REPORT n. 15

ON THE BICRITERIA MAXIMIZATION PROBLEM(*)

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(*) This reaearch was supported in part by the Ministry of Public Education.

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1. INTRODUCTION

The bicriteria maximization problem P has been studied by several authors [3,4,5,6,7,8] mainly with the aim of establising the connectedness of the set of all efficient points E. In this paper we will introduce a parametric real-valued function $z(\theta)$ which allows us to derive a parametric representation of E in a more general form than the one given in [8]; furthermore we will point out that the properties of the function $z(\theta)$ are strictly related to the connectedness of E and that they can play an important role in finding sequential methods for generating E [7].
The aforesaid approach allows us to obtain new results and known ones.

2. A PARAMETRIC REPRESENTATION OF THE SET OF ALL EFFICIENT POINTS.

Let us consider the bicriteria maximization problem

$$P: \max\ (f_1(x),\ f_2(x)),\quad x \in R$$

where f_1, f_2 are real-valued continuous functions on the non-empty compact set $R \subset \mathbb{R}^n$.

A point $x^\circ \in R$ is said to be efficient for P if there does not exist a point $x \in R$ such that $f_i(x) \geq f_i(x^\circ)$ $i=1,2$ where at least one of these inequalities is strict.

Several authors [3,4,5,8] have pointed out that set E of all efficient points of P is related to the optimal solutions of a suitable parametric problem.Following this idea,we will consider the parametric problem stated in a form which allows us to establish some important relations and to obtain known results and new ones.

Set:

$$P(\theta):\ z(\theta) \triangleq \max\ f_2(x),\quad x \in R(\theta) \triangleq \{x \in R:\ f_1(x) \geq M-\theta,\ \theta \geq 0\}$$

where $M \triangleq \max f_1(x),\ x \in R$.

Let us note that $P(\theta)$ has optimal solutions for any $\theta \geq 0$, since the feasible region $R(\theta)$ is a compact set.

Denote with $S(\theta)$ the set of optimal solutions for $P(\theta)$ and with $S'(\theta)$ the set of optimal solutions to the problem

$$(2.1)\qquad \max\ f_1(x),\quad x \in S(\theta).$$

The following theorem gives the relation between E and the optimal solutions to the parametric problem $P(\theta)$.

THEOREM 2.1. *Consider the bicriteria maximization problem* P. *Then*

(2.2) $$E = \bigsqcup_{\theta \in [0, M-m]} S'(\theta)$$

where $m \triangleq \min f_1(x),\ x \in R$.

Proof. If $\bar{x}$ is an efficient point for problem P then it is also an optimal solution for $P(\bar{\theta})$ with $\bar{\theta} = M - f_1(\bar{x})$ and it is easy to verify that $\bar{x} \in S'(\theta)$; consequently $E \subseteq \bigsqcup_{\theta \in [M-m]} S'(\theta)$. Now we must show that $E \supseteq \bigsqcup_{\theta - [0, M-m]} S'(\theta)$. Let $\bar{x}$ be an optimal solution for problem (2.1) with $\theta = \theta' \in [0, M-m]$; $\bar{x}$ is also an optimal solution for problem $P(\theta'')$ where $\theta'' = M - f_1(\bar{x})$ and $\theta'' < \theta'$. Suppose, ab absurdo, that $\bar{x}$ is not an efficient point for P, that is, there exists $x^o \in R$ such that the following inequalities:

(2.3a) $$f_1(x^o) \geqq f_1(\bar{x})$$

(2.3b) $$f_2(x^o) \geqq f_2(\bar{x})$$

hold where at least one of these is strict. Since (2.3a) implies $x^o \in R(\theta'')$, we have $f_2(x^o) \leqq f_2(\bar{x})$, that is, from (2.3b),

$f_2(x^o)=f_2(\bar{x})$. As a consequence $x^o \in S(\theta')$, so that (2.3a) can hold only as an equality and this is absurd.
This completes the proof.

The previous theorem points out that an optimal solution for $P(\theta)$ is not necessarily an efficient point for P, so that, in general, $E \neq \bigcup_{\theta \in [0,M-m]} S(\theta)$, nevertheless E can be characterized as the union of suitable sets $S(\theta)$.
As we will see, the function $z(\theta)$ plays an important role in finding this characterization.
The following theorem states some important properties of the function $z(\theta)$.

<u>THEOREM 2.2</u>. <u>Consider the parametric function</u> $z:[0,M-m] \to \mathbb{R}$, $z(\theta) = \max_{x \in R(\theta)} f_2(x)$. <u>Then</u> i) <u>and</u> ii) <u>hold</u>.

i) $z(\theta)$ <u>is a nondecreasing function</u>;

ii) $z(\theta)$ <u>is an upper semicontinuous function</u>.

<u>Proof</u>. i) It follows by noting that $\theta_1 < \theta_2$ implies $R(\theta_1) \subseteq R(\theta_2)$ so that $z(\theta_1) \leq z(\theta_2)$.

ii) The thesis follows immediately from theorem in [13]. For sake of simplicity we will give a direct proof.
Set $\theta_o \in [0,M-m]$; we must prove that the upperlimit of $z(\theta)$, $\theta \to \theta_o$, is not greater than $z(\theta_o)$ that is $\overline{\lim_{\theta \to \theta_o}} z(\theta) \leq z(\theta_o)$.

Let $\{\theta_n\} \subset [0,M-m]$ be such that $\theta_n \to \theta_o$ and let x_n be an optimal solution for problem $P(\theta_n)$. Since R is a compact set there exists a subsequence $\{x_s\} \subset \{x_n\}$ such that $x_s \to x^o \in R$. Taking into account that f_1 is continuous and $f_1(x_s) \geq M-\theta_s$ $\forall s$, we have, $f_1(x^o) \geq M-\theta_o$, so that $x^o \in R(\theta_o)$.
On the other hand $\overline{\lim\limits_{\theta \to \theta_o}}\ z(\theta) = \overline{\lim\limits_{x_s \to x^o}}\ f_2(x_s) = f_2(x^o) \leq z(\theta_o)$.
This completes the proof.

As a consequence of theorem 2.1 there exists a suitable set of indices I, $0 \in I$, such that $[0,M-m] = \bigcup\limits_{i \in I} [\theta_i, \theta_{i+1}]$, $\theta_o = 0$; furthermore, if $z(\theta)$ is constant in $]\theta_i, \theta_{i+1}[$ then it is either increasing in $]\theta_{i+1}, \theta_{i+2}[$ or constant in $]\theta_{i+1}, \theta_{i+2}[$ with $\lim\limits_{\theta \to \theta_{i+1}^+} z(\theta) \neq \lim\limits_{\theta \to \theta_{i+1}^-} z(\theta)$; on the contrary if $z(\theta)$ is increasing in $]\theta_i, \theta_{i+1}[$ then either $\theta_{i+1} = M-m$ or $z(\theta)$ is constant in $]\theta_{i+1}, \theta_{i+2}[$.

Let $I_1 \subseteq I$ be such that z is increasing in $]\theta_i, \theta_{i+1}[$, $i \in I_1$ and let $J \subseteq I$ be such that $i \in J$ implies

$$z(\theta_i) = \lim_{\theta \to \theta_i^+} z(\theta) \neq \lim_{\theta \to \theta_i^-} z(\theta).$$

Now we are able to establish the following results:

<u>LEMMA 2.1</u>. Let $\theta' \in]0,M-m[$ be such that

(2.4) $\qquad z(\theta) < z(\theta'), \quad \forall\, \theta < \theta'.$

Then every optimal solution for $P(\theta')$ is binding at the constraint $f_1(x) \geq M-\theta'$ and, furthermore, it is also an efficient point for problem P.

Proof. Let x' be an optimal solution for $P(\theta')$ and, suppose, ab absurdo, that $f_1(x') > M-\theta'$. Then x' is optimal for problem $P(\theta_1)$, $\theta_1=M-f_1(x') < \theta'$ so that $z(\theta_1)=z(\theta')=f_2(x')$ and this contradicts (2.4).
Now we must show that x' is an efficient point for P; to this aim it is sufficient to prove that both a) and b) hold:

a) $f_1(x) \geq f_1(x')$ implies $f_2(x) \leq f_2(x')$, $\forall x \in R$;

b) $f_1(x) > f_1(x')$ implies $f_2(x) < f_2(x')$, $\forall x \in R$.

If $f_1(x) \geq f_1(x')$ we necessarily have $f_2(x) \leq f_2(x')$ since $x \in R(\theta')$ and a) is satisfied. On the other hand $f_1(x) > f_1(x')$ implies that x is an interior point for $R(\theta')$ so that it is not an optimal solution for $P(\theta')$, i.e. $f_2(x) < f_2(x')$. This completes the proof.

LEMMA 2.2. Suppose that $z(\theta)$ is constant in $[a,b]$. Then any optimal solution for $P(\theta)$, $\theta \in]a,b[$ binding at the constraint $f_1(x) \geq M-\theta$ is not an efficient point for P.

Proof. Let x' be an optimal solution for $P(\theta)$, with $f_1(x') = M-\theta$ and let x° be an optimal solution for P(a). Since $z(a) = z(\theta)$ we have $f_2(x^\circ) = f_2(x')$ and furthermore $f_1(x^\circ) \geq M-a > M-\theta = f_1(x')$, so that x' is not an efficient point for P and this completes the proof.

THEOREM 2.3. Consider the bicriteria maximization problem P. Then

$$(2.5) \qquad E = \left(\bigsqcup_{\substack{\theta \in]\theta_i, \theta_{i+1}[\\ i \in I_1}} S(\theta) \right) \cup \left(\bigsqcup_{i \in J \cup \{0\}} S(\theta_i) \right)$$

Proof. Let us note that $\theta' \in]\theta_i, \theta_{i+1}[$, $i \in I_1$ is such that $z(\theta) < z(\theta')$ $\forall \theta < \theta'$ and that the same property holds for any θ_i, $i \in J$.

From Lemma 2.1 we have $E \supseteq \left(\bigsqcup_{\substack{\theta \in]\theta_i, \theta_{i+1}[\\ i \in I_1}} S(\theta) \right) \cup \left(\bigsqcup_{i \in J \cup \{0\}} S(\theta_i) \right)$

On the other hand, an efficient point $x' \in E$ is an optimal solution for the problem

$$\begin{cases} \max f_2(x), & x \in R \\ f_1(x) \geq f_1(x') = M - \theta' \end{cases}$$

Since x' is binding at the constraint $f_1(x) \geq M - \theta'$, we have, from Lemma 2.2, that $\theta' \in \left(\bigsqcup_{i \in I_1}]\theta_i, \theta_{i+1}[\right) \cup \left(\bigsqcup_{i \in J \cup \{0\}} \theta_i \right)$.

This completes the proof.

COROLLARY 2.1. Suppose that $z(\theta)$ is a semi-strictly quasi-concave function[1] and set $\theta^* = \max H(\theta)$, where $H(\theta) \triangleq$ $\triangleq \{\theta\colon z(\theta) \text{ is increasing in } [0,\theta]\}$.
Then

$$(2.6) \qquad E = \bigcup_{\theta\in[0,\theta^*]} S(\theta)$$

Proof. It is sufficient to note that the semi-strictly quasi-concave function $z(\theta)$ is either a constant function or an increasing function or a function which is increasing in $[0,\theta^*]$ and constant in $]\theta^*, M-m]$, so that the thesis follows from theorem 2.2.
Let us note that if $z(\theta)$ is constant then $E=S(0)$, while if $z(\theta)$ is increasing then $E = \bigcup_{\theta\in[0,M-m]} S(\theta)$.
This completes the proof.

The characterization of E as the union of suitable sets $S(\theta)$ plays an important role in finding sequential methods to generate E [4,7] and also in studying the connectedness of E [5,8] .
As pointed out in corollary 2.1, E assumes a simple form

(1) A real-valued function f defined on a convex set X is called semi-strictly quasiconcave if for all $x_1, x_2 \in X$ such that $f(x_1) \neq f(x_2)$ the inequality $f(x) > \min(f(x_1), f(x_2))$ holds for all x on the open line-segment $]x_1,x_2[$.

when $z(\theta)$ is a semi-strictly quasiconcave function.
A class of problems which ensures the semi-strictly quasiconcavity of the function $z(\theta)$ is one where at least one of the objective functions is semi-strictly quasiconcave; this is shown in the following theorem:

THEOREM 2.4. Let us consider problem P where the feasible region is a convex set and let us suppose that f_2 is a semi-strictly quasiconcave function. Then $z(\theta)$ turns out to be a semi-strictly quasiconcave function.

Proof. Ab absurdo, let us suppose that the nondecreasing function $z(\theta)$ is not semi-strictly quasiconcave, then there exist $\theta_1, \theta^*, \theta_2$ with $\theta_1 < \theta^* < \theta_2$ such that $z(\theta^*) = z(\theta_1) < z(\theta_2)$. Let x_1 and x_2 be optimal solutions for $P(\theta_1)$ and $P(\theta_2)$ respectively; since x_1, x_2 are not necessarily binding at the parametric constraint, we set

$$\theta' \triangleq M-f_1(x_1) \leq \theta_1 \quad , \quad \theta'' \triangleq M-f_1(x_2) \leq \theta_2 .$$

Now we will prove that $\theta^* \in]\theta', \theta''[$, that is $\theta^* < \theta''$, since $\theta^* > \theta'$ is obvious.
If $\theta^* \geq \theta''$, we have $f_1(x_2)=M-\theta'' \geq M-\theta^*$, so that $x_2 \in R(\theta^*)$; consequently $f_2(x_2) \triangleq z(\theta_2) \leq z(\theta^*)$ and this is absurd.
The continuity of the function f_1 implies the existence of $x^* \in]x_1,x_2[$ such that

$$f_1(x^*) = M-\theta^* \in]M-\theta', \quad M-\theta''[.$$

Consider now the restriction of the function f_2 on the line-segment $[x_1,x_2]$. We have

$f_2(x^*) \leqq z(\theta^*) = z(\theta_1) \triangleq f_2(x_1) < f_2(x_2)$ and this contradicts

the semi-strictly quasiconcavity of f_2.

The proof is complete.

For the subclass of bicriteria maximization problems where at least one of the objective functions is semi-strictly quasiconcave, we obtain, as a direct consequence of corollary 2.1 and theorem 2.4, the following result given in [8].

COROLLARY 2.2. Suppose f_2 is semi-strictly quasiconcave and the feasible region is a convex set. Then (2.6) holds.

3. ON THE CONNECTEDNESS OF E.

In the previous section we outlined the role played by the function $z(\theta)$ in order to achieve a characterization of the set of all efficient points of the bicriteria maximization problem P.

Now we will point out that $z(\theta)$ is also related to the connectedness(2) of E, in the sense that the semi-strictly

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(2) Connectedness of a set is referred to arcwise-connectedness.

quasiconcavity of z is a necessary condition for E to be connected.
This last property is established in the following theorem:

<u>THEOREM 3.1</u>. <u>Consider the bicriteria maximization problem</u> P <u>and assume that</u> E <u>is connected</u>.
<u>Then the function</u> $z(\theta)$ <u>is semi-strictly quasiconcave</u>.

<u>Proof</u>. Ab absurdo, let us suppose that the nondecreasing function $z(\theta)$ is not semi-strictly quasiconcave, then there exist $\theta_1, \theta^*, \theta_2$, with $\theta_1 < \theta^* < \theta_2$ such that $z(\theta^*)=z(\theta_1) < z(\theta_2)$. Let x_1 and x_2 be optimal solutions for $P(\theta_1)$ and $P(\theta_2)$ respectively and also efficient points for P.
Set: $\theta' \triangleq M-f_1(x_1) \leq \theta_1$, $\theta'' \triangleq M-f_1(x_2) \leq \theta_2$; we have (see proof of theorem 2.4) $\theta' < \theta^* < \theta''$.
Since E is connected, there exists a continuous function $\Phi: [0,1] \to \mathbb{R}^n$ such that $\Phi(0) = x_1$, $\Phi(1) = x_2$ and $\Phi(t)$ is an efficient point for P $\forall t \in]0,1[$. Consider the function $\phi = f_1 \circ \Phi : [0,1] \to \mathbb{R}$; since ϕ is a continuous function, we have $\phi([0,1]) \triangleq A \supseteq [f_1(x_2), f_1(x_1)] = [M-\theta'', M-\theta']$, so that there exists $t^* \in]0,1[$ such that $\phi(t^*) = M-\theta^*$. It follows that $x^* = \Phi(t^*)$ is an efficient point which is binding at the constraint $f_1(x) \geq M-\theta^*$ and this is absurd (see lemma 2.2).
This completes the proof.

The following example shows that the semi-strictly quasi-concavity of the function $z(\theta)$ is not, in general, a sufficient condition for E to be connected.

<u>EXAMPLE 3.1</u>. Consider the following bicriteria maximization problem

$$P: \begin{cases} \max\ (-x_1, x_1^2 + x_2^2) \\ 1 \leq x_1 \leq 4, \quad -1 \leq x_2 \leq 1 \end{cases}$$

It is easy to verify that the set of all efficient points is $E = [a_1, a_2] \cup [b_1, b_2]$, where $a_1 = (1,1)$, $a_2 = (4,1)$, $b_1 = (1,-1)$, $b_2 = (4,-1)$, so that E is disconnected.
On the other hand, $z(\theta) = (1-\theta)^2 + 1$, $\theta \in [0,3]$, and thus $z(\theta)$ turns out to be a semi-strictly quasiconcave function because it is increasing in $[0,3]$.

<u>REMARK 3.1</u>. Let us consider the parametric problem

$$P_1(\theta): \ z_1(\theta) \triangleq \max f_1(x), \ x \in R, \ f_2(x) \geq M_2 - \theta, \ \theta \geq 0$$

where $M_2 \triangleq \max f_2(x)$, $x \in R$.
Obviously, we can apply all the previously stated results to problem $P_1(\theta)$. Consequently a necessary condition for E to be connected is that both $z(\theta)$ and $z_1(\theta)$ are semi-strictly

quasiconcave functions.

In Example 3.1 the function $z_1(\theta)$ is semi-strictly quasiconcave because $f_1(x) = -x_1$ is also semi-strictly quasiconcave, so that the semi-strictly quasiconcavity of $z_1(\theta)$ and $z(\theta)$ is not sufficient to guarantee the connectedness of E. Taking into account (2.6), the loss of connectedness is related to the disconnectedness of sets $S(\theta)$. A particular subclass of bicriteria maximization problems for which E turns out to be connected is one where both of the objective functions are semi-strictly quasiconcave [8].

4. CONCLUDING REMARKS.

In this paper we have pointed out the role played by function $z(\theta)$ in characterizing the set of all efficient points of a bicriteria maximization problem as the union of suitable sets of optimal solutions to the parametric problem $P(\theta)$. Obviously the possibility of finding an efficient sequential method which is able to solve $P(\theta)$ is strictly related to the structure of problem P, that is, to the properties of the objective functions and the constraints.
In [7] a simple algorithm is given which generates the set of all efficient points of a bicriteria linear fractional problem for any feasible region (bounded or unbounded); such an algorithm points out that, when one of the objective functions is linear, then E is connected even if the feasible

region is unbounded.

The obtained results suggest the possibility of generating E for particular subclasses of bicriteria maximization problems by means of a suitable revision of the parametric methods given in [9,10,11,12].

This will be the subject of a forthcoming paper.

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