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ON SOLVING A LINEAR PROGRAM WITH ONE QUADRATIC CONSTRAINT

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KEY WORDS: Fractional programming, quadratic programming

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(*) The paper has been discussed jointly by the Authors; all sections have been developed by MARTEIN.

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INTRODUCTION

There are various decision problems that give rise to a linear program with an additional quadratic constraint.
Such a problem arises for instance when in a linear model the optimization is further restricted by a nonlinear constraint which can be approximated sufficiently well by a quadratic constraint. On the other hand, such an additional quadratic constraint may arise in a natural way. This is the case in portfolio theory when the expected return is maximized subject to a bounded variance of the return.
Similarly in stochastic linear programming certain deterministic equivalents give rise to a linear program with an additional quadratic constraint that involves the variance of coefficients [3].
Moreover, it was shown in [7,8] when defining the dual of a quadratic-linear fractional program that such a dual reduces to a linear programs with one quadratic constraint.
In fact it was the analysis of fractional programs and their duals which motivated the research in this paper.
We point out that the problem under consideration can also be viewed as the reciprocal of a quadratic program [4] in which the quadratic constraint has changed place with the objective function.
Our solution procedure is parametric in nature. The quadratic constraint $Q(x) \leqq 0$ is relaxed to $Q(x) \leqq \xi$ where $\xi \geqq 0$ is a parameter. There are other methods for our problem that are parametric as well, however the parameter introduced in such algorithms is used differently, see for example [5,9].

1. SOME THEORETICAL RESULTS

Consider the problem

$$P:\ \max\ c^T x,\ x \in R \overset{\Delta}{=} \{x \in \mathbb{R}^n:\ Ax \leq b;\ Q(x) \leq 0,\ x \geq 0\}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, A is a real $m \times n$ matrix and $Q(x)$ is a strictly convex quadratic function, that is $Q(x) \overset{\Delta}{=} 1/2\ x^T Q x + q^T x + q_0$, where Q is a symmetric positive definite matrix of order n.

The aim of the paper is to establish a sequential method for solving problem P in a finite number of iterations.

For this purpose we state, first of all, some theoretical results.

Consider the related linear programs

$$P_L:\ \max\ c^T x,\ x \in R_L \overset{\Delta}{=} \{x \in \mathbb{R}^n:\ Ax \leq b,\ x \geq 0\}\ .$$

We will assume that degeneracy does not occur.

We introduce the following notations:

S° the set of optimal solutions of problem P,

S_L° the set of optimal solutions of the linear problem P_L,

$$K \overset{\Delta}{=} \{x \in \mathbb{R}^n:\ Q(x) \leq 0\}\ ,$$

$$K(\xi) \overset{\Delta}{=} \{x \in \mathbb{R}^n:\ Q(x) \leq \xi\}\ ,\quad \xi \in \mathbb{R}\ .$$

The following theorem holds:

<u>THEOREM</u> 1.1. i) <u>We have</u> $R = \emptyset$ <u>or</u> $S^\circ \neq \emptyset$.

ii) <u>If</u> $S_L^\circ \cap K \neq \emptyset$, <u>then</u> $S_L^\circ \cap K = S^\circ$.

iii) <u>Assume</u> $R \neq \emptyset$. <u>If</u> $S_L^\circ = \emptyset$ <u>or</u> $S_L^\circ \cap K = \emptyset$, <u>then problem</u> P <u>has a unique optimal solution</u> $\bar{x}$ <u>which is binding at the quadratic constraint</u>, i.e. $Q(\bar{x}) = 0$.

<u>Proof</u>. i) It is sufficient to note that R is a compact set since Q(x) is strictly convex.

ii) Since $R \subseteq R_L$, we have

$$(1.1) \qquad \max_{x \in R} c^T x \leq \max_{x \in R_L} c^T x .$$

Any $\bar{x} \in S_L^\circ \cap K \subset R$ satisfies (1.1) as an equality so that $x^\circ \in S^\circ$, and thus $S_L^\circ \cap K \subset S^\circ$. On the other hand, $\bar{x} \in S^\circ$ implies, necessarily, $\bar{x} \in S_L^\circ \cap K$, otherwise (1.1) holds as a strict inequality and this is absurd.

iii) If $\bar{x} \in S^\circ \cap \text{int } K$ (1), then $\bar{x} \in S_L^\circ$ which contradicts our assumption. Hence $S^\circ \subseteq \text{fr } K$. Suppose now that $x^{(1)}, x^{(2)} \in S^\circ$. Since R is convex, the segment $[x^{(1)}, x^{(2)}] \subset S^\circ \cap K$ so that there exists $\hat{x} = \alpha x^{(1)} + (1-\alpha)x^{(2)}$, $\alpha \in (0,1)$, such that $\hat{x} \in S^\circ \cap \text{int } K$. However this is not possible since $S^\circ \subseteq \text{fr } K$.

This completes the proof.

Consider now the parametric problem:

$$P(\xi) : \qquad \max_{x \in R_L \cap K(\xi)} c^T x$$

(1) Let A be a subset of $\mathbb{R}^n$; int A and fr A denote the interior and the frontier of A, respectively.

and let $\bar{x}$ be an optimal solution of $P(\xi)$ with $Q(\bar{x}) = \xi$. The Karush-Kuhn-Tucker conditions applied at $\bar{x}$ establish the existence of multipliers $\lambda \in \mathbb{R}_+^{m+n}$, $\mu \in \mathbb{R}_+$, with $(\lambda,\mu) \neq 0$ such that:

$$(1.2a) \qquad c = \sum_{i=1}^{m+n} \lambda_i a^{(i)} + \mu \nabla Q(\bar{x})$$

$$(1.2b) \qquad \lambda_i((a^{(i)})^T \bar{x} - b_i) = 0, \quad i=1,\ldots,m$$

$$(1.2c) \qquad \lambda_{m+j} \bar{x}_j = 0 , \quad j=1,\ldots,n$$

$$(1.2d) \qquad \mu(Q(\bar{x})-\xi) = 0$$

$$(1.2e) \qquad A\bar{x} \leq b , \quad Q(\bar{x}) \leq \xi , \quad \bar{x} \geq 0$$

where $a^{(i)}$, $i=1,\ldots,m$ and $a^{(m+j)}$, $j=1,\ldots,n$, denote the transpose of the i-th row of A and the transpose of the j-th row of matrix $-I_n$, respectively.

The following theorem holds:

THEOREM 1.2. *Let* $\bar{x}$ *be an optimal solution of* $P(\xi)$, *with* $Q(\bar{x}) = \xi$. *Then there exist* $\bar{\lambda} \in \mathbb{R}_+^{m+n}$, $\bar{\mu} > 0$ *satisfying* (1.1), *if conditions* i) *or* ii) *hold*:

i) $\bar{x} \notin S_L^o$;

ii) $S_L^o = \{ \bar{x} \}$.

<u>Proof</u>. Suppose that $\bar{x} \notin S_L^o$. If conditions (1.2) are true for $\mu = 0$, then $\bar{x}$ satisfies the Karush-Kuhn-Tucker conditions for the linear problem P_L, so that $\bar{x} \in S_L^o$, but this contradicts our assumption. On the other hand, if $\bar{x}$ is the unique solution of P_L, then $\bar{x}$ is necessarily a vertex of R_L and we have

$$(1.3) \qquad c = \sum_{j \in J} \tilde{\lambda}_j \, a^{(j)} \quad , \quad \tilde{\lambda}_j > 0 \qquad \forall j \in J$$

where J is the set of indices associated with the active constraints at $\bar{x}$.

With respect to the binding constraints at $\bar{x}$, (1.2a) reduces to:

$$(1.4) \qquad c = \sum_{j \in J} \lambda_j \, a^{(j)} + \mu \nabla Q(\bar{x}) \ .$$

Let us note that (1.4) is true for $\lambda_j = \tilde{\lambda}_j$ and $\mu = 0$; we will show that there exist $\bar{\lambda}_j \geq 0$, $j \in J$ and $\bar{\mu} > 0$ satisfying (1.4). Since $a^{(j)}$, $j \in J$ are a basis of $\mathbb{R}^n$, then there exist α_j, $j \in J$, such that: $\nabla Q(\bar{x}) = \sum_{j \in J} \alpha_j \, a^{(j)}$. Thus we have:

$$(1.5a) \qquad c = \sum_{j \in J} (\lambda_j + \mu_j \alpha_j) a^{(j)}$$

$$(1.5b) \qquad \tilde{\lambda}_j = \lambda_j + \mu \alpha_j \ , \ j \in J \ .$$

Set $J_1 \triangleq \{j \in J \ : \ \alpha_j \geq 0\}$; the following cases arise:

I) $J_1 = \emptyset$; then (1.4) is satisfied for any $\mu > 0$ and $\bar{\lambda}_j = \tilde{\lambda}_j - \mu\alpha_j$, $j \in J$.

II) $J_1 \neq \emptyset$; set

$$\bar{\mu} = \min_{j \in J_1} \tilde{\lambda}_j / \alpha_j \triangleq \tilde{\lambda}_k / \alpha_k \tag{1.6}$$

Then (1.4) is satisfied for $\bar{\mu} = \dfrac{\tilde{\lambda}_k}{\alpha_k}$ and $\bar{\lambda}_j = \tilde{\lambda}_j - \bar{\mu}\alpha_j$, $j \in J$.
This completes the proof.

<u>REMARK 1.1</u> Taking into account Theorem 1.2, condition (1.2a) can be equivalently rewritten in the form

$$\lambda_o c = \nabla Q(\bar{x}) + \sum_{i=1}^{m+n} \lambda_i a^{(i)}, \quad \lambda_o > 0 . \tag{1.7}$$

2. BASIS THEORETICAL RESULTS FOR THE ALGORITHM

Consider the linear problem P_L. Suppose that $\bar{x}$ is the unique solution of P_L (the other cases will be discussed in Section 5). If $\bar{x}$ satisfies the quadratic constraint, then $\bar{x}$ is optimal for problem P. Otherwise we consider the parametric problem $P(\xi)$, with $\xi \in [0, \bar{\xi}]$, $\bar{\xi} = Q(\bar{x})$. The idea of the sequential method that we are going to describe is to generate a finite sequence of $\bar{x}^{(k)}$ where $\bar{x}^{(k)}$ is an optimal solution for $P(\xi_k)$ (see fig. 1 pag.26). An optimal solution corresponding to $\xi_k = 0$ is then an optimal solution of P.
With this aim in mind, consider problem $P(\xi_k)$, $\xi_k \in [0, \bar{\xi}]$ and its optimal solution $\bar{x}^{(k)}$.

Let $\bar{B}x = \bar{b}$ be the system of linear equations corresponding to the set of the constraints binding at $\bar{x}^{(k)}$, i.e. $\bar{B}\bar{x}^{(k)} = \bar{b}$, where $\bar{B}$ is a $k \times n$ real matrix of rank k. We suppose $k < n$; the case $k = n$ will be analyzed in Remark 3.1.
From the Karush-Kuhn-Tucker conditions applied to problem $P(\xi_k)$, taking into account Theorem 1.2 and Remark 1.1, we have:

$$(2.1a) \qquad Qx + \bar{B}^T \lambda = \lambda_o c - q$$

$$(2.1b) \qquad \bar{B}x = \bar{b}$$

that is

$$(2.2) \qquad H \begin{pmatrix} x \\ \lambda \end{pmatrix} = \lambda_o \begin{pmatrix} c \\ 0 \end{pmatrix} + \begin{pmatrix} -q \\ \bar{b} \end{pmatrix}$$

where $H \triangleq \begin{bmatrix} Q & \bar{B}^T \\ \bar{B} & 0 \end{bmatrix}$.

Since $\bar{B}$ has full rank, the matrix H is non singular so that, from (2.2), we obtain

$$(2.3a) \qquad x = \lambda_o \tilde{u} + \tilde{v}$$

$$(2.3b) \qquad \lambda = \lambda_o \bar{u} + \bar{v}$$

where $u \triangleq \begin{pmatrix} \tilde{u} \\ \bar{u} \end{pmatrix} = H^{-1} \begin{pmatrix} c \\ 0 \end{pmatrix}$ and $v \triangleq \begin{pmatrix} \tilde{v} \\ \bar{v} \end{pmatrix} = H^{-1} \begin{pmatrix} -q \\ \bar{b} \end{pmatrix}$.

Consider now the quadratic constraint in the parametric form:

$$(2.4) \qquad 1/2\; x^T Q x + q^T x + q_o = \xi \quad , \quad \xi \in [0, \xi_k]$$

and substitute (2.3a) in (2.4); we obtain the following second order equation:

$$(2.5) \qquad 1/2\,\alpha\lambda_o^2 + \beta\lambda_o + \gamma - \xi = 0$$

where

$$(2.6) \qquad \alpha = \tilde{u}^T Q\tilde{u} \;;\; \beta = \tilde{u}^T Q\tilde{v} + q^T\tilde{u}; \quad \gamma = 1/2\tilde{v}^T Q\tilde{v} + q^T\tilde{v} + q_o .$$

Set (2)

$$(2.6) \qquad \lambda_o(\xi) = \frac{-\beta + \sqrt{\beta^2 - 2\alpha\gamma + 2\alpha\xi}}{\alpha} .$$

We are interested in decreasing the value of ξ, starting from ξ_k, in such a way that $x(\xi)=\lambda_o(\xi)\tilde{u}+\tilde{v}$ is the optimal solution of problem $P(\xi)$. To this end, let us note that $\lambda_o(\xi)$ is an increasing function so that $\xi < \xi_k$ implies $\lambda_o(\xi) < \bar{\lambda}_o \triangleq \lambda_o(\xi_k)$. Then λ_o plays the role of a parameter in (2.3) and we can study the stability of the solution of problem $P(\xi_k)$ with respect to λ_o. In other words we want to find the values of $\lambda_o \in [0,\bar{\lambda}_o]$ which

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(2) Let us note that equation (2.5) must have, for $\xi = \xi_k$ a positive root since problem $P(\xi_k)$ has optimal solutions; on the other hand, (2.1a) collapses to the Karush-Kuhn-Tucker conditions applied to problem $\min c^T x;\ x \in \{x \in \mathbb{R}^n ;\ \bar{B}x \geq \bar{b},\ Q(x) \leq \xi_k\}$ so that (2.5) has also a negative root.

not only guarantee the nonnegativity of $x(\lambda_o)$ and $\lambda(\lambda_o)$ in (2.3), but also the feasibility of $x(\lambda_o)$ with respect to the constraints $\tilde{B}x \leq \tilde{b}$ which are nonactive at $\bar{x}^{(k)}$.

Set: $\hat{u} \triangleq \tilde{B}\tilde{u}$; $\hat{v} \triangleq \tilde{b}-\tilde{B}\tilde{v}$;

$$J_1=\{j:\tilde{v}_j < 0\};\; J_2=\{j:\bar{v}_j < 0\};\; J_3=\{j:\hat{v}_j < 0\} .$$

The following theorem holds:

<u>THEOREM</u> 2.1 <u>The vector</u> $x(\lambda_o)$ <u>is the optimal solution for the problem</u> $P(\xi)$, $\xi=Q(x(\lambda_o))$, <u>for any</u> $\lambda_o \in [\lambda_o^*,\bar{\lambda}_o]$, <u>where</u>

(2.7a) $$\lambda_o^* = \max\{\lambda_{o1},\lambda_{o2},\lambda_{o3}\} ;$$

(2.7b) $$\lambda_{o1} = \begin{cases} \max\limits_{j\in J_1} -\dfrac{\tilde{v}_j}{\tilde{u}_j} , & \text{if } J_1 \neq \emptyset \\ 0 & \text{otherwise} \end{cases} ;$$

(2.7c) $$\lambda_{o2} = \begin{cases} \max\limits_{j\in J_2} -\dfrac{\bar{v}_j}{\bar{u}_j} & \text{if } J_2 \neq \emptyset \\ 0 & \text{otherwise} \end{cases} ;$$

(2.7d) $$\lambda_{o3} = \begin{cases} \max\limits_{j\in J_3} -\dfrac{\hat{v}_j}{\hat{u}_j} & \text{if } J_3 \neq \emptyset \\ 0 & \text{otherwise} \end{cases} .$$

Proof. From the Karush-Kuhn-Tucker conditions applied to problem $P(\xi_k)$, taking into account of (2.3), $x(\lambda_o)$ is optimal for $P(\xi)$, $\xi=Q(x(\lambda_o))$, if i), ii), iii) hold:

i) $x(\lambda_o) = \lambda_o\tilde{u}+\tilde{v} \geq 0$; ii) $\lambda(\lambda_o) = \lambda_o\bar{u}+\bar{v} \geq 0$; iii) $\tilde{B}(x(\lambda_o)) =$
$= \lambda_o\tilde{B}\tilde{u}+\tilde{B}\tilde{v} \leq \tilde{b}$ that is $\lambda_o\hat{u} \leq \hat{v}$.

Now we will show that i) holds for any $\lambda_o \in [\lambda_{o1},\bar{\lambda}_o]$. If $\tilde{v}_j \geq 0$ and $\tilde{u}_j \geq 0$, then $x_j(\lambda_o) \geq 0$ for any $\lambda_o \geq 0$. If $\tilde{v}_j > 0$ and $\tilde{u}_j < 0$, then $x_j(\lambda_o) \geq 0$ is true for any $\lambda_o \leq -\frac{\tilde{v}_j}{\tilde{u}_j}$; since $\bar{x}^{(k)}$ is optimal for $P(\xi_k)$, then i) is satisfied for $\lambda_o = \bar{\lambda}_o$, so that $\bar{\lambda}_o \leq -\frac{\tilde{v}_j}{\tilde{u}_j}$ and thus $x_j(\lambda_o) \geq 0$ for any $\lambda_o \leq \bar{\lambda}_o$. Consider now the case $\tilde{v}_j \leq 0$; since $x(\bar{\lambda}_o) \geq 0$, necessarily we have $\tilde{u}_j > 0$, so that $x_j(\lambda_o) \geq 0$ for any $\lambda_o \geq -\frac{\tilde{v}_j}{\tilde{u}_j}$.
As a consequence i) is satisfied for any $\lambda_o \in [\lambda_{o1},\bar{\lambda}_o]$.
In a similar way we can prove that ii) and iii) hold for any $\lambda_o \in [\lambda_{o2},\bar{\lambda}_o]$ and for any $\lambda_o \in [\lambda_{o3},\bar{\lambda}_o]$, respectively.
Obviously, all the conditions i), ii) and iii) are satisfied for any $\lambda_o \in [\lambda_o^*, \bar{\lambda}_o]$.
This completes the proof.

We have just seen that starting from the optimal solution $\bar{x}^{(k)}$ of problem $P(\xi^k)$, we have found a new optimal solution $\bar{x}^{(k+1)}= =x(\lambda_o^*)$ of problem $P(\xi_{k+1})$, $\xi_{k+1}=Q(\bar{x}^{(k+1)})$. If $\xi_{k+1} > 0$, we must still decrease the value of the parameter ξ in order to reach the value zero. This will be analyzed in the next section. Some special cases will be studied in the following theorem.

THEOREM 2.2. Let λ_o^* be the value defined by (2.7a).
i) If $\lambda_o^*=0$ and $Q(x(0))>0$, then the feasible region of problem P is empty.
ii) If $Q(x(\lambda_o^*)) = 0$ then $x(\lambda_o^*)$ is an optimal solution of problem P.
iii) If $Q(x(\lambda_o^*))<0$, there $x(\hat{\lambda}_o)$ is the optimal solution of problem P, where $\hat{\lambda}_o$ is the positive root of the equation $Q(x(\lambda_o))=0$.

Proof. i) From the Karush-Kuhn-Tucker conditions (1.2) and taking into account of (1.7), $\lambda_o^* = 0$ implies that $x(0)$ is the optimal solution of the problem

$$\min Q(x) \quad , \quad x \in R_L \quad .$$

Since $Q(x(0))>0$, the feasible region of the problem P is empty.
ii) Obvious.
iii) Since $Q(x(\bar{\lambda}_o))>0$, the optimal solution of problem P is binding at the quadratic constraint (see Theorem 1.1 iii)); on the other hand $Q(x(\lambda_o^*))<0$ implies the existence of $\hat{\lambda}_o \in [\lambda_o^*, \bar{\lambda}_o]$ such that $Q(x(\hat{\lambda}_o)) = 0$, then $\hat{\lambda}_o$ is necessarily the positive root of the equation $Q(x(\lambda_o)) = 0$.
This completes the proof.

3. ADDING AND DELETING A CONSTRAINT

Consider again the optimal solution $x(\lambda_o^*) = \bar{x}^{(k+1)}$ of the problem $P(\xi_{k+1})$, $\xi_{k+1} = Q(\bar{x}^{(k+1)})$ and suppose tha $\xi_{k+1} > 0$.Since $\bar{x}^{(k+1)}$ is not optimal for P, we must generate a new optimal solution of $P(\xi)$, corresponding to a value of ξ lower than ξ_{k+1}. Let us note that, for $\lambda_o = \lambda_o^*$, one of the components of the vectors $x(\lambda_o)$, $\lambda(\lambda_o)$, $\tilde{B}x(\lambda_o)-\tilde{b}$ becomes zero, so that the set of the active constraints changes with respect to the parameter ξ. More precisely, if $\lambda_o^* = \lambda_{o1}$ or $\lambda_o^* = \lambda_{o2}$, an active constraint must be deleted, while if $\lambda_o^* = \lambda_{o3}$ one new constraint must be added to the set of active constraints. Thus we are interested to update system (2.3) in order to find the new value of λ_o^* and repeat the procedure.

ADDING A CONSTRAINT :

Suppose that $\alpha^T x = \beta$ becomes an active constraint, then system (2.1) can be updated in the following way:

$$(3.1) \quad \begin{cases} Qx + \bar{B}^T\lambda + \alpha\bar{\lambda} = \lambda_o c - q \\ \bar{B}x = \bar{b} \\ \alpha^T x = \beta \end{cases}$$

where $\bar{\lambda}$ is the multiplier associated with $\alpha^T x = \beta$.
System (3.1) can be rewritten in the following way:

$$(3.2)\qquad H'\begin{pmatrix} x \\ \lambda \\ \bar{\lambda} \end{pmatrix} = \lambda_o \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -q \\ \bar{b} \\ \beta \end{pmatrix}$$

where $H' = \left[\begin{array}{c|c} H & h \\ \hline h^T & 0 \end{array}\right]$, $h^T = (\alpha^T, 0)$

It is easy to show that the inverse of H' can be obtained from H^{-1} by the following formula:

$$(3.3)\quad (H')^{-1} = \left[\begin{array}{c|c} H^{-1} - \dfrac{(H^{-1}h)(h^T H^{-1})}{h^T H^{-1} h} & \dfrac{H^{-1}h}{h^T H^{-1} h} \\ \hline \dfrac{h^T H^{-1}}{h^T H^{-1} h} & -\dfrac{1}{h^T H^{-1} h} \end{array}\right]$$

By means of the new inverse, it is easily possible to update (2.3).

DELETING A CONSTRAINT :

Suppose that the constraint $\alpha^T x = \beta$ must be deleted from the set of the active ones; we can assume, without loss of generality, that the matrix and the right-hand side in (2.1b) are of the form:

$$\bar{B} = \begin{bmatrix} \tilde{B} \\ \alpha^T \end{bmatrix} , \qquad \bar{b} = \begin{pmatrix} \tilde{b} \\ \beta \end{pmatrix}$$

so that system (2.1) reduces to

$$\begin{cases} Qx + \tilde{B}^{T}\tilde{\lambda} = \lambda_o c - q & (3.4a) \\ \tilde{B}x = \tilde{b} & (3.4b) \end{cases}$$

that is

$$H'\begin{pmatrix} x \\ \tilde{\lambda} \end{pmatrix} = \lambda_o \begin{pmatrix} c \\ 0 \end{pmatrix} + \begin{pmatrix} -q \\ \tilde{b} \end{pmatrix}, \text{ where } H' \triangleq \begin{bmatrix} Q & \tilde{B}^{T} \\ \tilde{B} & 0 \end{bmatrix} .$$

It is easy to show that the inverse of H' can be obtained from $H^{-1} \triangleq \left[\begin{array}{c|c} C & d \\ \hline d^{T} & a \end{array}\right]$ by the following formula:

$$(3.5) \qquad (H')^{-1} = C - \frac{1}{a}(d \cdot d^{T})$$

REMARK 3.1. When, in system (2.1), $\bar{B}$ is a square matrix of order n, (2.3) reduces to

$$(3.6a) \qquad x = \tilde{v}$$

$$(3.6b) \qquad \lambda = \lambda_o \bar{u} + \bar{v}$$

where $\tilde{v} = \bar{B}^{-1}\bar{b}$; $\bar{u} = (\bar{B}^{T})^{-1}c$; $\bar{v} = -(\bar{B}^{T})^{-1}q - (\bar{B}Q^{-1}\bar{B}^{T})^{-1}\bar{b}$, so that, starting from the optimal solution of the linear problem P_L $\bar{x} = \bar{B}^{-1}\bar{b}$, we must only choose, by means of (2.7c), the constraint that must be deleted. Let us note that when in (2.7c) $\lambda_{02} = 0$ the feasible region of problem P is empty, since, from (1.5), $\bar{x}$ is the optimal solution of the problem $\min \{Q(x): x \in R_L\}$.

4. THE ALGORITHM

The theoretical results, established in the previous sections, allow us to suggest the following algorithm to solve problem P, when the linear problem P_L has a unique solution.

STEP.0 (Not iterative). Solve problem P_L and let $\bar{x}$ be the unique solution of P_L. If $Q(\bar{x}) \leq 0$, STOP: $\bar{x}$ is the optimal solution of P; otherwise set $\bar{x} = \bar{x}^{(k)}$ and go to step 1.

STEP.1 Let $\bar{B}$ ($\tilde{B}$) be the matrix associated with the active (non active) constraints at $\bar{x}^{(k)}$. Calculate $\lambda_{01}, \lambda_{02}, \lambda_{03}$ and λ_0^*. If $\lambda_0^* = 0$ go to step 7; otherwise $\bar{x}^{(k+1)} = x(\lambda_0^*)$ becomes the current solution; go to step 2.

STEP.2 If $Q(\bar{x}^{(k+1)}) = 0$, STOP: $\bar{x}^{(k+1)}$ is the optimal solution of P. If $Q(\bar{x}^{(k+1)}) < 0$, then go to step 6; otherwise go to step 3.

STEP.3 If $\lambda_0^* \neq \lambda_{01}$ go to step 4; otherwise a nonnegativity constraint is deleted (see procedure "deleting a constraint"); set k = k+1 and go to step 1.

STEP.4 If $\lambda_o^* = \lambda_{o3}$, go to step 5; otherwise the constraint corresponding to the multiplier which becomes zero is deleted (see procedure "deleting a constraint"); set k = k+1 and go to step 1.

STEP.5 A new constraint is added (see procedure "adding a constraint"); set k = k+1 and go to step 1.

STEP.6 Calculate $\hat{\lambda}_o$ and $x(\hat{\lambda}_o)$, STOP: $x(\hat{\lambda}_o)$ is the optimal solution of P.

STEP.7 If $Q(x(0)) > 0$, STOP: the feasible region of P is empty. If $Q(x(0)) = 0$, STOP: x(0) is the optimal solution of P; otherwise go to step 6.

5. SPECIAL CASES

Let us note that the sequential method which we have described in section 4 is based on the following assumptions:

1] Problem P_L has an optimal solution $\bar{x}$;

2] The Lagrange multiplier associated with the quadratic constraint in problem $P(\bar{\xi})$, $\bar{\xi} = Q(\bar{x})$, is strictly positive.

Thus in order to establish a sequential method for solving problem P in the general case, we must analyse what happens when 1] or 2] do not hold.
When the objective function of the linear problem P_L is not bounded from above, we solve first the problem:

(5.1) $$\max_{Q(x)\leq 0} c^T x \triangleq c^T x^\circ$$

and then the problem

(5.2) $$\max_{c^T x \leq c^T x^\circ,\ x \in R} c^T x \triangleq c^T x^*$$

This allows us to find a feasible level $\xi^* = Q(x^*)$ of the quadratic function, in such a way that we can apply the sequential method, described in section 4, starting from the problem $P(\xi^*)$.
When the linear problem P_L has alternate optimal solutions, it can happen that the only Lagrange multiplier μ, satisfying (1.2a), is zero, so that we cannot start with our procedure.
In this case the idea is to perturbate the objective function in such a way that the new linear problem has a unique optimal solution and to apply the sequential method of section 4 until we can restore the original objective function.
More precisely, let $\bar{x}$ be an optimal solution (not unique) of P_L and consider the problem

$$P_\varepsilon : \max_{x \in R} (c+\varepsilon c')^T x$$

where[3] c' is such that the linear problem

$$(5.3) \qquad \max_{x \in R_L} (c+\varepsilon c')^T x$$

has the unique solution $\bar{x}$; (2.3) becomes

$$(5.4a) \qquad x = \lambda_o(\tilde{u}+\varepsilon\tilde{h}) + \tilde{v}$$

$$(5.4b) \qquad \lambda = \lambda_o(\bar{u}+\varepsilon\bar{h}) + \bar{v}$$

where $h \triangleq \begin{pmatrix} \tilde{h} \\ \bar{h} \end{pmatrix} = H^{-1} \begin{pmatrix} c' \\ 0 \end{pmatrix}$.

Let us note that (2.7) cannot be applied since the coefficients of λ_o depend on ε, for this reason we consider the following set of indeces:

$J'_1 = \{j:\ \tilde{u}_j=0,\ \tilde{h}_j > 0,\ \tilde{v}_j < 0\}$; $J'_2 = \{j:\ \bar{u}_j=0,\ \bar{h}_j > 0,\ \bar{v}_j < 0\}$;

$J'_3 = \{j:\ \hat{u}_j=0,\ \hat{h}_j > 0,\ \hat{v}_j < 0\}$,

where $\hat{h} \triangleq \tilde{B}\tilde{h}$.

Theorem 2.1 can be reformulated in the following way:

---

(3) A suitable choise for c' may be $c' = \sum_{i=1}^{n} a^{(i)}$, where $a^{(i)}$ denotes the gradient of the i-th linear constraint binding at $\bar{x}$.

<u>THEOREM</u> 5.1. <u>The vector</u> $x(\lambda_o)$ <u>is the optimal solution for the problem</u> $P_\varepsilon(\xi)$, $\xi = Q(x(\lambda_o))$, <u>for any</u> $\lambda_o \in [\tilde{\lambda}_o/\varepsilon, \bar{\lambda}_o/\varepsilon]$, <u>where</u>

$$\tilde{\lambda}_o = \max\{\lambda'_{o1}, \lambda'_{o2}, \lambda'_{o3}\} \tag{5.5a}$$

$$\lambda'_{o1} = \begin{cases} \max\limits_{j \in J'_1} -\dfrac{\tilde{v}_j}{\tilde{h}_j}, & \text{if } J'_1 \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{5.5b}$$

$$\lambda'_{o2} = \begin{cases} \max\limits_{j \in J'_2} -\dfrac{\bar{v}_j}{\bar{h}_j}, & \text{if } J'_2 \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{5.5c}$$

$$\lambda'_{o3} = \begin{cases} \max\limits_{j \in J'_3} -\dfrac{\hat{v}_j}{\hat{h}_j}, & \text{if } J'_3 \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{5.5d}$$

<u>Proof</u>. It is sufficient to note, taking into account of the arbitrarity of ε, that $\max\limits_{j \in J_1} -\dfrac{\tilde{v}_j}{\tilde{u}_j + \varepsilon \tilde{h}_j}$ is reached when $\tilde{u}_j = 0$.

COROLLARY 5.1. If $\tilde{\lambda}_o = 0$, then $x(\lambda_o)$ is the optimal solution for problem $P(\xi)$, $\xi = Q(x(\lambda_o))$, for any $\lambda_o \in [\lambda_o^*, \bar{\lambda}_o]$.

Proof. The statement follows immediately from Theorem 5.1 and Theorem 2.1.

REMARK 5.1. If $\tilde{\lambda}_o > 0$, we must add or delete a constraint according to $\tilde{\lambda}_o = \lambda'_{o3}$ or not (see Section 3). As a consequence of Corollary 5.1, if $\tilde{\lambda}_o = 0$, then we can set $\varepsilon = 0$ and restore problem $P(\xi)$.

6. NUMERICAL EXAMPLES

EXAMPLE 6.1. (P_L has a unique solution).

Consider the problem

$$P : \begin{cases} \max\ x_1 + 2x_2 \\ 3 \leq x_1 \leq 8,\ 2 \leq x_2 \leq 7 \\ Q(x) \triangleq x_1^2 + x_2^2 - 25 \leq 0 \end{cases}$$

The linear problem

$$P_L : \begin{cases} \max\ x_1 + 2x_2 \\ \\ 3 \leq x_1 \leq 8,\ 2 \leq x_2 \leq 7 \end{cases}$$

has the unique solution $\bar{x} = (8,7)$, which is not optimal for P since $Q(\bar{x}) = 88 > 0$.

The constraints which are active at $\bar{x}$ are $x_1 \leq 8$, $x_2 \leq 7$, so that we have

$$H = \left[\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] , \quad H^{-1} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{array}\right] .$$

Hence (2.3) turns out to be:

$$x_1 = 8$$
$$x_2 = 7$$
$$\lambda_1 = \lambda_o - 16$$
$$\lambda_2 = 2\lambda_o - 14$$

Since $x(\lambda_o) = (8,7)$ is indipendent from λ_o, we have $J_1 = J_3 = \emptyset$ so that $\lambda_o^* = \lambda_{o2} = \max\{16, \frac{14}{2}\} = 16$; as a consequence the constraint $x_1 \leq 8$ associated with the multiplier λ_1 must be deleted. We apply the procedure "deleting a constraint" and find the new inverse

$$(H')^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \quad ; \quad \text{(2.3) becomes}$$

$$x_1 = 1/2\,\lambda_o$$

$$x_2 = 7$$

$$\lambda_2 = 2\,\lambda_o - 14$$

With respect to the nonnegative constraints we have:

$$\tilde{B} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad , \quad \tilde{b} = \begin{pmatrix} -3 \\ -2 \end{pmatrix} \quad ; \quad \hat{u} = \begin{pmatrix} -1/2 \\ 0 \end{pmatrix} \quad ; \quad \hat{v} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$$

So that $\lambda_{o1} = 0$; $\lambda_{o2} = 7$, $\lambda_{o3} = 6$ and $\lambda_o^* = \lambda_{o2} = 7$ implies that the current solution is $\bar{x}^{(1)} = (7/2, 7)$. Since $Q(\bar{x}^{(1)}) = \frac{145}{4} > 0$, the value of λ_o must be decreased, the constraint $x_2 \leq 7$ associated with the multiplier λ_2 must be deleted. The new inverse is

$$(H')^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$ and (2.3) becomes:

$$x_1 = 1/2\,\lambda_o$$

$$x_2 = \lambda_o \; .$$

Regarding the nonactive constraints we have

$$\tilde{B} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \quad ; \quad \tilde{b} = \begin{pmatrix} 8 \\ -3 \\ -2 \end{pmatrix} \quad ; \quad \hat{u} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix} \quad ; \quad \hat{v} = \begin{pmatrix} 8 \\ -3 \\ -2 \end{pmatrix} ;$$

so that $\lambda_{o1} = 0$; $\lambda_{o2} = 0$; $\lambda_{o3} = 6$ and $\lambda_o^* = \lambda_{o3} = 6$ gives the solution $\bar{x}^{(2)} = (3,6)$. Since $Q(\bar{x}^{(2)}) = 20 > 0$, the value of λ_o must be decreased; the constraint $-x_1 \leq -3$ must be added. We apply the procedure "adding a constraint" and we find the new inverse:

$$(H')^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1/2 & 0 \\ -1 & 0 & -2 \end{bmatrix} \qquad \text{and (2.3) becomes:}$$

$$x_1 = 3$$

$$x_2 = \lambda_o$$

$$\lambda_1 = -\lambda_o + 6$$

With regard to the nonactive constraints we have

$$\tilde{B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \quad \tilde{b} = \begin{pmatrix} 8 \\ -2 \\ 7 \end{pmatrix} \quad ; \quad \hat{u} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad ; \quad \hat{v} = \begin{pmatrix} 5 \\ -2 \\ 7 \end{pmatrix}$$

so that $\lambda_{o1} = 0$, $\lambda_{o2} = 0$; $\lambda_{o3} = 2$ and $\lambda_o^* = \lambda_{o3} = 2$ implies that the current solution is $\bar{x}^{(3)} = (3,2)$. Since $Q(\bar{x}^{(3)}) = -12 < 0$, then, for Theorem 2.2 iii), we calculate the positive root of $Q(x(\lambda_o)) = \lambda_o^2 - 16 = 0$, so that $\bar{x}^{(4)} = (3,4)$ is the optimal solution of problem P.

The following picture outlines the finite sequence of optimal level solution[4] generated in solving the problem by means of the sequential method

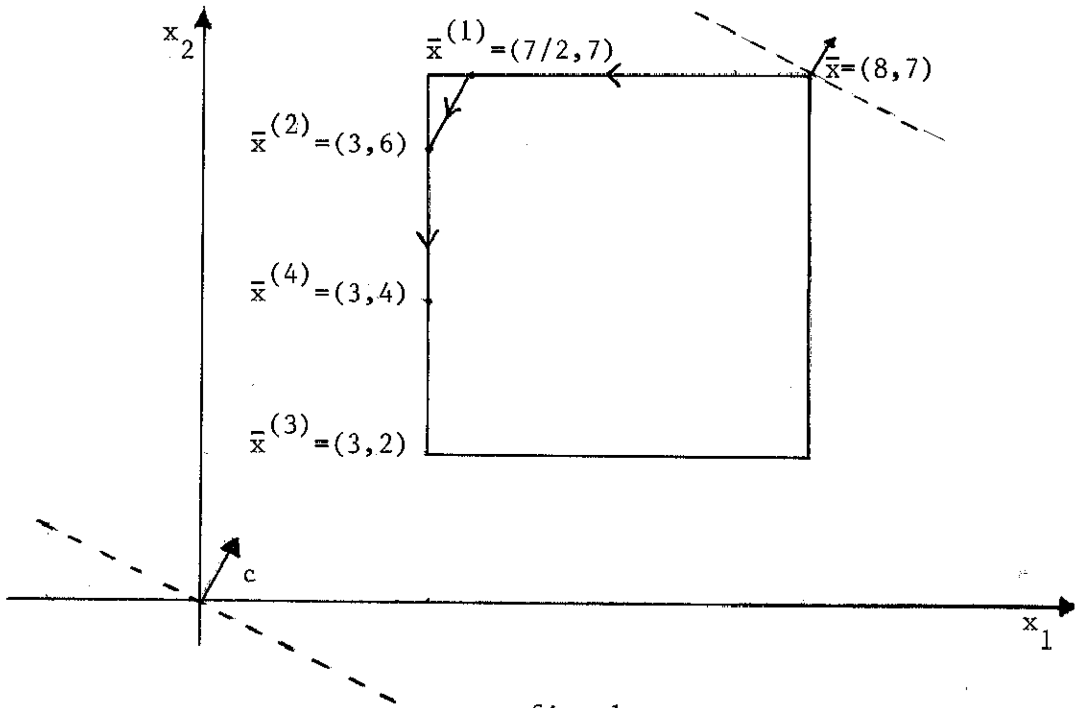

fig. 1

Optimal trajectory $\bar{x} \to \bar{x}^{(1)} \to \bar{x}^{(2)} \to \bar{x}^{(3)} \to \bar{x}^{(4)}$.

(4) Let $\bar{x}^{(k)}$ be the optimal solution of problem $P(\xi_k)$; we refer to $\bar{x}^{(k)}$ as as <u>optimal level solution</u>.

<u>EXAMPLE</u> 6.2. (The feasible region of P is empty).
Consider the problem:

$$P : \begin{cases} \max\ (x_1 + 2x_2) \\ 3 \leq x_1 \leq 8,\ 2 \leq x_2 \leq 8 \\ Q(x) = x_1^2 + x_2^2 - 1 \leq 0 \end{cases}$$

Since the linear problem P_L associated with P is the same as the one of example 6.1 and the quadratic function differs from the previous one only in a constant, the same calculation of example 6.2 are valid until we reach the optimal level solution $\bar{x}^{(3)} = (3,2)$.
At $\bar{x}^{(3)}$, we have $Q(\bar{x}^{(3)}) > 0$. Since $\lambda_o^* = \lambda_{o3}$, the constraint $-x_2 \leq -2$ must be added. We apply the procedure "adding a constraint"and we find the new inverse:

$$(H')^{-1} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix}$$ and (2.3) becomes

$$x_1 = 3$$

$$x_2 = 2$$

$$\lambda_1 = -\lambda_o + 6$$

$$\lambda_2 = -2\lambda_o + 4$$

With respect to the nonactive constraint we have

$$\tilde{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad ; \quad \tilde{b} = \begin{pmatrix} 8 \\ 7 \end{pmatrix} \quad ; \quad \hat{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad ; \quad \hat{v} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

so that $\lambda_{o1} = \lambda_{o2} = \lambda_{o3} = 0$. Since $\lambda_o^* = 0$ and $Q(\bar{x}^{(3)}) > 0$, the feasible region of P is empty.

<u>EXAMPLE</u> 6.3. (P_L has alternate optimal solutions).
Consider the problem

$$P : \quad \begin{cases} \max\ x_2 \\ 0 \leq x_1 \leq 4,\ 0 \leq x_2 \leq 4 \\ Q(x) = x_1^2 - x_2^2 - 8x_1 + 12 \leq 0 \ . \end{cases}$$

In this case the linear problem

$$P_L : \quad \begin{cases} \max\ x_2 \\ 0 \leq x_1 \leq 4,\ 0 \leq x_2 \leq 4 \end{cases}$$

has alternate optimal solutions, one of which is $\bar{x} = (0,4)$; taking into account that the constraints binding at $\bar{x}$ are $-x_1 \leq 0$, $x_2 \leq 4$, we consider the problem

$$P_\varepsilon : \begin{cases} \max\ (-\varepsilon x_1 + (1+\varepsilon)x_2) \\ 0 \leq x_1 \leq 4,\ 0 \leq x_2 \leq 4 \\ x_1^2 + x_2^2 - 8x_1 + 12 \leq 0 \end{cases}$$

Now the linear problem associated with P_ε ($\varepsilon > 0$ arbitrarily chosen) has the unique solution $\bar{x} = (0,4)$ so that we can apply the algorithm of section 4. Since $Q(\bar{x}) = 28 > 0$, $\bar{x}$ is not optimal for P_ε.

The constraints which are active at $\bar{x}$ are $-x_1 \leq 0$, $x_2 \leq 4$ so that we have:

$$H = \left[\begin{array}{cc|cc} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \quad \text{and} \quad H^{-1} = \left[\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{array}\right]$$

Hence (5.4) becomes:

$$x_1 = 0$$

$$x_2 = 4$$

$$\lambda_1 = \varepsilon \lambda_o - 8$$

$$\lambda_2 = (1+\varepsilon)\lambda_o - 8 \quad .$$

Since $x(\lambda_o) = (0,4)$ is independent of λ_o, we have $J_1' = J_3' = \emptyset$, so that $\lambda_o = \lambda_{o2}' = 8$; as a consequence the constraint $-x_1 \leq 0$ must be deleted. We apply the procedure "deleting a constraint" and find the new inverse:

$$(H')^{-1} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \quad ; \text{ (5.4) becomes:}$$

$$x_1 = -1/2\,\varepsilon\lambda_o + 4$$

$$x_2 = 4$$

$$\lambda_2 = (1+\varepsilon)\lambda_o - 8$$

With respect to the non active constraints, we have

$$\tilde{B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \; ; \quad \tilde{b} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \; ; \quad \hat{h} = \begin{pmatrix} -1/2 \\ 0 \end{pmatrix} \; ; \quad \hat{v} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \; ; \quad \hat{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so that $J_1' = J_2' = J_3' = \emptyset$; according to Remark 5.1, we can set $\varepsilon = 0$ and restore problem P. We have:

$$x_1 = 4$$

$$x_2 = 4$$

$$\lambda_2 = \lambda_o - 8$$

and with respect to the non active constraints we find $J_3 = \emptyset$. Since $\lambda_o^* = \lambda_{o2} = 8$, $\overline{x}^{(1)} = (4,4)$ is the current solution, with $Q(\overline{x}^{(1)}) = 12 > 0$; the constraint $x_2 \leq 4$ must be deleted; the new inverse is

$$(H')^{-1} = \begin{bmatrix} 1/2 & 0 \\ & \\ 0 & 1/2 \end{bmatrix} \qquad \text{and (2.3) becomes}$$

$$x_1 = 4$$
$$x_2 = 1/2\,\lambda_o \quad .$$

With respect to the non active constraints we have

$$\tilde{B} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \; ; \; \tilde{b} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} \; ; \; \hat{u} = \begin{pmatrix} 0 \\ 0 \\ -1/2 \end{pmatrix} ; \; \hat{v} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \; ,$$

so that $\lambda_{o1} = \lambda_{o2} = \lambda_{o3} = \lambda_o^* = 0$. The current solution is $\bar{x}^{(2)} = (4,0)$ and $Q(\bar{x}^{(2)}) = -4 < 0$, then, for Theorem 2.2 iii), we calculate the positive root of $Q(x(\lambda_o)) = 1/4\lambda_o^2 - 4 = 0$, so that $\bar{x}^{(3)} = (4,2)$ is the optimal solution of P.

<u>EXAMPLE</u> 6.4 (P_L has no optimal solution).
Consider the problem

$$P \quad : \quad \begin{cases} \max\ x_2 \\ 0 \leq x_1 \leq 4, \; x_2 \geq 0 \\ Q(x) = x_1^2 + x_2^2 + 8x_1 + 2x_2 - 8 \leq 0 \end{cases}$$

The corresponding linear problem

$$P_L : \begin{cases} \max x_2 \\ 0 \leqq x_1 \leqq 4, \; x_2 \geqq 0 \end{cases}$$

has not optimal solution, so that we solve first the problem

$$\begin{cases} \max x_2 \\ x_1^2 + x_2^2 + 8x_1 + 2x_2 - 8 \leqq 0 \end{cases}$$

whose optimal solution is $x° = (-4,4)$ and then the problem:

$$\begin{cases} \max x_2 \\ 0 \leqq x_1 \leqq 4, \; x_2 \geqq 0 \\ x_1^2 + x_2^2 + 8x_1 + 2x_2 - 8 \leqq 0 \\ x_2 \leqq 4 \end{cases}$$

The linear problem $\max \{ x_2 : 0 \leqq x_2 \leqq 4, \; 0 \leqq x_2 \leqq 4\}$ has alternate optimal solutions, one of which is $\bar{x} = (0,4)$; taking into account that the constraints binding at $\bar{x}$ are $-x_1 \leqq 0$, $x_2 \leqq 4$, we consider the problem

$$P_\varepsilon : \begin{cases} \max -\varepsilon x_1 + (1+\varepsilon)x_2 \\ 0 \leqq x_1 \leqq 4, \; 0 \leqq x_2 \leqq 4 \\ x_1^2 + x_2^2 + 8x_1 + 2x_2 - 8 \leqq 0 \end{cases}$$

The linear problem associated with P_ε ($\varepsilon > 0$ arbitrarily chosen) has the unique solution $\bar{x} = (0,4)$, so we can apply the algorithm of section 4. Since $Q(\bar{x}) = 16 > 0$, $\bar{x}$ is not optimal for P_ε. We have

$$H = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad H^{-1} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix} \text{; hence}$$

(5.4) becomes:

$$x_1 = 0$$
$$x_2 = 4$$
$$\lambda_1 = \varepsilon\lambda_o + 8$$
$$\lambda_2 = (1+\varepsilon)\lambda_o - 10 \quad .$$

Since $x(\lambda_o) = (0,4)$ is independent of λ_o, we find $J_1' = J_3' = \emptyset$ and $\tilde{\lambda}_o = \lambda_{o2} = 10$; as a consequence the constraint $x_2 \leq 4$ must be deleted.

The new inverse is

$$(H')^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1/2 & 0 \\ -1 & 0 & -2 \end{bmatrix} \quad \text{and (5.4) becomes}$$

$$x_1 = 0$$

$$x_2 = (1/2 + \varepsilon/2)\lambda_o - 1$$

$$\lambda_1 = \varepsilon\lambda_o + 8 \quad .$$

With respect to the non active constraints, we have

$$\tilde{B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \tilde{b} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} ; \quad \hat{h} = \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} ; \quad \hat{u} = \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} ; \quad \hat{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

so that $J'_1 = J'_2 = J'_3 = \emptyset$; according to Remark 5.1, we can set $\varepsilon = 0$ and restore the problem. We have

$$x_1 = 0$$

$$x_2 = 1/2\lambda_o - 1$$

$$\lambda_1 = 8$$

so that $\lambda_{o1} = 2$, $J_2 = \emptyset$ and, with respect to the nonactive constraint we find $J_3 = \emptyset$. The current solution is $\bar{x}^{(1)}=(0,0)$ with $Q(\bar{x}^{(1)}) = -8 < 0$. We calculate the positive root of the equation $Q(x(\lambda_o)) = (1/2\lambda_o-1)^2 + 2(1/2\lambda_o-1) - 8 = 0$, that is $\lambda_o = 3/2$ and we find the optimal solution of P, namely $\bar{x}^{(2)}=(0,2)$.

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