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On Parametric Linear Fractional Programs

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On Parametric Linear Fractional Programs (*)

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Abstract

In this paper we analyse the linear fractional program $\sup f(x) = [(c^Tx + c_0)/(d^Tx + d_0)]$, $x \in S = \{x \in R^n : Ax=b, x \geq 0\}$ with respect to the variation of the vector (c, c_0). Further an algorithm to solve the parametric linear fractional program $z(\theta) = \sup [((c + \theta u)^Tx + c_0 + \theta u_0)/(d^Tx + d_0)]$, $x \in S$ is proposed for any feasible region (bounded or unbounded).

Key words : linear fractional program, sensitivity analysis.

1. Introduction

Sensitivity and parametric analysis for a linear fractional program whose feasible region is a compact set, is very similar to the linear program one [1,2,8]. The aim of the paper is to deep the parametric analysis to the case where the feasible region is unbounded; in such a case some difficulties arise in studying the supremum of the problem as a function of a parameter. These difficulties are pointed out from a geometrical point of view by means of the study of suitable cones which allows us to describe a sequential method to solve a parametric linear fractional program for any feasible region.

2. Statement of the problem

Let us consider the linear fractional program :

$$(2.1) \quad \sup f(x) = [(c^Tx + c_0)/(d^Tx + d_0)], \; x \in S = \{x \in R^n : Ax=b, x \geq 0\} \neq \phi$$

where A is an m x n matrix, $c, d \in R^n$, $b \in R^m$, $c_0, d_0 \in R$ and $d^Tx + d_0 > 0$, $\forall x \in S$.

(*) The paper has been discussed jointly by the Authors. Cambini has developed section 3 ; Sodini has developed sections 1-2-4-5-6.

The aim of the paper is to study the function

$$z(\theta) = \sup [((c + \theta u)^T x + c_0 + \theta u_0)/(d^T x + d_0)], \ x \in S ;$$

where (u,u_0) is a fixed vector of R^{n+1}. With this purpose let us note that for problem (2.1) the following cases arise :

1) there exists an $x' \in S$ such that $\max_{x \in S} f(x) = f(x')$; in this case at least a vertex of S is an optimal solution;

2) it results $\sup_{x \in S} f(x) = +\infty$; i.e. there exists a vertex $x' \in S$ and an extreme ray $W_r = \{ x : x = x' + t r, t \geq 0 \}$(1) , $W_r \subset S$, such that

$$\lim_{t \to +\infty} f(x' + t r) = +\infty ;$$

3) 1) does not hold and it results $\sup_{x \in S} f(x) = L < +\infty$; i.e. there exists a vertex $x' \in S$ and an extreme ray r such that $\lim_{t \to +\infty} f(x' + t r) = L$.

Clearly only case 1) can happen when the feasible region is bounded; furthermore let us notice that linear fractional program and linear program differ only for case 3), which cannot happen for the linear case. This fact introduce some difficulties in sensitivity and parametric analysis of linear fractional program with respect to the linear program. In the next section some theoretical results concerning this aspect are considered.

3. On the variation of vector (c, c_0).

In this section we present some theoretical properties of problem (2.1) with respect to the variation of vector (c,c_0), which allows us to give a geometrical interpretation of a parametric analysis.

First we introduce the following notation :

- $V(S)$, set of vertices of S ;

(1) In the following, for sake of notation, we identify W_r with r.

- $R(x^i)$, set of extreme rays starting from $x^i \in V(S)$;
- $C(x^i) = \{(c,c_0) \in R^{n+1} : \max_{x\in S} f(x) = f(x^i),\ x^i \in V(S)\}$;
- $C(r_{ij}) = \{(c,c_0) \in R^{n+1} : \sup_{x\in S} f(x) = \lim_{t\to+\infty} f(x^i + t\, r_{ij}) < +\infty,\ x^i \in V(S),\ r_{ij} \in R(x^i)\}$;
- $C^*(x^i) = C(x^i) \cup (\bigcup_{r_{ij}\in R(x^i)} C(r_{ij}))$;

- $C = \bigcup_{x^i\in V(S)} C(x^i) = \{(c,c_0) \in R^{n+1} : \exists\ x' \in V(S) : \max_{x\in S} f(x) = f(x')\}$

- $C^* = \bigcup_{x^i\in V(S)} C^*(x^i) = \{(c,c_0) \in R^{n+1} : \sup_{x\in S} f(x) < +\infty\}$.

The following theorem holds :

Theorem 3.1

i) $C(x^i)$ is a convex polyhedral cone ;
ii) $C(r_{ij})$ is a convex polyhedral cone ;
iii) $C^*(x^i)$ is a cone (not necessarily convex) ;
iv) C is a cone (not necessarily convex);
v) C^* is a convex cone ;
vi) $C^* = R^{n+1}$ iff S is linearly bounded with respect to $d^Tx + d_0$ (2) .

Proof . i) $x^i \in V(S)$ is an optimal solution iff (*), (**) hold :

$$(*)\quad [(c^Tx^i + c_0)/(d^Tx^i + d_0)] \geq [(c^Tx^j + c_0)/(d^Tx^j + d_0)],\ x^j \in V(S)$$

$$(**)\quad [(c^Tx^i + c_0)/(d^Tx^i + d_0)] \geq \lim_{t\to+\infty} [(c^T(x^j + t\, r_{jk}) + c_0)/(d^T(x^j + t\, r_{jk}) + d_0)] =$$

$$= (c^Tr_{jk}/d^Tr_{jk}),\ x^j \in V(S),\ r_{jk} \in R(x^j).$$

Since (*),(**) are linear inequalities then $C(x^i)$ is a convex polyhedral cone; ii) the same proof of i) ; iii) - iv) it follows directly from the definitions of $C^*(x^i)$ and C ; v) C^* is a cone as union of cones; it remains to prove that C^* is convex ; let $(c',c'_0),(c'',c''_0) \in C^*$, it results

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(2) S is linearly bounded with respect to $d^Tx + d_0$ if the set $S \cap \{x \in R^n : d^Tx + d_0 = k\}$ is a compact set for any $k \in R$.

$$\sup_{x \in S} [(c'^T x + c'_0)/(d^T x + d_0)] = L_1,$$

$$\sup_{x \in S} [(c''^T x + c''_0)/(d^T x + d_0)] = L_2 \ ,$$

$$\sup_{x \in S} [((\lambda c' + (1-\lambda)c'')^T x + \lambda c'_0 + (1-\lambda)c''_0)/(d^T x + d_0)] =$$

$$\sup_{x \in S} [(\lambda(c'^T x + c'_0)+(1-\lambda)(c''^T x + c''_0))/(d^T x + d_0)] \le \lambda L_1 + (1-\lambda)L_2 \ , \ 0 \le \lambda \le 1 \ ;$$

vi) (if part) Ab absurdo suppose that $C^* \neq R^{n+1}$; then there exists $(c',c'_0) \in R^{n+1}$, $x^i \in V(S)$, $r_{ij} \in R(x^i)$ such that :

$$\lim_{t \to +\infty} [(c'^T(x^i + t\, r_{ij}) + c'_0)/(d^T(x^i + t\, r_{ij}) + d_0)] = +\infty \ ; \text{ it follows that}$$

$d^T r_{ij} = 0$ and $d^T(x^i + t\, r_{ij}) + d_0 = d^T x^i + d_0 =$ constant for each t, and this contradicts the hypothesis .(only if part) Ab absurdo suppose that there exists $x^i \in V(S)$, $r_{ij} \in R(x^i)$ such that $d^T(x^i + t\, r_{ij}) =$ constant for each t ; then it follows that there exists $(c',c'_0) \in R^{n+1}$ such that

$$\lim_{t \to +\infty} [(c'^T(x^i + t\, r_{ij}) + c'_0)/(d^T(x^i + t\, r_{ij}) + d_0)] = +\infty .$$

The contradiction of the hypothesis is obtained.

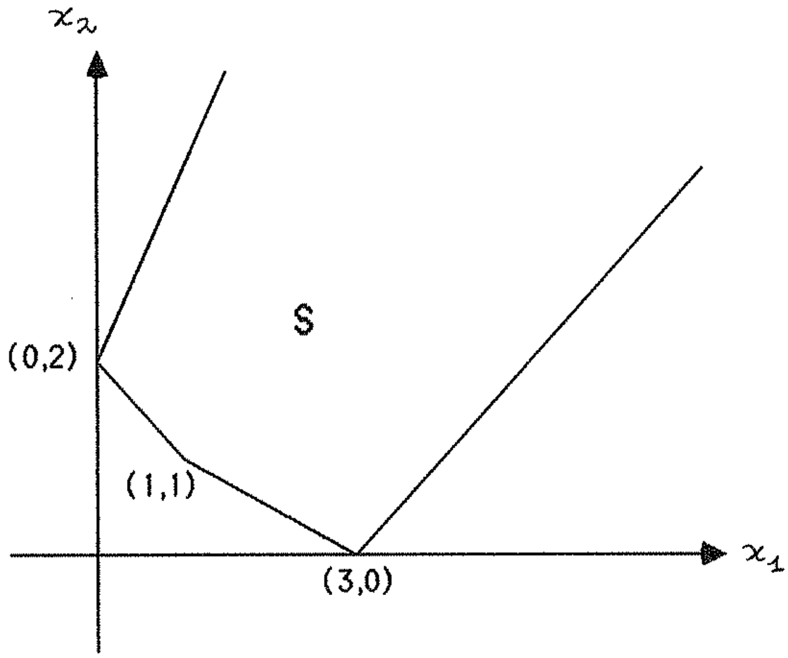
Remark

The properties of Theorem 3.1 does not hold for the linear program; in fact for the linear program it results $C(x^i) = C^*(x^i)$, $C = C^*$ and C is a convex cone.

Let us consider the following numerical example :

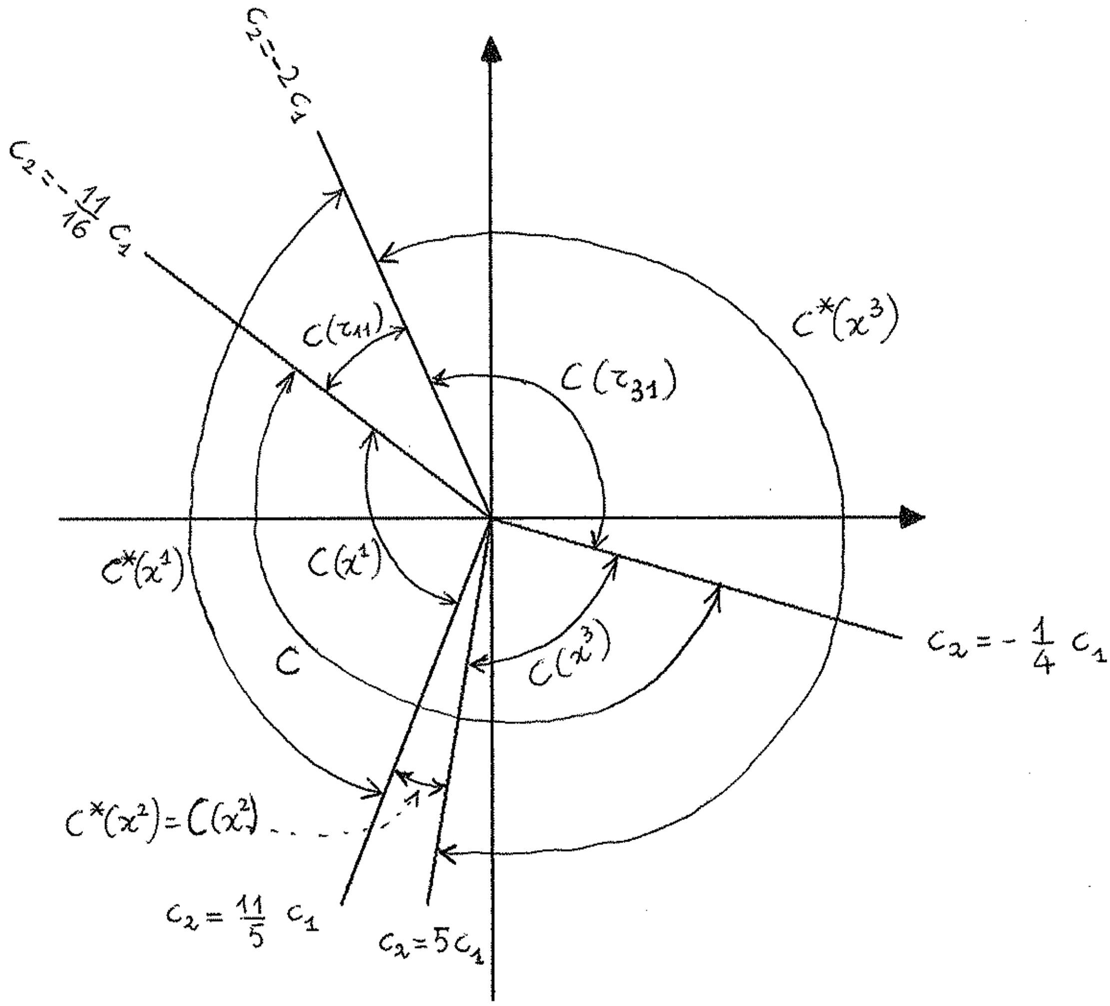
$$\text{(3.2)} \quad \begin{array}{l} \sup f(x) = (c_1 x_1 + c_2 x_2)/(-x_1 + 2x_2 + 7) \\ -2x_1 + x_2 \le 2 \ , x_1 - x_2 \le 3 \ , x_1 + x_2 \ge 2 \ , x_1 + 2x_2 \ge 3 \\ x_1, x_2 \ge 0 \end{array}$$

The feasible region is described in fig.1 where $x^1=(0,2)$, $x^2=(1,1)$, $x^3=(3,0)$, $r_{11}=(1,2)$, $r_{31}=(1,1)$.



x_2
S
(0,2)
(1,1)
(3,0)
x_1

fig. 1



$c_2 = -2c_1$
$c_2 = -\frac{11}{16} c_1$
$C(\tau_{11})$
$C(\tau_{31})$
$C^*(x^3)$
$C^*(x^1)$
$C(x^1)$
C
$C(x^3)$
$c_2 = -\frac{1}{4} c_1$
$C^*(x^2) = C(x^2)$
$c_2 = \frac{11}{5} c_1$
$c_2 = 5c_1$

fig. 2

The cones $C(x^1)$, $C(x^2)$, $C(x^3)$, C, $C(r_{11})$, $C(r_{31})$, $C^*(x^1)$, $C^*(x^2)$, $C^*(x^3)$ and C^* are described in fig. 2. Clearly $C(x^1)$, $C(x^2)$, $C(x^3)$, $C(r_{11})$, $C(r_{31})$, $C^*(x^1)$, $C^*(x^2)$ are convex cones ; $C^*(x^3)$ and C are not convex cones and $C^* = R^2$. Now let us suppose that the numerator of the objective function in (3.2) is parameterized in the following way : $(c_1,c_2) = (-3,2) + \theta\ (7,-3)$, $\theta \in R$. This ray, together with the cones $C(x^1)$, $C(x^2)$, $C(x^3)$, $C(r_{11})$, $C(r_{31})$ is decribed in fig. 3.

From the analysis of fig. 3 and with simple calculations, we find that for :

- $\theta \le 1/29$, the vertex $x^1 = (0,2)$ is a maximum ;
- $1/29 \le \theta \le 4/11$, the supremum is obtained on the extreme ray $r_{11} = (1,2)$ starting from vertex x^1 ;
- $4/11 \le \theta \le 1$, the supremum is obtained on the extreme ray $r_{31} = (1,1)$ starting from vertex x^3;
- $1 \le \theta$, the vertex $x^3 = (3,0)$ is a maximum.

Notice that the adjacent intervals [1/29,4/11] and [4/11,1] correspond to extreme rays starting from the vertices x^1 and x^3 which are not adjacent. It follows that the study of a linear fractional program as a function of a parameter in the numerator of the objective function, requires the capability of going from an extreme ray r_{ik} starting from the vertex x^i to a new extreme ray r_{jv} starting from the vertex x^j which may be not adjacent to the vertex x^i.

4. A modified version of Martos algorithm

In this section a modified version of Martos algorithm is described [5]. The algorithm,called MVM, is able to work also when the feasible region is unbounded.Let us consider the linear fractional program:

$$\text{(4.1)} \quad \begin{aligned} &\max f(x) = (c^T + c_0)/(d^Tx + d_0) \\ &x \in X = \{ x \in R^n : Ax=b \ , x \ge 0 \} \end{aligned}$$

The solution $x_0 \in X$ is said to be an "optimal level solution" if it is an optimal solution of the following linear program:

$$P(\zeta_0): \quad \begin{aligned} &1/\zeta_0 \ \max c^Tx + c_0 \\ &Ax = b\ ,\ d^Tx + d_0 = \zeta_0\ ,\ x \ge 0 \end{aligned}$$

where $\zeta_0 = d^T x_0 + d_0$.

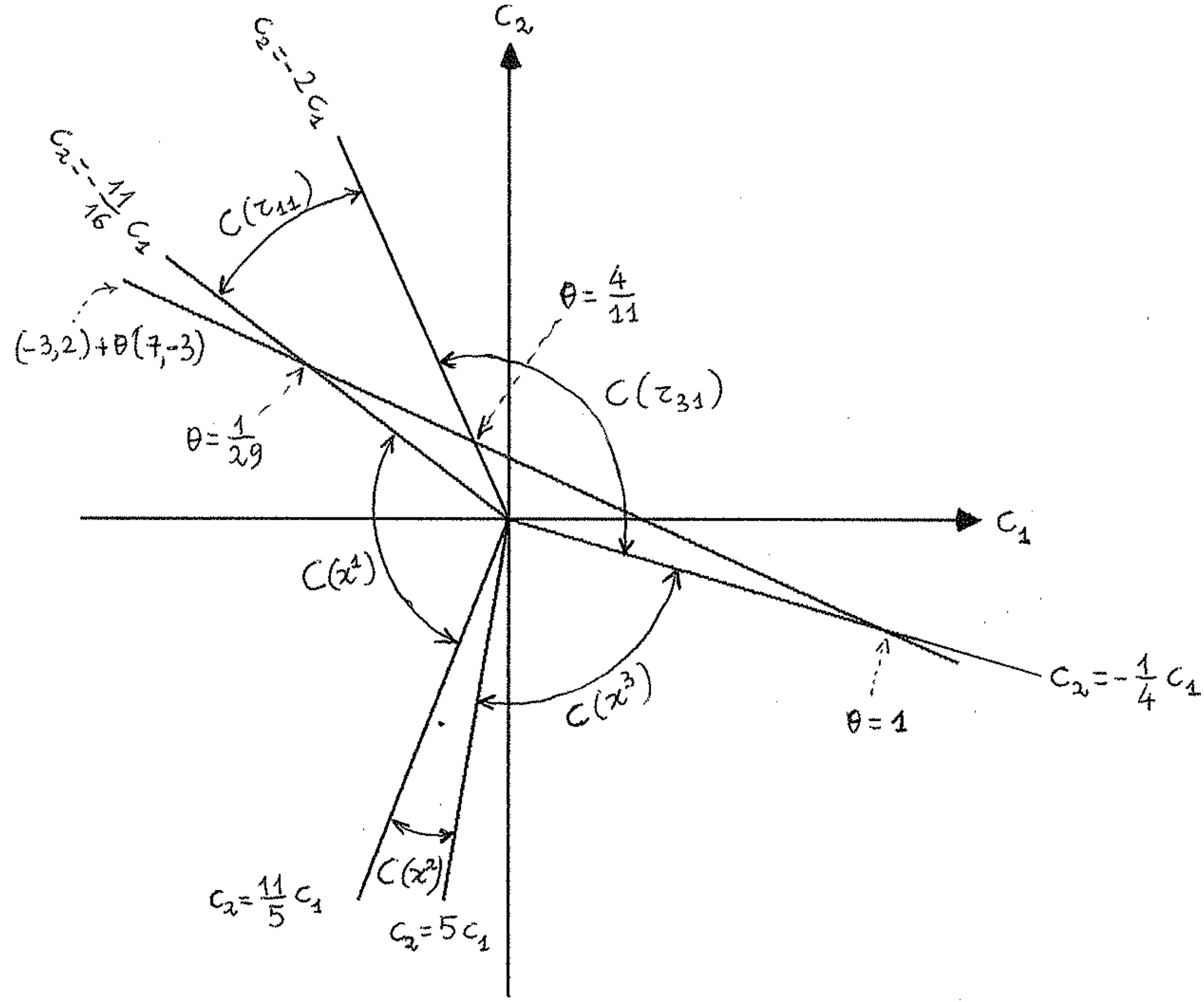


fig. 3

Algorithm MVM differs from Martos algorithm in the fact that the optimal solution (if any) is obtained by examining only optimal level vertices. Let x' be a vertex of X and B, N the sets of indices of basic variables and non basic variables corresponding to x'.

Set $c'_N{}^T = c_N{}^T - c_B{}^T A_B{}^{-1} A_N$, $d'_N{}^T = d_N{}^T - d_B{}^T A_B{}^{-1} A_N$, $\beta' = c'_N - f(x')\, d'_N$.

<u>Algorithm MVM</u>

<u>Step 0</u> Solve problem $P_0 : \min\ d^T x + d_0$, $x \in X$. If the optimal solution x' of P_0 is unique then go to **Step 1**; otherwise solve problem $P_1 : \max\ c^T x + c_0$, $x \in X' = X \cap \{ x : d^T x = d^T x' \}$; if

$\sup_{x \in X'} c^T x + c_0 = +\infty$ then $\sup_{x \in X} f(x) = +\infty$; otherwise go to **Step 1** with x' equal to the optimal solution of P_1;

Step 1 Compute c'_N, d'_N, $\beta' = c'_N - f(x')\, d'_N$ and $J = \{ j : \beta'_j > 0 \}$. If $J = \phi$ then STOP (x' is an optimal solution) ; otherwise select k such that $c'_{Nk} / d'_{Nk} = \max_{j \in J} \{ c'_{Nj} / d'_{Nj} \}$, go to **Step2** ;

Step 2 Compute $y^k = A_B^{-1} A_N^k$. If $y^k \leq 0$ then STOP ($\sup f(x) = c'_{Nk}/d'_{Nk}$) ; otherwise do a simplex iteration with x_{Nk} as entering variable. Let x' be the new basic solution ; go to **Step1**.

In **Step 0** a starting optimal level vertex is determined; condition $\max_{j \in J} \{ c'_{Nj} / d'_{Nj} \}$ in **Step1** guarantees that if x' is an optimal level vertex also the new vertex is an optimal level vertex.

5. An algorithm for the parametric linear fractional program

In this section an algorithm for the parametric linear fractional program

$$\text{PLF}(\theta): \quad \begin{aligned} & z(\theta) = \sup[((c+\theta u)^T x + c_0 + \theta u_0)/(d^T x + d_0)] \\ & Ax = b \,,\ x \geq 0 \end{aligned}$$

is proposed. The algorithm is similar to that for the linear case

$$\begin{aligned} & \varphi(\theta) = \max\, (c + \theta u)^T x \\ & Ax = b \,,\ x \geq 0 \end{aligned}$$

where the function $\varphi(\theta)$ is obtained by a vertex following procedure. Suppose that problem PLF(0) is solved by means of algorithm MVM. Let A_B the basis at the end of the procedure and $x'_B = A_B^{-1} b$, $c'^T_N = c_N^T - c_B^T A_B^{-1} A_N$, $c'_0 = c_0 + c_B^T A_B^{-1} b$, $d'^T_N = d_N^T - d_B^T A_B^{-1} A_N$, $d'_0 = d_0 + d_B^T A_B^{-1} b$, $\gamma = c'_0/d'_0$, $\beta' = c'_N - \gamma\, d'_N$.

Clearly $x' = (x'_B, 0)$ is an optimal basic solution of the subproblem PL(0) where

$$\max\ (c+\theta u)^T x + c_0 + \theta u_0$$

PL(θ)

$$Ax=b\ ,\ d^T x = d^T x'\ ,\ x \geq 0\ ;$$

and hence there exists a basis $A^*_{B'} = \begin{bmatrix} A_B & A_N^k \\ d_B^T & d_{Nk} \end{bmatrix}$ such that

$c^*_{N'}{}^T = c_{N'}{}^T - c_{B'}{}^T A^*_{B'}{}^{-1} A^*_{N'} \leq 0$.

If $\beta' \leq 0$ then x' is a maximum of PLF(0) ; if β' is not ≤ 0 then there exists an index k such that $y^k = A_B^{-1} A_N^k \leq 0$ and $c'_{Nk}/d'_{Nk} = \max\ \{ c'_{Nj}/d'_{Nj}\ ,\ j{:}\ \beta'_j > 0 \}$; in this case PLF (0) has a supremum (equal to c'_{Nk}/d'_{Nk}) on the extreme ray from x' along y^k.

The set of values of θ such that x' is a maximum or the supremum is obtained on an extreme ray starting from x' is referred as "the stability set" of vertex x'.

In order to find the stability set of x' it is necessary to introduce the parameter θ in β', c'_N, γ, $c^*_{N'}$, c'_0.

It results:

- $c'_0(\theta) = c_0 + \theta u_0 + (c_B + \theta u_B)^T A_B^{-1} b$

$$= c'_0 + \theta u'_0\ , \quad \text{where } u'_0 = u_0 + u_B^T A_B^{-1} b\ ;$$

- $\gamma(\theta) = c'_0(\theta)/d'_0 = \gamma + \theta\,(u'_0/d'_0)$;

- $c'_N{}^T(\theta) = (c_N + \theta u_N)^T - (c_B + \theta u_B)^T A_B^{-1} A_N$

$$= c'_N{}^T + \theta u'_N{}^T, \text{ where } u'_N{}^T = u_N{}^T - u_B^T A_B^{-1} A_N\ ;$$

- $\beta'(\theta) = c'_N(\theta) - \gamma(\theta)\, d'_N = c'_N + \theta u'_N - \gamma\, d'_N - \theta\,(u'_0/d'_0)\, d'_N$

$$= \beta' + \theta\,(u'_N - (u'_0/d'_0)\, d'_N)$$

- $c^*_{N'}{}^T(\theta) = c^*_{N'}{}^T + \theta\, u^*_{N'}{}^T$, where $u^*_{N'}{}^T = u_{N'}{}^T - u_{B'}{}^T A^*_{B'}{}^{-1} A^*_{N'}$.

Let us define:

$$H(\theta) = \{\ \theta \in R :\ \beta'(\theta) \leq 0\ \}\ ;$$

$H'(\theta) = \{ \theta \in R : c^*_{N'}(\theta) \leq 0 \}$;

clearly $H'(\theta) \supseteq H(\theta)$.

If $\beta' \leq 0$ then $H(\theta) \neq \phi$ and $\theta \in H(\theta)$ implies that x' is a maximum of PLF(θ) with $z(\theta) = \gamma(\theta)$. If the supremum of PLF(0) is obtained along y^k then consider the set :

$H''(\theta) = \{ \theta : c'_{Nk}(\theta)/d'_{Nk} = \max \{ c'_{Nj}(\theta)/ d'_{Nj}\ ,\ j: \beta'_j > 0 \}\}$ if $d'_{Nk} > 0$; or the set

$H''(\theta) = \{ \theta : c'_{Nk}(\theta)/d'_{Nk} = \min \{ c'_{Nj}(\theta)/ d'_{Nj}\ ,\ j: \beta'_j > 0 \}\}$ if $d'_{Nk} < 0$;

clearly for $\theta \in H''(\theta) \cap H'(\theta)$ the supremum is obtained along y^k and $z(\theta) = c'_{Nk}(\theta)/d'_{Nk}$.

Let us suppose that $H'(\theta) = \{ \theta : \theta' \leq \theta \leq \theta'' \}$ where it is possible that $\theta' = -\infty$ or $\theta'' = +\infty$. Clearly for $\theta \notin H'(\theta)$ x' is not yet an optimal level solution and it is necessary to determine the optimal solution of PL(θ). If θ' (θ'') is finite set $\theta^* = \theta' - \varepsilon$ ($\theta^* = \theta'' + \varepsilon$), $\varepsilon > 0$ and small and solve PL(θ^*). The optimal solution is obtained starting from the basis $A^*_{B'}$ by one simplex iteration inserting into the basis the variable $x_{N'v}$ such that $c^*_{N'v}(\theta') = 0$ ($c^*_{N'v}(\theta'') = 0$). Let x^* be the optimal solution of PL(θ^*) and $A^*_{B''}$ the corresponding basis. x^* lies on an edge of S and is not in general a vertex of S.

Let us define the set :

$E(t) = \{ t : x^*_{B''} + t\ A^{*\,-1}_{B''} (0, 0, \ldots\ldots, 1)^T = x^*_{B''} + ty \in S \}$.

The set $\{ (x^*_{B''} + ty, 0) : t \in E(t) \}$ is the edge containing x^*. Clearly E(t) can be a segment or an halfline ; in the first case the edge is bounded and in the second case it is unbounded. If the edge is bounded then $E(t) = [t', t'']$ and $x^1 = (x^*_{B''} + t'y)$, $x^2 = (x^*_{B''} + t''y)$ are the extreme points of the edge. If the edge is unbounded then $E(t) = (-\infty, t']$ or $E(t) = [t', +\infty)$ and $x'' = (x^*_{B''} + t'y, 0)$ is the origin of the halfline. In the bounded case we go to the vertex x^1 (x^2) such that $[((c+\theta^*u)^T x^1 + c_0 + \theta^* u_0)/(d^T x^1 + d_0)] > [((c+\theta^*u)^T x^2 + c_0 + \theta^* u_0)/(d^T x^2 + d_0)]$ ($[((c+\theta^*u)^T x^2 + c_0 + \theta^* u_0)/(d^T x^2 + d_0)] > [((c+\theta^*u)^T x^1 + c_0 + \theta^* u_0)/(d^T x^1 + d_0)]$) ; in the unbounded case we go to the vertex x''. In this way an optimal level vertex for PLF(θ^*) is obtained. The stability set of the new vertex is adjacent to the stability set of the vertex x' while the two vertices may be not adjacent. In this way we can go from a vertex to another having adjacent stability sets. Clearly this allows us to describe the function $z(\theta)$ by a vertex following procedure.

The following theorem states the properties of function $z(\theta)$.

Theorem 5.1

$z(\theta)$ is a convex piecewise linear function.

Proof. The domain of $z(\theta)$ is the union of adjacent stability sets and hence is a convex set. $z(\theta)$ is piecewise linear as a stability set is the union of sets of type $H(\theta)$ and $H''(\theta)$ which are related to the same vertex of S ; it folllows that for $\theta \in H(\theta)$ ($\theta \in H''(\theta)$) the denominator is constant and hence $z(\theta)$ is a linear function. For the convexity we must show that

$$\lambda z(\theta_1) + (1-\lambda) z(\theta_2) \geq z(\lambda \theta_1 + (1-\lambda)\theta_2), \quad 0 \leq \lambda \leq 1 .$$

Taking into account that

$$z(\theta_1) = \sup_{x \in S} [(c^T x + c_0 + \theta_1 u^T x + \theta_1 u_o)/(d^T x + d_o)] ;$$

$$z(\theta_1) = \sup_{x \in S} [(c^T x + c_0 + \theta_2 u^T x + \theta_2 u_o)/(d^T x + d_o)] ;$$

$$z(\lambda \theta_1 + (1-\lambda)\theta_2) = \sup_{x \in S} [(c^T x + c_0 + (\lambda \theta_1 + (1-\lambda)\theta_2)(u^T x + u_o))/(d^T x + d_o)] ;$$

it results

$$z(\lambda \theta_1 + (1-\lambda)\theta_2) = \sup_{x \in S} [(\lambda(c^T x + c_0 + \theta_1 u^T x + \theta_1 u_o) + (1-\lambda)(c^T x + c_0 + \theta_2 u^T x$$

$$+ \theta_2 u_o))/(d^T x + d_o))] \leq \lambda z(\theta_1) + (1-\lambda) z(\theta_2), \quad 0 \leq \lambda \leq 1 .$$

The proof is completed.

6. Numerical example

Let us consider the example of section 3 :

$$\sup f(x) = ((-3 + 7\theta) x_1 + (2-3\theta) x_2)/(-x_1 + 2x_2 + 7)$$

(6.1) $\quad -2x_1 + x_2 + x_3 = 2 , x_1 - x_2 + x_4 = 3 , x_1 + x_2 - x_5 = 2 , x_1 + 2x_2 - x_6 = 3 ,$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

where x_3, x_4, x_5, x_6 are the slack variables. If we solve problem (6.1) for

$\theta=0$ by means of algorithm MVM we obtain the following results :

$$B=\{2,4,5,6\},\ N=\{1,3\},\ A_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 2 & 0 & 0 & -1 \end{bmatrix},\quad A_B^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 2 & 0 & 0 & -1 \end{bmatrix},$$

$x'_B = A_B^{-1}b = (2,5,0,1)^T$, $c'^T_N = (1,-2)$, $d'^T_N = (3,-2)$, $c'_0 = 4$, $d'_0 = 11$, $\gamma=4/11$, $u'^T_N = (1,3)$, $u'_0 = -6$, $\beta'^T = (-1/11,-14/11)$; clearly $x' = (0,2,0,5,0,1)$ is an optimal solution as $\beta' < 0$. It results $H(\theta) = \{ \theta : \theta \leq 1/29 \}$ and hence x' is optimal for $\theta \leq 1/29$ with $z(\theta) = (4-6\theta)/11$. In order to determine the set $H'(\theta)$ it is necessary to solve problem $PL(\theta)$; it results $B'=\{2,4,5,6,1\}$, $N'=\{3\}$,

$$A^*_{B'} = \begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ -1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 \\ 2 & 0 & 0 & -1 & 1 \\ 2 & 0 & 0 & 0 & -1 \end{bmatrix},\quad A^{*\,-1}_{B'} = \begin{bmatrix} -1/3 & 0 & 0 & 0 & 2/3 \\ 1/3 & 1 & 0 & 0 & 1/3 \\ -1 & 0 & -1 & 0 & 1 \\ -4/3 & 0 & 0 & -1 & 5/3 \\ -2/3 & 0 & 0 & 0 & 1/3 \end{bmatrix},$$

$x^*_{B'} = A^{*\,-1}_{B'}b^* = (2,5,0,1,0)^T$, $c^*_{N'} = -4/3$, $u^*_{N'} = 11/3$ and hence from $(-4/3) + (11/3)\theta \leq 0$ we obtain $H'(\theta) = \{ \theta : \theta \leq 4/11 \}$. For $\theta = (1/29) + \varepsilon$, $c'_{N1} > 0$ and $y^1 = A_B^{-1}A_N^1 = (-2,-1,-3,-5)^T$; it follows that for $1/29 \leq \theta \leq 4/11$ the supremum is obtained along y^1 and $z(\theta) = (1+\theta)/3$. When $\theta = (4/11) + \varepsilon$, $c^*_{N'}(\theta) > 0$ and x' is not an optimal level vertex. A simplex iteration with x_3 as entering variable must be made. It results $B'=\{2,3,5,6,1\}$, $N'=\{4\}$,

$$A^*_{B'} = \begin{bmatrix} 1 & 1 & 0 & 0 & -2 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 \\ 2 & 0 & 0 & -1 & 1 \\ 2 & 0 & 0 & 0 & -1 \end{bmatrix},\quad A^{*\,-1}_{B'} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 & 1 \\ 0 & 3 & -1 & 0 & 2 \\ 0 & 4 & 0 & -1 & 3 \\ 0 & 2 & 0 & 0 & 1 \end{bmatrix},$$

$x^*_{B'} = A^{*\,-1}_{B'}b^* = (7,15,15,21,10)^T$,

$E(t) = \{ t : (7,15,15,21,10)^T + t\,(1,1,2,3,1)^T \geq (0,0,0,0,0)^T \} = \{ t : t \geq -7 \}$. For $t=-7$ we obtain the new vertex $x^1=(3,0,8,0,1,0)$ which corresponds to the basis $B=\{3,5,6,1\}$, $N=\{2,4\}$,

$$A_B = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad A_B^{-1} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad x'_B = A_B^{-1}b = (8,1,0,3)^T,$$

$c'_N{}^T = (-1,3)$, $d'_N{}^T = (1,1)$, $c'_0 = -9$, $d'_0 = 4$, $u'_N{}^T = (4,-7)$, $u'_0 = 21$, $\beta'^T = (5/4, 21/4)$, $\beta'^T(\theta) = ((5/4) - (5/4)\theta, (21/4) - (49/4)\theta)$, $H(\theta) = \{ \theta : \theta \geq 1 \}$, $H'(\theta) = \{ \theta : \theta \geq 4/11 \}$; in fact it results $B' = \{3,5,6,1,2\}$, $N' = \{4\}$,

$$A^*_{B'} = \begin{bmatrix} 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}, \quad A^*_{B'}{}^{-1} = \begin{bmatrix} 1 & 5/3 & 0 & 0 & 1/3 \\ 0 & 1/3 & -1 & 0 & 2/3 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 2/3 & 0 & 0 & 1/3 \\ 0 & -1/3 & 0 & 0 & 1/3 \end{bmatrix},$$

$x^*_{B'} = A^*_{B'}{}^{-1}b^* = (7,15,15,21,10)^T$, $c^*_{N'} = 4$, $u^*_{N'} = -11$, $c^*_{N'}(\theta) = 4 - 11\theta$. For $4/11 \leq \theta \leq 1$ $(c'_{N1}(\theta))/d'_{N1} > (c'_{N2}(\theta))/d'_{N2}$; as $y^1 = A_B^{-1}A_N^1 = (-1,-2,-3,-1)^T$ then the supremum is obtained along y^1 with $z(\theta) = -1+4\theta$. Finally for $\theta \geq 1$ the vertex x^3 is optimal and $z(\theta) = (-9 + 21\theta)/4$.

The function $z(\theta)$ is the following :

$$z(\theta) = \begin{cases} (4-6\theta)/11 , & \theta \leq 1/29 ; \\ (1+\theta)/3 , & 1/29 \leq \theta \leq 4/11 ; \\ -1+4\theta , & 4/11 \leq \theta \leq 1 ; \\ (-9 + 21\theta)/4 , & \theta \geq 1 . \end{cases}$$

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