$x_k{}^j(\theta) > 0$ iff $y_k{}^j(t) > 0$</u> ,
<u>$x_k{}^j(\theta) = 0$ iff $y_k{}^j(t) = 0$ where θ and t satisfies (4.3) and $k = 1,\dots, n$</u> .

proof :

a) By (4.1) ,taking into account the value of ρ^j in (3.2) , of ψ^j and λ^j in (2.2) and

noting that $\xi_j = 1/t_j = d^t x^j + d_0$, we have $\rho^j = c_B{}^t \eta^j + c_0 = c_B{}^t (x^j - \mu^j / t_j) + c_0 =$
$= c_B{}^t x^j + c_0 - c_B{}^t \mu^j / t_j = \xi_j \psi^j - \lambda^j / t_j = \xi_j (\psi^j - \lambda^j)$, so that $\rho^j \geq 0$ iff $\psi^j \geq \lambda^j$.
b), c) Let us note that $K^j = \{ \theta : x^j(\theta) \geq 0 , \theta \geq 0 \}$ and, by (4.2),
$H^j = \{ t : y^j(t) \geq 0 , t \in [0,t_j] \} = \{ t : t (x^j + \theta \mu^j) \geq 0 , t \in [0,t_j] \}$: since (4.3) , $t \neq 0$ and c) follows immediately ; furthermore $K^j = \{0\}$ and $H^j = \{0\}$ iff $\mu^j < 0$.

Remark 4.1

Set $\theta^* = \sup_{\theta \geq 0} K^j(\theta)$, $t^* = \inf_{0 \leq t \leq t_j} H^j(t)$, by (4.3) we have $\theta^* = \dfrac{t_j - t^*}{t_j t^*}$

As a direct consequence of Theorem 4.1 and by (4.3) $\theta^* > 0$ iff $t^* < t_j$ and $\theta^* = +\infty$ iff $t^* = 0$

The following cases arise :

a) θ^* is finite and different from zero (or equivalently $0 < t^* < t_j$).
Then there exists an index h such that $x_h{}^j(\theta^*) = y_h{}^j(t^*) = 0$, so that the vertices $x^{j+1} = x^j(\theta^*)$, $y^{j+1} = y^j(t^*)$ generated by the two parametric methods are the same vertex in the sense that $x^{j+1} = y^{j+1} / t_{j+1}$ where $t_{j+1} = t^*$.

b) $\theta^* = +\infty$ (or equivalently $t^* = 0$).
In such a case $z^* = \sup_{x \in r} z(x)$ where r is the half-line whose equation is $x(\theta) = x^j + \theta \mu^j$; by the third identity of relation (4.2) the optimal solution of the problem P^*_L is $(\mu^j, 0)$; viceversa suppose that $(y^*, 0)$ is the optimal solution of P^*_L , by the third identity of relation (4.2) , $y^* = \mu^j$ and taking into account the value of x^j in (4.1) we obtain that

$$(4.4) \qquad x(\theta) = \left(\eta^j + \frac{y^*}{t_j} \right) + \theta\, y^*$$

is the equation of the half-line $r \in S_F$ along which the supremum is reached.

Remark 4.2 :

One more possibility is that neither in the first algorithm nor in the second one an optimal level solution can be reached (at Step 0). This means that neither P_F nor P^*_L has solution and that $z^* = w^* = +\infty$.

Let us give now two examples of fractional problems, one with solution, the other without solution but having finite supremum.

Example 1

Let us consider the problem P_F :

$$\sup\ [z(x)= (3x_1-x_2-22)\ (x_1+2x_2+2)^{-1}]$$

$$x \in S_F \equiv \begin{cases} x_1-2x_2 \leq 3 \\ 5x_1+3x_2 \leq 54 \\ x_2 \leq 8 \\ -2x_1+x_2 \leq 4 \\ x_1, x_2 \geq 0 \end{cases}$$

by applying the first algorithm to the problem $P(\theta)$

$$\psi(\theta) = (\xi_j+\theta)^{-1} \sup\ (3x_1-x_2-22)$$

$$\begin{cases} x_1-2x_2 \leq 3 \\ 5x_1+3x_2 \leq 54 \\ x_2 \leq 8 \\ -2x_1+x_2 \leq 4 \\ x_1+2x_2 = (\xi_j-2)+\theta \end{cases} \qquad x_1, x_2 \geq 0\ ,\ \ \theta \geq 0$$

we obtain :

Step 0 $\quad$ j=0 ; since $x_0=(0,0)$ is the unique optimal solution of $\min\limits_{x\in S_F} (x_1+2x_2+2)$,

set $\xi_0=2$ and $\xi_1=\xi_0$

$x^{1t}= (x_3,x_4,x_5,x_6,x_1)^t = (3-\theta,54-5\theta,8,4+2\theta,\theta)^t =$

$= (x^1+\theta\mu^1)^t = (3,54,8,4,0)^t+\theta(-1,-5,0,2,1)^t$

Step 1 $\quad$ j=1 ; in the problem $P(\theta)$:

$$c_B{}^t = (0,0,0,0,3) \qquad\qquad B^{-1} = \begin{pmatrix} 1&0&0&0&-1 \\ 0&1&0&0&-5 \\ 0&0&1&0&0 \\ 0&0&0&1&2 \\ 0&0&0&0&1 \end{pmatrix}$$

$x^{1t}= (x_3,x_4,x_5,x_6,x_1)^t = (3-\theta,54-5\theta,8,4+2\theta,\theta)^t =$

$= (x^1+\theta\mu^1)^t = (3,54,8,4,0)^t+\theta(-1,-5,0,2,1)^t$

$\psi^1(\theta) = (\psi^1 + \theta\, \lambda^1) / \xi_1 = (-22 + 3\theta) / 2\ :\ \theta^* = 3$

Step 3 $\quad$ since $\lambda^1 = 3 > \psi^1 = -11$ and $\theta^* = 3$, set $\xi_2 = \xi_1+3 = 5$, j=2

Step 2 change basis in $P(\theta)$

$$c_B^t = (-1,0,0,0,3) \qquad B^{-1} = \begin{pmatrix} -1/4 & 0 & 0 & 0 & 1/4 \\ -7/4 & 1 & 0 & 0 & -13/4 \\ 1/4 & 0 & 1 & 0 & -1/4 \\ 5/4 & 0 & 0 & 1 & 3/4 \\ 1/2 & 0 & 0 & 1 & 1/2 \end{pmatrix}$$

$x^{2t} = (x_2,x_4,x_5,x_6,x_1)^t = (\theta/4, 39-13\theta/4, 8-\theta/4, 10+3\theta/4, 3+\theta/2)^t =$
$= (x^2+\theta\mu^2)^t = (0,39,8,10,3)^t + \theta(1/4,-13/4,-1/4,3/4,1/2)^t$
$\psi^2(\theta) = (\psi^2 + \theta\,\lambda^2)\,/\,\xi_2 = (-13+5\theta/4)\,/\,5 \quad : \quad \theta^* = 12$

Step 3 since $\lambda^2 = 5/4 > \psi^2 = -13/5$ and $\theta^* = 12$, set $\xi_3 = \xi_2 + 12 = 17$, $j=3$

Step 2 change basis in $P(\theta)$

$$c_B^t = (-1,0,0,0,3) \qquad B^{-1} = \begin{pmatrix} 0 & -1/7 & 0 & 0 & 5/7 \\ 1 & -4/7 & 0 & 0 & 13/7 \\ 0 & 1/7 & 1 & 0 & -5/7 \\ 0 & 5/7 & 0 & 1 & -11/7 \\ 0 & 2/7 & 0 & 0 & -3/7 \end{pmatrix}$$

$x^{3t} = (x_2,x_3,x_5,x_6,x_1)^t = (3+5\theta/7, 13\theta/7, 5-5\theta/7, 19-11\theta/7, 9-3\theta/7)^t =$
$= (x^3+\theta\mu^3)^t = (3,0,5,19,9)^t + \theta(5/7,13/7,-5/7,-11/7,-3/7)^t$
$\psi^3(\theta) = (\psi^3 + \theta\,\lambda^3) = (2-2\theta)\,/\,17 \;:\; \theta^* = 7$

Step 1 since $\lambda^3 = -2 < \psi^3 = 2/17$,then $x^3(0) = (9,3,0,0,5,19)$ is the optimal solution for P_F , $z^* = 2/17$

Now by applying the second algorithm to the problem $P(t)$

$$\varphi(t) = \sup\,(3y_1 - y_2 - 22t)$$
$$\begin{cases} y_1 - 2y_2 \le 3t \\ 5y_1 + 3y_2 \le 54t \\ y_2 \le 8t \\ -2y_1 + y_2 \le 4t \\ y_1 + 2y_2 = 1-2t \end{cases} \qquad y_1, y_2 \ge 0 \ , \quad t \ge 0$$

we obtain :

Step 0 $j=0$; since $x_0=(0,0)$ is the unique optimal solution of $\min\limits_{x \in S_F} (x_1+2x_2+2)$,

set $t_0=1/2$ and $t_1=t_0$

Step 1   j=1 ; in the problem P(t) :

$$c_B{}^t = (0,0,0,0,3) \qquad B^{-1} = \begin{pmatrix} 1&0&0&0&-1 \\ 0&1&0&0&-5 \\ 0&0&1&0&0 \\ 0&0&0&1&2 \\ 0&0&0&0&1 \end{pmatrix}$$

$y^{1t} = (y_3,y_4,y_5,y_6,y_1)^t = (y^1 + t\,\eta^1)^t = (-1,-5,0,2,1)^t + t\,(5,64,8,0,-2)^t$

$\varphi^1(t) = \varphi^1 + t\,\rho^1 = 3 - 28\,t \ :\ t^* = 1/5$

Step 3   since $\rho^1 = -28 < 0$ and $t^* = 1/5$, set $t_2 = t^* = 1/5$, j=2

Step 2   change basis in P(t)

$$c_B{}^t = (-1,0,0,0,3) \qquad B^{-1} = \begin{pmatrix} -1/4&0&0&0&1/4 \\ -7/4&1&0&0&-13/4 \\ 1/4&0&1&0&-1/4 \\ 5/4&0&0&1&3/4 \\ 1/2&0&0&1&1/2 \end{pmatrix}$$

$y^{2t} = (y_2,y_4,y_5,y_6,y_1)^t = (y^2 + t\,\eta^2)^t =$

$= (1/4,-13/4,-1/4,3/4,1/2)^t + t\,(-5/4,221/4,37/4,25/4,1/2)^t$

$\varphi^2(t) = \varphi^2 + t\,\rho^2 = 5/4 - 77\,t/4 \ :\ t^* = 1/17$

Step 3   since $\rho^2 = -77/4 < 0$ and $t^* = 1/17$, set $t_3 = t^* = 1/17$, j=3

Step 2   change basis in P(t)

$$c_B{}^t = (-1,0,0,0,3) \qquad B^{-1} = \begin{pmatrix} 0&-1/7&0&0&5/7 \\ 1&-4/7&0&0&13/7 \\ 0&1/7&1&0&-5/7 \\ 0&5/7&0&1&-11/7 \\ 0&2/7&0&0&-3/7 \end{pmatrix}$$

$y^{3t} = (y_2,y_3,y_5,y_6,y_1)^t = (y^3 + t\,\eta^3)^t =$

$= (5/7,13/7,-5/7,-11/7,-3/7)^t + t\,(-64/7,-221/7,120/7,320/7,114/7)^t$

$\varphi^3(t) = \varphi^3 + t\,\rho^3 = -2 + 36\,t \ :\ t^* = 1/24$

Step 1   since $\rho^3 = 36 > 0$, then $(y^3(0),t_3) = (9/17,3/17,0,0,5/17,19/17,1/17)$ is the optimal solution for P^*_L, $w^* = 2/17$

Example 2

Let us consider the problem P_F :

$$\sup\,[z(x) = (-x_1-2)\,(3x_1+x_2+1)^{-1}]$$
$$x \in S_F \equiv \{\ -x_1 + x_2 \le 4\ ,\ x_1, x_2 \ge 0\ \}$$

by applying the first algorithm to the problem $P(\theta)$

$$\psi(\theta) = (\xi_j+\theta)^{-1} \sup (-x_1-2) \quad \begin{cases} -x_1+x_2 \le 4 \\ 3x_1+x_2 = (\xi_i-1)+\theta \\ x_1, x_2 \ge 0 \ , \quad \theta \ge 0 \end{cases}$$

we obtain :

Step 0 $\quad$ j=0 ; since $x_0=(0,0)$ is the unique optimal solution of $\min_{x\in S_F} (3x_1+x_2+1)$,

set $\xi_0=1$ and $\xi_1=\xi_0$

$x^{1t}= (x_3,x_2)^t = (4-\theta,\theta)^t = (x^1+\theta\mu^1)^t = (4,0)^t+\theta(-1,1)^t$

Step 1 $\quad$ j=1 ; in the problem $P(\theta)$:

$$c_B{}^t = (0,0) \qquad B^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$x^{1t}= (x_3,x_2)^t = (4-\theta,\theta)^t = (x^1+\theta\mu^1)^t = (4,0)^t+\theta(-1,1)^t$

$\psi^1(\theta) = (\psi^1 + \theta\, \lambda^1) / \xi_1 = (-2 + 0\theta) / 1 = -2$: $\theta^* = 4$

Step 3 $\quad$ since $\lambda^1 = 0 > \psi^1 = -2/1 = -2$ and $\theta^* = 4$, set $\xi_2 = \xi_1+4 = 5$, j=2

Step 2 $\quad$ change basis in $P(\theta)$

$$c_B{}^t = (-1,0) \qquad B^{-1} = \begin{pmatrix} -1/4 & 1/4 \\ 3/4 & 1/4 \end{pmatrix}$$

$x^{2t}= (x_1,x_2)^t = (\theta/4,4+\theta/4)^t = (x^2+\theta\mu^2) = (0,4)^t+\theta(1/4,1/4)^t$

$\psi^2(\theta) = (\psi^2 + \theta\, \lambda^2) / \xi_2 = (-2-\theta/4) / 5$: $\theta^* = +\infty$

Step 3 $\quad$ since $\lambda^2 = -1/4 > \psi^2 = -2/5$ and $\theta^* = +\infty$,

P_F does not have optimal solution : $z^* = \lambda^2 = -1/4$

The half-line $r \in S_F$, along which supremum is reached, is given in the parametric form by the equations : $x_1 = \theta/4$, $x_2 = 4+\theta/4$.

Now by applying the second algorithm to the problem $P(t)$

$$\varphi(t) = \sup (-y_1-2t) \quad \begin{cases} -y_1+y_2 \le 4t \\ 3y_1+y_2 = 1-t \\ y_1, y_2 \ge 0 \ , \quad t \ge 0 \end{cases}$$

we obtain :

Step 0 $\quad$ j=0 ; since $x_0=(0,0)$ is the unique optimal solution of $\min_{x \in S_F} (3x_1+x_2+1)$,

set $t_0=1$ and $t_1=t_0$

Step 1 $\quad$ j=1 ; in the problem P(t) :

$$c_B{}^t = (0,0) \qquad B^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$y^{1t} = (y_3,y_2)^t = (y^1 + t\,\eta^1)^t = (-1,1)^t + t\,(5,-1)^t$

$\varphi^1(t) = \varphi^1 + t\,\rho^1 = -2t \;:\; t^* = 1/5$

Step 3 $\quad$ since $\rho^1 = -2 < 0$ and $t^* = 1/5$, set $t_2 = t^* = 1/5$, j=2

Step 2 $\quad$ change basis in P(t)

$$c_B{}^t = (-1,0) \qquad B^{-1} = \begin{pmatrix} -1/4 & 1/4 \\ 3/4 & 1/4 \end{pmatrix}$$

$y^{2t} = (y_1,y_2)^t = (y^2 + t\,\eta^2)^t = (1/4,1/4)^t + t\,(-5/4,11/4)^t$

$\varphi^2(t) = \varphi^2 + t\,\rho^2 = -1/4 - 3t/4 \;:\; t^* = 0$

Step 3 $\quad$ since $\rho^2 = -3/4 < 0$ and $t^* = 0$

$(y^1(t^*),0) = (1/4,1/4,0)$ is the optimal solution for problem P^*_L :

$w^* = w(y(t^*),0) = \varphi(t^*) = -1/4$

Noting that $t^* = 0$ and according to the relation (4.4) we have $x(\theta) = (-5/4,11/4)^t + 5\,(1/4,1/4)^t + \theta\,(1/4,1/4)^t = (0,4)^t + \theta\,(1/4,1/4)^t$ so the equations of the half-line $r \in S_F$ can now be rewritten in the same form as before : $x_1 = \theta/4$, $x_2 = 4+\theta/4$.

Conclusions :

In this paper we have proven that it is possible to assert the equivalence between the parametric method applied to the problem $P(\theta)$ and the parametric version of the Charnes-Cooper algorithm applied to the problem P(t) in the sense that starting from the same optimal level solution for problems P_F and P^*_L they generate, in the same number of steps, the same sequence of optimal level solutions .

These methods give the optimal solution for problem P_F, if it exists . If it does not exist, they both give the possibility of finding a finite value for the $\sup z = z^*$, if it exists, or to state that the objective function is not upper bounded, that is $\sup z = +\infty$.

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