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On generating the set of all efficient points of a bicriteria linear fractional problem

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Introduction

In this paper, we will study the multiobjective mathematical programming problems with two linear fractional objective functions; problems like these are important and have potentially broad applications. In fact, linear fractional objective functions occur frequently in optimization problems involving criteria that are rates or ratios, such as return on investment, dividend coverage, margin on sales, residential density. Furthermore linear fractional functions are widely used as performance measures in many management situations such as production planning and scheduling, educational administration and analysis of financial enterprises.
For the bicriteria linear fractional problems, the characterization of efficiency and some global properties, such as connectedness of set E of all efficient points have been investigated [5, 6, 9, 10] and some algorithms for generating E have been proposed when the feasible region R is a compact set. When R is unbounded, set E is not necessarily connected; one of the main findings in this paper is to show that E is connected for any feasible region when, at least, one of the objective functions is linear and to suggest a sequential method for generating E.
Furthemore the general bicriteria linear fractional problem can be reduced, by means of the Charnes-Cooper trasnsformation, to one having at least one linear objective function; so that the suggested aforesaid algorithm allows us to generate set E even if it is not necessarily connected.

2.Some properties of the set of all efficient points.

The properties of set E of all efficient points of a bicriteria linear fractional problem have been studied by several authors [5, 6, 9, 10] which have pointed out that E is connected when the feasible region R is a compact set.
The compactness of R is a crucial assumption, since it is easy to show by means of simple examples that E may be disconnected when R is unbounded. One of the main findings of this paper is to show that the set of all efficient points is still connected when R is not necessarily bounded for the class of bicriteria linear fractional problems where one of the objective functions is linear. With this aim, let us consider the following problem

$$P \;:\quad \max\ (f_1(x), f_2(x))\ ,\ x \in R = \{\, x \in IR^n : Ax \leq b \,\}$$

where

$$f_1(x) = a^T x \qquad , \qquad f_2(x) = \frac{c^T x + c_0}{d^T x + d_0}$$

and A is a **m** x **n** real matrix and $b \in IR^m$.

A point $x^0 \in R$ is said to be efficient for P if there does not exist a point $x \in R$ such that $f_i(x) \leq f_i(x^0)$ $(i = 1,2)$ where at least one of these inequalities is strict.
As outlined in [5, 10] the set E is related to the set of optimal solution of the scalar problem

$$P_\alpha \qquad \sup f_2(x) \quad , x \in R \quad , f_1(x) \geq \alpha \quad , \alpha \in IR$$

In order to characterize set E, consider the set $H \triangleq \{\alpha : P_\alpha$ has optimal solutions$\}$ and the parametric problem :

$P_{\alpha_0}(\theta)$: $z(\theta) \triangleq \max f_2(x)$, $x \in R(\theta) \triangleq \{x \in R : f_1(x) \geq \alpha_0 - \theta, \theta \geq 0\}$
where

$\alpha_0 = \max H$ if $H \neq \emptyset$.

Let us note that $\theta_1 < \theta_2$ implies $R(\theta_1) \subseteq R(\theta_2)$ so that $z(\theta)$ turns out to be a non - decreasing function[1] .

The following theorem holds :

Theorem 2.1 Assume $H \neq \emptyset$. Then i) and ii) hold.

i) Set $S(\theta)$ of the optimal solutions of $P_{\alpha_0}(\theta)$ is non-empty for any $\theta \geq 0$.

ii) $z(\theta)$ is an increasing function in $[0, +\infty[$ or there exists $\tilde{\theta} \geq 0$ such that $z(\theta)$ is increasing in $[0, \tilde{\theta}]$ and $z(\theta) = z(\tilde{\theta})$ $\forall$ $\theta \geq \tilde{\theta}$.

Proof.

i) The thesis is obvious if R is a compact set . Let x^0 be an optimal solution to the problem $P_{\alpha_0}(0)$ and suppose, ab absurdo, that there exists $\bar{\theta} > 0$ such that $S(\bar{\theta}) = \emptyset$.

Since $P_{\alpha_0}(\theta)$ is a linear fractional problem, there exists a halfline, whose equation is of the form $x = \bar{x} + t u$, $t \geq 0$, contained in $R(\bar{\theta})$, such that $z(\bar{\theta}) = \lim_{t \to +\infty} f_2(\bar{x} + t u)$.

Since $z(\theta)$ is a non-decreasing function and $S(\bar{\theta}) = \emptyset$, necessarily we have $z(\bar{\theta}) > z(0)$. It is easy to verify[2] that the halfline $x = x^0 + t u$, $t \geq 0$, is contained in $R(0)$ so that

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[1] The function $f : IR^n \rightarrow IR$ is said non-decreasing (increasing) iff $\forall x_1, x_2$ $x_1 > x_2$ $f(x_1) \geq f(x_2)$. ($f(x_1) > f(x_2)$)

[2] Let us note that $\{ x : \bar{x} + t u, t \geq 0 \} \cap R(\bar{\theta})$ implies $Au \leq 0$ and $f_1(u) \geq 0$, otherwise the inequalities $A\bar{x} + t Au \leq b$, $f_1(\bar{x}) + t f_1(u) \geq \alpha_0 - \bar{\theta}$ are not verified for $t \rightarrow +\infty$. As a consequence $Ax^0 + t Au \leq b$ and $f(x^0) + t f_1(u) \geq \alpha_0$; $\forall t \geq 0$, so that $x = x^0 + t u \in R(0)$, $\forall$ $t \geq 0$.

$$z(\bar{\theta}) = \lim_{t\to+\infty} f_2(\bar{x} + t u) = \lim_{t\to+\infty} f_2 (x^0 + t u) > z(0)$$

and this contradicts the optimality of x^0.

ii) If $z(\theta)$ is non-increasing in $[0, +\infty[$, set $\tilde{\theta} \triangleq \max \{ \theta: z(\theta) \text{ is increasing in } [0, \theta] \}$. Suppose, ab absurdo, that there exist θ_1 and θ_2 such that $\tilde{\theta} < \theta_1 < \theta_2$ with $z(\tilde{\theta}) = z(\theta_1) < z(\theta_2)$. Let $\tilde{x}$ and x_2 be optimal solutions for $P_{\alpha_0}(\tilde{\theta})$ and $P_{\alpha_0}(\theta_2)$, respectively. Since $f_2(\tilde{x}) < f_2(x_2)$, the restriction of f_2 on the segment $[\tilde{x}, x_2] \subset R(\theta_2)$ is an increasing linear fractional function, so that $f_2(x) > f_2(\tilde{x})\ \forall\ x \in [\tilde{x}, x_2] \cap R(\theta_1)$ which contradicts the equality $z(\tilde{\theta}) = z(\theta_1)$.
This completes the proof.

The following theorem points out the relationship between the optimal solutions of the parametric scalar problem $P_{\alpha_0}(\theta)$ and set E of all efficient points of the bicriteria problem P.

Theorem 2.2 If $H = \emptyset$ then $E = \emptyset$, otherwise

$$E = \bigcup_{\theta \in [0,\tilde{\theta}]} S(\theta).$$

Proof. If $\bar{x}$ is an efficient point for problem P, then it is also an optimal solution for the problem $P_{\alpha_0}(\bar{\theta})$, $\bar{\theta} = \alpha_0 - f_1(\bar{x})$, so that $E \subseteq \bigcup_{\theta \in [0,\tilde{\theta}]} S(\theta)$.

Now we must show that $E \supseteq \bigcup_{\theta \in [0,\tilde{\theta}]} S(\theta)$.

Let $\bar{x}$ be an optimal solution for $P_{\alpha_0}(\bar{\theta})$, $\bar{\theta} \in [0,\tilde{\theta}]$. It is obvious that $\bar{x}$ is an efficient point for P if a) and b) hold:

a) $f_1(x) \geq f_1(\bar{x})$ implies $f_2(x) \leq f_2(\bar{x})$, $\forall\ x \in R$;

b) $f_1(x) > f_1(\bar{x})$ implies $f_2(x) < f_2(\bar{x})$, $\forall\ x \in R$.

If $f_1(x) \geq f_1(\bar{x})$, necessarily we have $f_2(x) \leq f_2(\bar{x})$ since $x \in R(\bar{\theta})$ and a) is satisfied. Suppose now that $f_1(x) > f_1(\bar{x})$ and set $\theta = \alpha_0 - f_1(x)$; we have $\bar{\theta} = \alpha_0 - f_1(\bar{x}) > \alpha_0 - f_1(x) = \theta$ so that for ii) of theorem 2.1, $f_2(x) \leq \sup_{x \in R(\theta)} f_2(x) < z(\bar{\theta}) = f_2(\bar{x})$ and b) holds.
This completes the proof.

The previous results point out that we can generate all efficient points of P if we are able to answer the following questions:

I) how to find α_0;

II) how to find $S(\theta)$ for any $\theta \in [0,\tilde{\theta}]$.
In the next section we will propose a sequential method which allows us to solve problem II), and, at the same time, to establish that E is connected. In section 4 we will solve problem I).

Let us note that since $P_{\alpha_0}(\theta)$ is a linear fractional problem, it is easy to find, in two-dimensional space, the set of optimal solutions $S(\theta)$, $\theta \geq 0$ from a geometrical point of view. As a consequence we are able to describe the set of all efficient points taking into account that $E = \bigcup_{\theta \in [0,\tilde{\theta}]} S(\theta)$ This will be done in the following example

Example 2.1 . Consider the following bicriteria linear fractional problem

P : $\max (f_1(x), f_2(x))$, $x \in R = \{x = (x_1,x_2) : x_1 - 2x_2 \leq 4, x_1,x_2 \geq 0\}$

where $f_1(x) = -x_1 + x_2$, $f_2(x) = (x_1 - 2)/(x_2 + 1)$

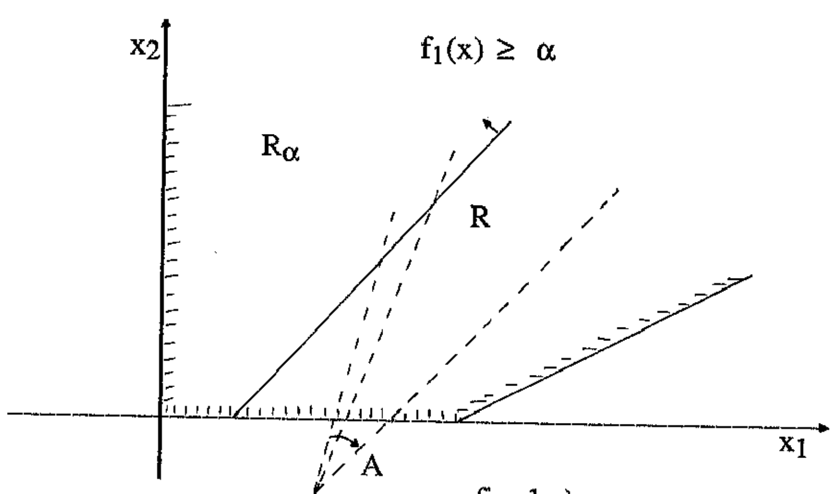

fig. 1 a)

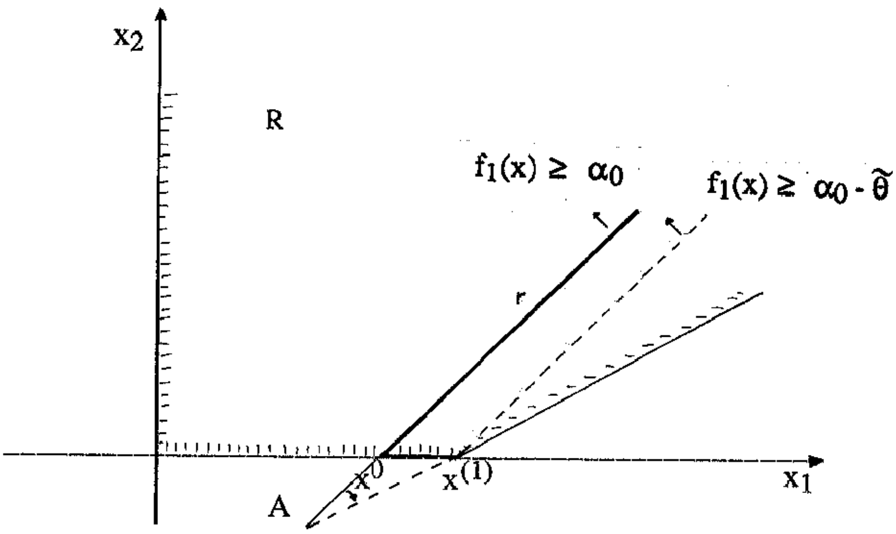

fig. 1 b)

Figure 1 a) points out that problem P_α has not optimal solutions for $\alpha > \alpha_0$ where $\alpha_0 = f_1(A) = f_2(A)$; figure 1 b) shows the level α_0, the value $\tilde{\theta}$ and the set of all efficient points $E = r \cup [\, x^0, x^{(1)}]$.

3 - A sequential method.

As we will point out in section 2, the set of all efficient points of problem P is related to the optimal solutions of the parametric problem $P_{\alpha_0}(\theta)$, $\theta \geq 0$. For this reason, we will give a sequential method for solving $P_{\alpha_0}(\theta)$, $\theta \in [\,0,\tilde{\theta}\,]$.

Suppose that α_0 is known. Let us note that $P_{\alpha_0}(\theta)$ is a linear fractional problem whose feasible region is not necessarily bounded, so that the modified version of Martos's algorithm[1] [3] can be used to solve it. The parametric problem $P_{\alpha_0}(\theta)$ will be solved by means of a suitable post-optimality analysis performed either by a dual-simplex like procedure or by the modified version od Martos's algorithm.

With this aim we will rewrite problem $P_{\alpha_0}(\theta)$ in the following standard form

1 See Appendix A.

$$P_{\alpha_0}(\theta) : \begin{cases} \sup\ f_2(x) = \dfrac{c^T x + c_0}{d^T x + d_0} \\ A\,x = b \\ a^T x = \alpha_0 - \theta \quad , \qquad x \geq 0 \end{cases}$$

Let $\hat{x} = (\hat{x}_B = \hat{b}\ , \hat{x}_N = 0)$ be a feasible basic solution for $P_{\alpha_0}(\hat{\theta})$, with corresponding basis $\hat{B}$. We partition the vectors c and d as $c = (c_B, c_N)$, $d = (d_B, d_N)$ and the matrix $\hat{A} = \begin{bmatrix} A \\ -- \\ a \end{bmatrix}$ as $\hat{A} = [\ \hat{B} : \hat{N}\ ]$.

Set

$$\bar{c}_N^T \triangleq c_N^T - c_B^T \hat{B}^{-1} \hat{N} \quad , \qquad \bar{d}_N^T \triangleq d_N^T - d_B^T \hat{B}^{-1} \hat{N} \quad , \qquad \bar{c}_0 \triangleq c^T \hat{x} + c_0$$

$$\bar{d}_0 \triangleq d^T \hat{x} + d_0 \quad , \qquad \gamma \triangleq \bar{d}_0\, \bar{c}_N - \bar{c}_0\, \bar{d}_N$$

and introduce the following notations:

$$\gamma(\theta) = \gamma - \theta\, w \quad , \qquad w = \lambda_0\, \bar{d}_N - \mu_0\, \bar{c}_N$$

where λ_0 and μ_0 are the last components of the vectors $c_B^T \hat{B}^{-1}$ and $d_B^T \hat{B}^{-1}$ respectively ;

$$\hat{x}_B(\theta) = \hat{x}_B - \theta\ h$$

where h is the last column of the matrix $\hat{B}^{-1}$.

The parametric analysis is performed by studying the optimality condition $\gamma(\theta) \leq 0$ and the feasibility condition $\hat{x}_B(\theta) \geq 0$. With regard to the optimality condition, let us set $I_1 = \{ i : w_i < 0 \}$; if $I_1 = \emptyset$, then $\gamma(\theta) \leq 0$, for any $\theta \geq 0$, otherwise $\gamma(\theta) \leq 0\ \ \forall\ \theta \in [0, \theta']$ where:

$$\text{(3.1)} \qquad \theta' = \min_{i \in I_1} \frac{\gamma_i}{w_i} \triangleq \frac{\gamma_k}{w_k}$$

With regard to the feasibilty condition, let us set $I_2=\{ i : h_i > 0 \}$; if $I_2=\emptyset$, then $\hat{x}_B(\theta) \geq 0$ for any $\theta \geq 0$, otherwise $\hat{x}_B(\theta) \geq 0 \ \forall \ \theta \in [0, \theta'']$, where

$$(3.2) \qquad \theta'' = \min_{i \in I_2} \frac{\hat{x}_{B_i}}{h_i} \triangleq \frac{\hat{x}_{B_j}}{h_j}$$

As a consequence for any $\theta \in [0, \bar{\theta}]$, where $\bar{\theta} \triangleq \min(\theta', \theta'')$; $\hat{x}_B(\theta)$ is the optimal solution of the problem $P_{\alpha_0}(\theta)$; when $\theta > \bar{\theta}$ and $\bar{\theta} = \theta''$, we can restore the feasibility by means of dual-simplex like algorithm; when $\theta > \bar{\theta}$ and $\bar{\theta} = \theta'$, we can restore the optimality by means of the modified version of Martos's algorithm.
Let us note that in this kind of case a pivot operation can always be performed for i) of theorem 2.1.

Now we will describe a sequential method for solving problem $P_{\alpha_0}(\theta)$, $\theta \in [0, \tilde{\theta}]$ starting from an optimal basic solution to P_{α_0}.

STEP 0 (not iterative) Solve $P_{\alpha_0}(\theta^0 = 0)$ and let $\hat{x}_B^{(0)}$ an optimal basic solution; set i=0 and go to step 1.

STEP 1 Consider $P_{\alpha_0}(\theta^i + \theta)$, $\theta \geq 0$; calculate $\gamma(\theta)$, $\hat{x}_B^{(i)}(\theta)$, $\bar{\theta}$ and set $\theta^{i+1} = \theta^i + \bar{\theta}$; $\hat{x}_B^{(i+1)} = \hat{x}_B^{(i)}(\bar{\theta})$ is an optimal basic solution for $P_{\alpha_0}(\theta^{i+1})$.
If $z(\theta^{i+1}) = z(\theta^i)$, then $\tilde{\theta}=\theta'$ STOP; otherwise go to step 2.

STEP 2 If $\bar{\theta}= \theta' < +\infty$, then x_{N_k} enters the basis by means of a simplex-like pivot operation; set i=i+1 and return to step 1.
If $\bar{\theta}=\theta'' < +\infty$, then x_{B_j} must leave the basis and a pivot operation is performed on a_{tj} such that $\gamma_t(\bar{\theta})/ a_{tj} \triangleq \min_{\substack{a_{ij}<0 \\ i \in I_2}} \gamma_i(\bar{\theta})/ a_{ij}$; set i = i+1 and return to step 1.
Otherwise $\hat{x}_B^{(i+1)}(\theta)$ is optimal for $P_{\alpha_0}(\theta^{i+1} + \theta)$, $\tilde{\theta} = +\infty$; STOP.

Remark 3.1 Let us note that if a value $\bar{\alpha} > \alpha_0$ is known such that problem $P_{\bar{\alpha}}$ has optimal solutions, we must solve problem $P_{\bar{\alpha}}(\theta)$ $\forall \ \theta \in$ IR, in order to find the set of optimal solutions $S(\theta)$ $\forall \ \theta \in [0, \tilde{\theta}]$; this can be performed by means of the previous algorithm applied to the parametric problems $P_{\bar{\alpha}}(\theta)$ $\theta \geq 0$ and $P_{\bar{\alpha}}(\theta)$ $\theta \leq 0$.

4 - On generating the set of all efficient points of problem P.

We have just outlined that set $E = \bigcup_{\theta \in [0,\tilde{\theta}]} S(\theta)$ of all efficient points of problem P can be obtained, if $E \neq \emptyset$, by starting from the optimal solutions to problem P_{α_0}.

The algorithm, given in section 3, generates a finite sequence of parameters $0 = \theta^0 < \theta^1 < \dots < \theta^s = \tilde{\theta}$ and, as a consequence, a finite number of connected line segments whose end-points $x^{(0)}, x^{(1)}, \dots, x^{(s)}$ are optimal basic solutions to $P_{\alpha_0}(\theta^i)$ i=0,...., s and also efficient points for P. Taking into account that $\gamma(\theta)$ and $x_B(\theta)$ are linear functions and $S(\theta)$ is a convex set, the following theorem is established:

Theorem 4.1 . The set of all efficent points of problem P is connected.

Theorem 2.2 together with the sequential method, suggested in section 3, points out that, in order to describe set E, we must solve the following problem : how to find α_0 or to establish that $E = \emptyset$.

In order to answer the question consider the linear problem:

$$P_L : \max\ f_1(x)\ ,\ x \in R.$$

Let us note that if P_L has a unique solution x^0 or set S_L of the optimal solutions of P_L is compact, then we obviously have $\alpha_0 = f_1(x^0)$. In the general case let x^0 be an optimal basic solution to P_L , if one exists, or a vertex which is the end-point of a ray $r \subset R$ such that

$$\sup_{x \in r} f_1(x) = +\infty.$$

Consider the problem

$$\sup f_2(x) \stackrel{\Delta}{=} L \quad , \quad x \in R\ , \qquad f_1(x) \geq f_1(x^0)$$

The following cases arise:

a) L is reached as a maximum; in this case we can find α_0 by applying the algorithm to the problem $P_{\bar{\alpha}}(\theta)$, $\theta \leq 0$, $\bar{\alpha} = f_1(x^0)$ as outlined in remark 3.1.

b) $L < +\infty$ is not reached as a maximum; in this case in solving the linear fractional problem P_α, $\alpha = f_1(x^0)$, by means of the modified version of Martos's algorithm, we find a basic solution and an index t such that $\gamma_t > 0$ and the corresponding column has nonpositive coefficients.

Consider the parametric problem $P_{\bar{\alpha}}(\theta)$, $\theta \geq 0$, and calculate θ'', setting $\theta'' = +\infty$ if $I_2 = \emptyset$.

Let $\theta' > 0$ be the positive root of the linear equation $\gamma_t(\theta)$ and set $\theta' = +\infty$, if such a positive root does not exist. If $\theta' = \theta'' = +\infty$, then $H = \emptyset$ so that $E = \emptyset$. If $\theta'' < \theta'$, set $\theta = \theta''$, consider problem $P_{\bar{\alpha}}$, where $\bar{\alpha} = \alpha - \theta''$, and repeat the above considerations. If $\theta' \leq \theta''$, set $\theta = \theta'$ and consider problem $P_{\bar{\alpha}}$, $\bar{\alpha} = \alpha - \theta'$; if $P_{\bar{\alpha}}$ has optimal solutions, then $E \neq \emptyset$ and $\alpha_0 = \bar{\alpha}$, otherwise repeat the above considerations. After a finite number of such iterations we are able to find α_0 or to establish that $E = \emptyset$.

c) $L = +\infty$; in this kind of case $H = \emptyset$ so that $E = \emptyset$. We get this result after taking into account that $L = +\infty$ implies the existence of halfline $x = x^0 + t u$, $t \geq 0$, such that

$$L = \lim_{t \to +\infty} f_2(x^0 + t u).$$

Since any halfline of equation $x = \hat{x} + t u$, $t \geq 0$, is contained in the feasible region $R(\alpha)$ of problem P_α , if $\hat{x} \in R(\alpha)$ (see footnote 2), we have $\sup_{x \in R(\alpha)} f_2(x) = +\infty$, $\forall \alpha$.

5. Numerical Examples.

We will now propose some numerical examples with the aim of clarifying some theoretical aspects which have been pointed out in the previous sections.

Example 5.1 Consider problem P where

$$f_1(x) = -2x_1 + x_2 \, , \; f_2(x) = \frac{x_1}{x_1+x_2+4}$$

and

$$R = \{ \, x = (x_1,x_2) \in \mathbb{R}^2 : -x_1 + x_2 \leq 2, \; - x_1 + 2x_2 \leq 8, \; x_1, x_2 \geq 0 \, \}$$

First of all we will solve the linear problem P_L : max $f_1(x)$, $x \in R$, which has a unique optimal solution $x^0 = (0,2)$, so that x^0 turns out to be the unique solution to the linear fractional problem P_{α_0} where $\alpha_0 = f_1(x^0) = 2$.

Consider now the parametric problem

$$P_{\alpha_0}(\theta) : \begin{cases} \max f_2(x) = x_1 / (x_1 + x_2 + 4) \\ (x_1, x_2) \in R \\ -2x_1 + x_2 \geq 2 - \theta ,\ \theta \geq 0 \end{cases}$$

We solve the linear fractional problem $P_{\alpha_0}(0)$, by means of the modified version of Martos's algorithm; the optimal simplex-tableau is the following

$-\bar{c}_0$	0	0	0	-1	0	-1
$-\bar{d}_0$	-6	0	0	-3	0	-2
x_2	2	0	1	2	0	1
x_4	4	0	0	-3	1	-1
x_1	0	1	0	1	0	1

where the first and the second rows represent the reduced costs of the numerator and of the denominator of $f_2(x)$ respectively ; the optimal solution is $x^0 = (0,2)$.

Sensitivity analysis applied to $P_{\alpha_0}(0)$ gives the following simplex-tableau:

	$-\theta$	0	0	-1	0	-1
	$-6-2\theta$	0	0	-3	0	-2
x_2	$2+\theta$	0	1	2	0	1
x_4	$4-\theta$	0	0	-3	1	-1
x_1	θ	1	0	1	0	1

tableau 1

Since $\gamma(\theta) = (6 + 2\theta)(-1, -1) - \theta(-3, -2) = (-6 + \theta, -6)$ and the optimality condition requires $\gamma(\theta) \leq 0$, we have $\theta' = 6$; on the other hand the feasibility condition $x_B(\theta) \geq 0$ implies $\theta'' = 4$. Since $\bar{\theta} = \theta'' = 4$, $x(\theta) = (\theta, 2 + \theta)$ is optimal for $P_{\alpha_0}(\theta)$ $\theta \in [0,4]$, we have $\theta^1 = \theta^0 + 4 = 4$ and $x^{(1)} = (4,6)$.

It results that $z(\theta^0) = 0 < z(\theta^1) = 2/7$, so that we set $\theta = 4 + \theta$ in the tableau 1 and we restore the feasibility by performing a pivot operation on the circled number - 3.We have :

	$-4-\theta$	0	0	-1	0	-1
	$-14-2\theta$	0	0	-3	0	-2
x_2	$6+\theta$	0	1	2	0	1
x_4	$-\theta$	0	0	(-3)	1	-1
x_1	$4+\theta$	1	0	1	0	1

	$-4-(2/3)\theta$	0	0	0	-1/3	-2/3
	$-14-\theta$	0	0	0	-1	-1
x_2	$6+(1/3)\theta$	0	1	0	(2/3)	1/3
x_3	$(1/3)\theta$	0	0	1	-1/3	1/3
x_1	$4+(2/3)\theta$	1	0	0	1/3	2/3

tableau 2

Since $\gamma(\theta) = (14 + \theta)(-1/3, -2/3) - (4 + 2/3\theta)(-1, -1) = (-2/3 + (1/3)\theta, -16/3)$, we have $\theta' = 2$, while $\theta'' = +\infty$. As a consequence $x(\theta) = (4 + (2/3)\theta, 6 + (1/3)\theta)$ is optimal for $\theta \in [0, 2]$; we have $\theta^2 = \theta^1 + 2 = 6$, $x^{(2)} = (16/3, 20/3)$ and $z(\theta^1) = 2/7 < z(\theta^2) = 1/3$, so that we set $\theta = \theta + 2$ in tableau 2 and we restore the optimality performing a pivot operation on the circled number 2/3. We obtain

	$-2-(1/2)\theta$	0	1/2	0	0	-1/2
	$-6+(1/2)\theta$	0	3/2	0	0	1/2
x_4	$10+(1/2)\theta$	0	3/2	0	1	1/2
x_3	$4+(1/2)\theta$	0	1/2	1	0	1/2
x_1	$2+(1/2)\theta$	1	-1/2	0	0	1/2

tableau 3

Since $\gamma(\theta) = (6 - (1/2)\theta)(1/2, -1/2) - (2 + (1/2)\theta)(3/2, 1/2) = (-\theta, -4)$, we have $\theta' = +\infty$; on the other hand $\theta'' = +\infty$, so that $x(\theta) = (2 + (1/2)\theta, 0)$ is optimal for $\theta \geq 0$; we have $\tilde{\theta} = +\infty$. $\bar{x}^{(2)} = (2, 0)$ and the algorithm stops. The set of all efficient points is

$$E = [x^0, x^{(1)}] \cup [x^{(1)}, x^{(2)}] \cup [x^{(2)}, \bar{x}^{(2)}] \cup r,$$

where r is the halfline whose equation is $x = (2,0) + \theta\,(1/2,0)$, $\theta \geq 0$.

In figure 2, the feasible region and set E are drawn.

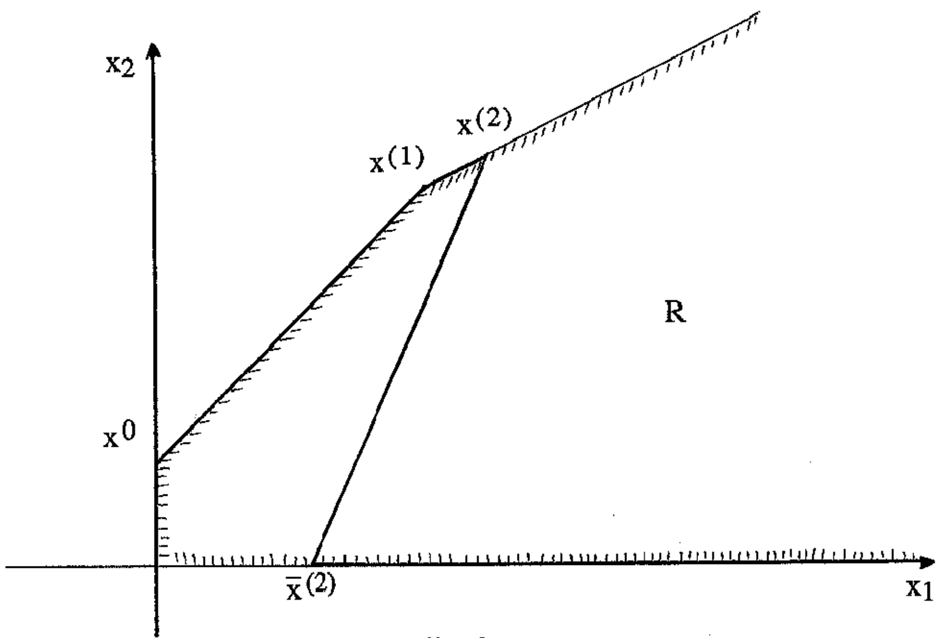

fig.2

Example 5.2 The following cases a) and b) show that when the set of optimal solutions to the linear problem P_L is unbounded, then E may be empty or not.

Case a) Consider problem P where now $f_1(x) = -x_1+x_2$, $f_2(x) = x_1/(x_1+x_2+4)$ and

$$R = \{\, x = (x_1, x_2) \in \mathbb{R}^2 : -x_1+x_2 \leq 2,\; x_1,x_2 \geq 0\}.$$

The set of optimal solutions to the linear problem P_L is the halfline $x = (0,2) + (1,1)\,t$, $t \geq 0$ so that we consider the linear fractional problem

$$(5.1)\quad \begin{cases} \sup \dfrac{x_1}{x_1 + x_2 + 4} \\ (x_1, x_2) \in R \\ -x_1 + x_2 \geq f_1(x^0) = 2 \end{cases}$$

The final tableau obtained by applying the modified version of Martos's algorithm is the following

$-\bar{c}_0$	0	1	0	0	0
$-\bar{d}_0$	-6	2	0	0	1
x_2	2	-1	1	0	-1
x_3	0	0	0	1	1

so that we can conclude that problem 5.1 does not have optimal solutions and

$$\sup f_2(x) = L = 1/2 < +\infty.$$

Set $\bar{\alpha} = f_1(x^0) = 2$ and consider the parametric problem

$$P_{\bar{\alpha}}(\theta)\quad \begin{cases} \sup \dfrac{x_1}{x_1 + x_2 + 4} \\ (x_1, x_2) \in R \\ -x_1 + x_2 \geq \bar{\alpha} - \theta \ , \quad \theta \geq 0 \end{cases}$$

starting from the following simplex-tableau

	0	1	0	0	0
	$-6+\theta$	2	0	0	1
x_2	$2-\theta$	(-1)	1	0	-1
x_3	θ	0	0	1	1

tableau 1

The feasibility condition implies $\theta'' = 2$, and the positive root of $\gamma_1(\theta) = 6 - \theta$ is $\theta' = 6$. Since $\theta'' < \theta'$, let us note that $P_{\bar{\alpha}}(\theta)$ does not have optimal solutions $\forall\ \theta \in [0, 2]$; set $\theta = 2 + \theta$ in tableau 1 and perform a pivot operation on the circled number. We obtain the tableau associated with parametric problem $P_{\bar{\alpha}}(\theta)$, where now $\bar{\alpha} = 2 - \theta'' = 0$.

	$-\theta$	0	1	0	-1
	$-4-\theta$	0	2	0	-1
x_1	θ	1	-1	0	1
x_3	$2+\theta$	0	0	1	1

tableau 2

The feasibility condition implies $\theta'' = +\infty$ and the positive root of the linear function $\gamma_2(\theta) = 4 - \theta$ is $\theta' = 4$. Let us note that $P_{\bar{\alpha}}(\theta)$ does not have optimal solution $\forall\ \theta \in [0, 4[$; set $\theta = \theta' = 4$ and consider the problem $P_{\bar{\alpha}}$; $\bar{\alpha} = 0 - \theta' = -4$.
The previous tableau becomes optimal for $\theta = 4$, so that $\alpha_0 = \bar{\alpha} = -4$ and the set of optimal solutions to P_{α_0} is the halfline r_1 whose equation is $x = (4,0) + (1,1)\,t$, $t \geq 0$.
According to theorem 2.1, set E of all efficient points can be obtained by solving problem $P_{\alpha_0}(\theta)$, $\theta \in [0, \tilde{\theta}]$. By setting $\theta = \theta + 4$ in the previous tableau we obtain:

	$-4-\theta$	0	1	0	-1
	$-8-\theta$	0	2	0	-1
x_1	$4+\theta$	1	-1	0	1
x_3	$6+\theta$	0	0	1	1

The feasibility condition implies $\theta'' = +\infty$, the optimality condition $\gamma(\theta) \leq 0$ implies $\theta' = +\infty$. So that $\tilde{\theta} = +\infty$ and $x(\theta) = (4 + \theta, 0)$ $\theta \geq 0$ is optimal $\forall\ \theta \geq 0$. The set of all efficient points is $E = r_1 \cup r_2$ where r_2 is the halfline whose equation is

$$x = (4,0) + \theta\ (1,0) \quad \theta \geq 0.$$

In figure 3, the feasible region and set E are drawn.

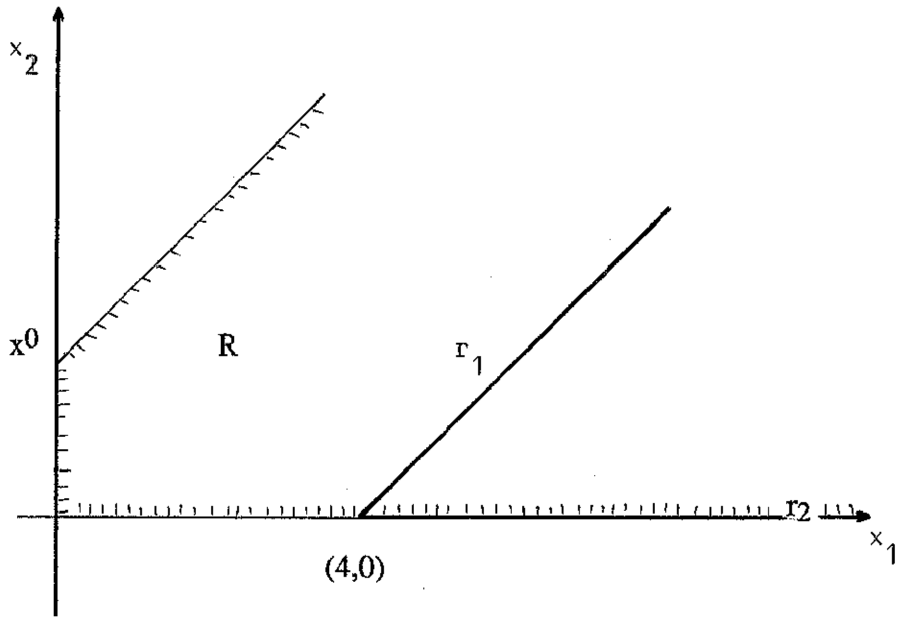

fig.3

Case b) Consider problem P where now

$$f_1(x) = x_1 - x_2 \quad , \qquad f_2(x) = \frac{x_1}{x_1 + x_2 + 4} \qquad \text{and}$$

$$R = \{ x = (x_1, x_2) \in \mathbb{R}^2 : -x_1 + x_2 \geq 2 ,\ x_1, x_2 \geq 0 \} .$$

The set of optimal solutions to the linear problem P_L is the halfline $x = (0,2) + (1,1)\, t$, $t \geq 0$, so that we consider the linear fractional problem

$$(5.2) \quad \begin{cases} \sup \dfrac{x_1}{x_1 + x_2 + 4} \\ (x_1, x_2) \in R \\ x_1 - x_2 \geq - f_1(x^0) = -2 \end{cases}$$

The final tableau obtained by applying the modified version of Martos's algorithm is the following:

	0	1	0	0	0
	-6	2	0	0	-1
x_2	2	-1	1	0	1
x_3	0	0	0	1	1

so that we can conclude that problem 5.2 does not have optimal solutions and $\sup f_2(x)=1/2<+\infty$. Set $\bar{\alpha}=f_1(x^0)=-2$ and consider the parametric problem

$$P_{\bar{\alpha}}(\theta)\quad\begin{cases}\sup \dfrac{x_1}{x_1+x_2+4}\\ (x_1,x_2)\in R\\ x_1-x_2\geq -2-\theta\ ,\quad \theta\geq 0\end{cases}$$

starting from the following tableau

	0	1	0	0	0
	$-6-\theta$	2	0	0	−1
x_2	$2+\theta$	−1	1	0	1
x_3	θ	0	0	1	1

The feasibility condition implies $\theta''=+\infty$; since the linear function $\gamma_1(\theta)=6+\theta$ does not have a positive root, we have $\theta'=+\infty$, so that $H=\emptyset$ and consequently $E=\emptyset$.

Example 5.3 Now we will apply the algorithm to the problem solved geometrically in example 2.1. Let us note that the linear problem P_L does not have optimal solutions; $x^0=(0, 0)$ is the end-point of the halfline r, whose parametric equation is $x_1=0$, $x_2\geq 0$, such that $\sup_{x\in r} f_1(x)=+\infty$.

Consider the problem

$$P_{\bar{\alpha}} \quad \begin{cases} \sup \dfrac{x_1 - 2}{x_2 + 1} \\ (x_1, x_2) \in R \\ -x_1 + x_2 \geq \bar{\alpha} = f_1(x^0) = 0 \end{cases}$$

and solve it by means of the modified version of Martos's algorithm ; the final tableau is the following :

	2	0	1	0	-1
	-1	0	1	0	0
x_3	4	0	-1	1	-1
x_1	0	1	-1	0	1

We have $\gamma = (3, -1)$; since $\gamma_1 > 0$ and any coefficient of the corresponding column is negative,then problem $P_{\bar{\alpha}}$ does not have an optimal solution.
Consider the parametric problem

$$P_{\bar{\alpha}}(\theta) : \begin{cases} \sup f_2(x) \\ x \in R \\ -x_1 + x_2 \geq -\theta \qquad \theta \geq 0 \end{cases}$$

starting from the following tableau

	2-θ	0	1	0	-1
	-1	0	1	0	0
x_3	4-θ	0	−1	1	−1
x_1	θ	1	-1	0	1

The feasibility condition implies $\theta'' = 4$ and the positive root of $\gamma_1(\theta) = 3 - \theta$ is $\theta' = 3$. Problem $P_{\bar{\alpha}}(\theta)$ does not have optimal solution $\forall\ \theta \in [\ 0,3\ [$; set $\theta = \theta' = 3$ and consider problem $P_{\bar{\alpha}}$, $\bar{\alpha} = 0 - \theta' = -3$.The previous tableau becomes optimal for $\theta = 3$, so that $\alpha_0 = \bar{\alpha} = -3$, $\theta^0 = 3$ and the set of optimal solution to P_{α_0} is the halfline r_1 whose equation is $x=(3,0) + (1,1)\,t,\ t \geq 0$.

Setting $\theta = \theta + 3$ in the previous tableau, we obtain

	$-1-\theta$	0	1	0	-1
	-1	0	1	0	0
x_3	$1-\theta$	0	(−1)	1	−1
x_1	$3+\theta$	1	-1	0	1

Since $\gamma_1(\theta) = -\theta$, we have $\theta' = +\infty$, while $\theta'' = 1$. We have $\bar{\theta} = \theta'' = 1$, so that $x(\theta) = (3+\theta, 0)$ is optimal for $P_{\alpha_0}(\theta)$, $\forall\, \theta \in [0, 1]$. Set $\theta^1 = \theta^0 + 1 = 4$, we have $x^{(1)} = (4, 6)$, $z(\theta^0) = 1 < z(\theta^1) = 2$ so that we set $\theta = \theta + 1$ in the previous table and we restore the feasibility by performing a pivot operation on the circled number. We obtain

	$-2-2\theta$	0	0	1	-2
	$-1-\theta$	0	0	1	−1
x_2	θ	0	1	−1	1
x_1	$4+2\theta$	1	0	-1	2

Since $\gamma(\theta) = (-1-\theta, 0)$, we have $\theta' = +\infty$ and $\theta'' = +\infty$; consequently $x(\theta) = (4+2\theta, \theta)$ is optimal for $P_{\alpha_0}(\theta)$, $\forall\, \theta \geq 0$, but it is not an efficient solution for P, since

$$z(\theta) = \frac{2+2\theta}{1+\theta} = 2 = z(\theta^1) \quad \forall\, \theta \geq 0$$

Thus $\tilde{\theta} = \theta^1 = 4$ and set E is given by $E = r_1 \cup [(3,0), (4,0)]$.

Example 5.4. Consider the following problem

$$P : \max\left(x_1 - x_2, \frac{x_1+3}{x_2+1}\right), \; x_1, x_2 \geq 0$$

Let us note that the linear problem P_L does not have optimal solutions and $x^0 = (0, 0)$ is the end-point of the halfline r, whose parametric equation is $x_1 = 0$, $x_2 \geq 0$, such that

$$\sup_{x \in r} f_1(x) = +\infty .$$

Consider the problem

$$P_{\bar{\alpha}} \quad \begin{cases} \sup f_2(x) = \dfrac{x_1 + 3}{x_2 + 1} \\ -x_1 + x_2 \geq \bar{\alpha} = f_1(x^0) = 0 \\ x_1, x_2 \geq 0 \end{cases}$$

and solve it by means of the modified version of Martos's algorithm; the final optimal tableau is the following:

	-3	1	0	0
	-1	1	0	1
x2	0	-1	1	-1

Since problem $P_{\bar{\alpha}}$ has optimal solutions, we must consider the parametric problems $P_{\bar{\alpha}}(\theta)$, $\theta \geq 0$, and $P_{\bar{\alpha}}(\theta)$, $\theta \leq 0$, in order to find $\tilde{\theta}$ and α_0 respectively (see remark 3.1) starting from the following tableau:

	-3	1	0	0
	-1+θ	1	0	1
x2	-θ	-1	1	-1

a) **Case** $\theta \geq 0$

$\gamma(\theta) = (-2-\theta, -3)$, so that $\theta' = +\infty$; since the feasibility condition implies $\theta'' = 0$, we will perform a pivot operation and obtain the following tableau

	-3-θ	0	1	-1
	-1	0	1	0
x2	θ	1	-1	1

The feasibility and the optimality conditions imply $\theta' = \theta'' = +\infty$ so that the halfline r_1 whose equation is $x(\theta) = (\theta, 0)$, $\theta \geq 0$, is optimal and such that $r_1 \subset E$.

b) **Case** $\theta \leq 0$

$\gamma(\theta) = (-2-\theta, -3)$, so that $\gamma(\theta) \leq 0$ for $\theta \leq -2$;taking into account that $x_2(\theta) = -\theta \geq 0$ $\forall\ \theta \leq 0$, $x(\theta) = (0, -\theta)$ is an optimal solution for $P_{\bar{\alpha}}(\theta)$ and also an efficient point for P for any $\theta \in [-2, 0]$. Set $\theta = \theta - 2$ in the tableau, we obtain

	-3	1	0	0
	-3+θ	1	0	1
x_2	2-θ	-1	1	-1

We have $\gamma(\theta) = (-\theta, -3)$ and $x_2(\theta) = 2-\theta$ .Since $\gamma_1(\theta) > 0$ for $\theta < 0$, and $x_2(\theta) > 0$ for $\theta < 0$ problem $P_{\bar{\alpha}}(\theta)$, $\bar{\alpha} = 2$, does not have optimal solutions so that $\alpha_0 = 2$. For $\theta = 0$, we have alternate optimal solutions which are also efficient points for P.

Set E is drawn is figure 4.

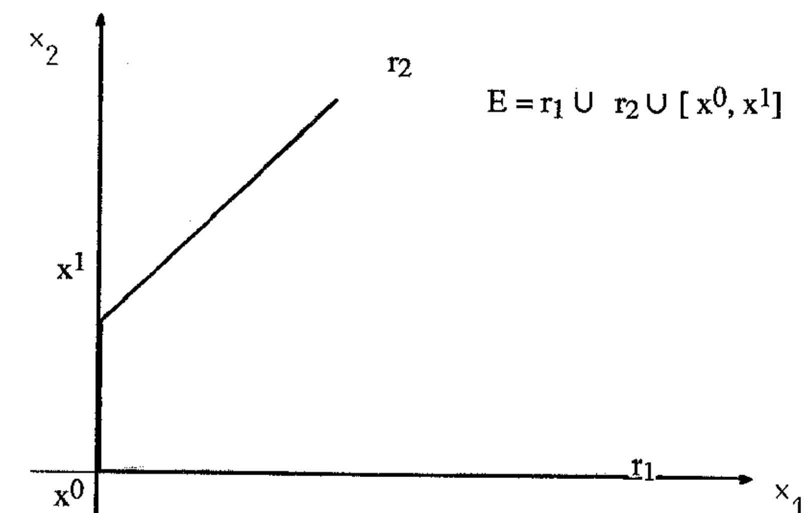

fig.4

Remark 5.1 The above examples have outlined the main features of the sequential method suggested in sections 3, 4; at the same time the examples show that set E of all efficient points may be non-empty even if one or both objective functions are not upperbounded on the feasible region.

6. The Bicriteria linear fractional problem.

In this section we will consider problem P where now f_1, f_2 are both linear fractional functions. We have just said that, in this case, set E of all efficient points is not necessarily connected when the feasible region is unbounded;nevertheless the sequential method suggested in the previous sections can be used in order to find E since, by means of the Charnes-Cooper transformation, problem P reduces to a problem where one of the objective functions is linear.
Suppose that

$$f_1(x) = \frac{a^T x + a_0}{b^T x + b_0}$$

and consider the Charnes-Cooper transformation

$$t = \frac{1}{b^T x + b_0} \quad , \qquad y = t x \ ;$$

problem P is transformed in the problem

$$P^* \quad \begin{cases} \max \left(f_1\left(\frac{y}{t}\right), f_2\left(\frac{y}{t}\right)\right) \\ \frac{y}{t} \in R \\ b^T y + b_0 t = 1 \ , \quad t \geq 0 \end{cases}$$

where f_1 (y/t) and f_2 (y/t) are a linear function and a linear fractional real-valued function respectively.
It is easy to prove the following theorem which establishes a relationship between the efficient points of the problems P and P*.

Theorem 6.1 i) If x^0 is an efficient point for P then (y^0, t_0) is an efficient point for P*, where

$$t_0 = \frac{1}{b^T x^0 + b_0} \quad , \qquad y^0 = t_0 x^0$$

ii) If (y^0, t_0) is an efficient point for P*, then $x^0 = y^0 / t_0$ is an efficient point for P.

Remark 6.1 Since P* is a bicriteria linear fractional problem having, at least, one linear objective function, the sequential method described in sections 3 and 4, can be used to solve

it. Taking into account ii) of theorem 6.1, we are able to find all efficient points of problem P.
Let us note that E turns out to be disconnected if and only if in the algorithm variable t leaves the basis and successively enters the basis again at a positive value.
In order to point out the results given in this section, we will propose a numerical example where the set of all efficient points is disconnected.
Consider the following bicriteria fractional problem

$$P : \max \left(f_1(x) = \frac{-x_1 + 2}{x_2 + 2} \quad , \quad f_2(x) = \frac{x_1 - 2}{x_1 + 3x_2 + 1} \right), \quad x = (x_1, x_2) \geq 0$$

By means of the Charnes-Cooper transformation

$$t = \frac{1}{x_2 + 2} \quad , \quad y = t\,x \quad \text{that is} \quad y_1 = t\,x_1 \ , \ y_2 = t\,x_2$$

we obtain a bicriteria problem where the first objective function is now linear

$$P^* \begin{cases} \max \left(-y_1 + 2t \ , \ \dfrac{y_1 - 2t}{y_1 + 3y_2 + t} \right) \\ y_2 + 2t = 1 \\ y_1, y_2 \geq 0 \qquad t \geq 0 \end{cases}$$

First of all we will solve the linear problem

$$P_L^* \begin{cases} \max \ (f_1(y_1, y_2, t) \triangleq -y_1 + 2t \\ y_2 + 2t = 1 \\ y_1, y_2 \geq 0 \ , \quad t \geq 0 \end{cases}$$

which has the unique optimal solution $w^0 = (y1 = 0, y2 = 0, t = 1/2)$, so that w^0 also turns out to be the unique solution to the linear fractional problem $P^*_{\alpha 0}$ where $\alpha_0 = f_1(w^0) = 1$.
Now consider the parametric problem

$$P^*_{\alpha_0}(\theta) : \begin{cases} \max f_2(y_1, y_2, t) = \dfrac{y_1 - 2t}{y_1 + 3y_2 + t} \\ y_2 + 2t = 1 \\ -y_1 + 2t \geq 1 - \theta \ , \quad \theta \geq 0 \\ y_1, y_2 \geq 0 \ , \qquad t \geq 0 \end{cases}$$

The following optimal tableau is obtained by applying the modified version of Martos's algorithm to the problem $P^*_{\alpha 0}$ (0).

	1	0	0	0	-1
	-1/2	-3/2	0	0	-5/2
t	1/2	-1/2	0	1	-1/2
y2	0	1	1	0	1

Sensitivity analysis applied to $P^*_{\alpha 0}$ (0) gives the following tableau

	1-θ	0	0	0	-1
	-1/2-(5/2)θ	-3/2	0	0	-5/2
t	1/2-(1/2)θ	-1/2	0	1	-1/2
y2	θ	1	1	0	1

Since $\gamma(\theta) = (-3/2 + (3/2)\theta)$, we have $\theta' = 1$; on the other hand the feasibility condition implies $\theta'' = 1$, so that $w(\theta) = (0, \theta, 1/2 - (1/2)\,\theta)$ is optimal for $P_{\alpha_0}(\theta)$ $\forall\, \theta \in [\,0\ \ 1]$.
Let us note that, in the original problem P, the solution $x(\theta)$ associated with $w(\theta)$ is

$$x(\theta) = \left(0\,,\ \frac{\theta}{\frac{1}{2} - \frac{1}{2}\theta}\right)\ ;$$

as a consequence the halfline r_1, whose parametric equation is

$$x_1 = 0\ ,\quad x_2 = \frac{\theta}{\frac{1}{2} - \frac{1}{2}\theta}\ ,\qquad \theta \in [\,0\,,1\,[$$

is contained in set E of all efficient solutions. Set $\theta = 1 + \theta$ in the previous tableau; in order to restore the feasibility and the optimality conditions we are obliged to perform a pivot operation so that t will have to leave the basis.

	-θ	0	0	0	-1
	-3-(5/2)θ	-3/2	0	0	-5/2
t	-(1/2)θ	(-1/2)	0	1	-1/2
y2	1+θ	1	1	0	1

	$-\theta$	0	0	0	-1
	$-3-\theta$	0	0	-3	-1
y_1	θ	1	0	-2	1
y_2	1	0	1	2	0

Since $\gamma(\theta) = 3\theta$, we have $\theta' = 0$ while $\theta'' = +\infty$. We restore the optimality by performing a pivot operation; we obtain the following tableau

	$-\theta$	0	0	0	-1
	$-3/2-\theta$	0	3/2	0	-1
y_1	$1+\theta$	1	1	0	1
t	1/2	0	1/2	1	0

Since $\gamma(\theta) = (-3/2)\,\theta$, we have $\theta' = +\infty$; the feasibility condition implies $\theta'' = +\infty$. so that $w(\theta) = (1+\theta, 0, 1/2)$ is optimal $\forall\ \theta \geq 0$. As a consequence, in the original problem P, any point of the halfline r_2, whose parametric equation is $x_1(\theta) = 2\,(1+\theta)$, $x_2 = 0$, $\theta \geq 0$, is an efficient point.

Let us note that for $\theta = 0$, $\gamma(\theta) = 0$, so that $w = (1-y_2, y_2, 1/2 - (1/2)y_2)$ is an alternative optimal solution for any $y_2 \geq 0$.

Consequently the halfline r_3, whose parametric equation is

$$x_1 = \frac{1-y_2}{\frac{1}{2} - \frac{1}{2}y_2} = 2 \quad , \quad x_2 = \frac{2\,y_2}{1-y_2} \quad , \quad y_2 \geq 0$$

is contained in E.

Set E of all efficient solutions to P is $E = r_1 \cup r_2 \cup r_3$. Since, in the suggested algorithm, variable t leaves the basis and successively enters the basis again at a positive value, set E is disconnected.

Appendix A (A modified version of Martos's algorithm) [3]

Consider the problem

$$\max \left(z(x) = \frac{c^T x + c_0}{d^T x + d_0}\right), \; x \in R = \{x \in \mathbb{R}^n;\; Ax = b,\; x \geq 0\}$$

where A is a m x n matrix, $b \in \mathbb{R}^m$ and $d^T x + d_0 > 0 \quad \forall x \in R$.

STEP 0 Find an optimal level solution [1] x'; if such a solution does not exist STOP ($\sup_{x \in R} z(x) = +\infty$); otherwise go to step 1.

STEP 1 Set $\bar{\gamma} = (\bar{d}_0 \bar{c}_N - \bar{c}_0 \bar{d}_N)$ and $J = \{ j : \bar{\gamma}_j > 0 \}$ where $\bar{c}_N$ and $\bar{d}_N$ are the reduced costs of numerator and denominator corresponding to x' respectively and $\bar{c}_0$ and $\bar{d}_0$ are the value of numerator and denominator at x = x' respectively. If $J = \varnothing$, STOP, x' is an optimal solution; otherwise set k such that

$$\bar{c}_{N_k} / \bar{d}_{N_k} = \max_{j \in J} (\bar{c}_{N_j} / \bar{d}_{N_j})$$

go to step 2.

STEP 2 The non-basic variable x_{N_k} enters the basis by means of a pivot operation, go to step 1. If an operation like this is not possible, STOP.
($\sup_{x \in R} z(x) = \bar{c}_{N_k} / \bar{d}_{N_k}$).

---

1 $x' \in R$ is said an optimal level solution if x' solves the parametric problem

$$\frac{1}{\xi'} \cdot \max (c^T x + c_0) \;, \; x \in R \;, \; d^T x + d_0 = \xi'$$

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