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Equivalence in Linear Fractional Programming

Alberto CAMBINI - Laura MARTEIN

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Equivalence in Linear Fractional Programming (*)

A. Cambini - L. Martein (**)

§ 1. Introduction

The linear fractional problem is the problem of maximizing the quotient of two affine functions subject to linear constraints. Because of the potentially broad applications of the linear fractional programming, in the last twenty years, several authors suggested algorithms for solving this problem. These studies have led to theoretical and computational analysis in order to compare the various methods.
Some algorithms (for example Martos [12] and Charnes-Cooper [8], Isbell-Marlow [9] and Bitran-Novaes [3] and Bhatt [1]), even if different in their approach, have been shown algorithmically equivalent in the sense that they generate the same finite sequence of points leading to an optimal solution.
This equivalence fails when the feasible region is unbounded. This fact has stimulated the Authors in trying to extend the result given by Wagner-Yuan [14], taking into account the experiments performed by Bitran [2] which have shown the superiority of Martos procedure over the others and taking into account the results given in [4] where it is suggested an algorithm which works efficiently even for the unbounded case.
Since such an algorithm has been shown to be an useful tool in solving other kinds of problems like as maximizing the sum of ratios [6] or

(*) *This research was partially supported by Public Education Ministry.*
(**) *Department of Statistics and Applied Mathematics. University of Pisa, Pisa, ITALY.*

in studying parametric analysis in linear fractional programming [7] or for generating the set of all efficient points of a bicriteria linear fractional problem [11], in this paper we will give a self-contained version of such an algorithm and, at the same time, we will point out the reason why Martos procedure does not process the problem in the not compact case; finally we will extend in a suitable way the results on the equivalence given in [14].

§ 2. Cambini-Martein algorithm

Consider the problem

$$P: \left[\sup f(x) = \frac{cx + c_o}{dx + d_o}\right] \triangleq L, x \in R = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$$

where c, d are $1 \times n$ vectors; $c_0, d_0 \in \mathbb{R}$; A is a $m \times n$ real matrix; $b \in \mathbb{R}^m$ and $dx + d_0 > 0, \forall x \in R$.

The following theorem states some fundamental properties of a linear fractional problem.

Theorem 2.1. Consider the linear fractional problem P. Then conditions i), ii) and iii) hold.

i) $L = \max_{x \in R} f(x)$ iff there exists a vertex $x^0 \in R$ such that $L = f(x^0)$.

ii) If $L \neq \max_{x \in R} f(x)$ then there exists an extremum ray, i.e., a ray r of equation $x = x^0 + tu, t \geq 0$, such that $L = \sup_{x \in r} f(x) = \lim_{t \to +\infty} f(x^0 + tu)$.

iii) Let D be the set of the optimal solutions of the linear problem: $\min_{x \in R}(dx + d_0)$.

Then $L = +\infty$ iff D is unbounded and $\sup_{x \in D}(cx + c_0) = +\infty$.

Proof: For the proofs of i) and ii) see Martos [12].
iii) Let us note that $D \neq \emptyset$ since the linear function $dx + d_0$ is lower bounded on R. For ii) we have $L = \sup_{x \in r} f(x) = \sup_{t \geq 0} \frac{cx^0 + c_0 + t(cu)}{dx^0 + d_0 + t(du)}$, so that $L = +\infty$ iff $du = 0$ and $cu > 0$.
It is easy to prove that the ray r' whose equation is $x = x^* + tu$, $t \geq 0$, $x^* \in D$, is contained in D and, furthermore,

$$\sup_{x \in D}(cx + c_0) \geq \sup_{x \in r'}(cx + c_0) = +\infty.$$

Suppose now that $\sup_{x \in D}(cx + c_0) = +\infty$. Then there exists a ray $r' \subset D$ of equation $x = x^* + tu$, $t \geq 0$, such that:

$$\sup_{x \in r'}(cx + c_0) = +\infty. \tag{2.1}$$

Since $r' \subset D$ implies $du = 0$ and (2.1) implies $cu > 0$, then we have $L = +\infty$. This completes the proof.

In order to clarify the main ideas of the sequential method that we are going to describe, we will give the following definitions.
We say that the feasible point x^* is an *optimal level solution* for the linear fractional problem P, if x^* is optimal for the linear problem:

$$\begin{cases} \max(cx + c_0) \\ dx = dx^*, \ x \in R \end{cases}$$

In particular, x^* is referred to as a *basic optimal level solution* if, in addition, x^* is a vertex of R.
Let us note that an optimal solution for P is also a basic optimal level solution and a ray having the property that any of its points is an optimal level solution is also an extremum ray.
This remark points out how the concept of optimal level solution is important in solving the linear fractional problem.

In order to characterize a basic optimal level solution, we introduce some notations.
Let x^0 be a vertex of the feasible region R with corresponding basis B; we partition the vectors c and d as $c = (c_B, c_N)$, $d = (d_B, d_N)$ and the matrix A as $A = [B : N]$. Set:

$\overline{c}_0 = c_B B^{-1} b + c_0$, $\overline{d}_0 = d_B B^{-1} b + d_0$,

$\overline{c}_N = c_N - c_B B^{-1} N$, $\overline{d}_N = d_N - d_B B^{-1} N$, $\overline{c}_j$ and $\overline{d}_j$ the j-th component of $\overline{c}_N$ and $\overline{d}_N$ respectively.
Consider the following sets of indices:

$$J_1 = \{j : \overline{d}_j > 0\}; \quad J_2 = \{j : \overline{d}_j < 0\}; \quad J_3 = \{j : \overline{d}_j = 0\}.$$

The following theorem holds:

Theorem 2.2. The vertex $x^0 \in R$ is an optimal level solution iff conditions i) and ii) both hold.
i) For any index $j \in J_3$ we have $\overline{c}_j \leq 0$.

ii) If J_1 and J_2 are not empty sets, then:

$$\overline{c}_k / \overline{d}_k \triangleq \max_{j \in J_1} \overline{c}_j / \overline{d}_j \leq \min_{j \in J_2} \overline{c}_j / \overline{d}_j \triangleq \overline{c}_h / \overline{d}_h.$$

Proof: Let us note that, if we refer to the simplex tableau associated to the vertex x^0, x^0 turns out to be an optimal level solution iff there exists an index s such that, by performing a pivot operation on $\overline{d}_s$, the corresponding reduced costs $c_j^* = \overline{c}_j - (\overline{c}_s / \overline{d}_s)\, \overline{d}_j$ are non-positive $\forall j$, and this is true iff (2.2) holds:

(2.2a) $$\overline{d}_j = 0 \quad \text{implies} \quad \overline{c}_j \leq 0$$

(2.2b) $$\overline{d}_j > 0 \quad \text{implies} \quad \overline{c}_s / \overline{d}_s \geq \overline{c}_j / \overline{d}_j$$

(2.2c) $\quad \bar{d}_j < 0 \quad \text{implies} \quad \bar{c}_s / \bar{d}_s \le \bar{c}_j / \bar{d}_j$

Consequently we have $\bar{c}_j / \bar{d}_j \le \bar{c}_s / \bar{d}_s \le \bar{c}_i / \bar{d}_i \quad \forall j \in J_1, \quad \forall i \in J_2$, so that condition (2.2) implies i) and ii).
Let us note that if J_1 and J_2 are not empty sets, then $s = k$ or $s = h$. This completes the proof.

Remark 2.1. Let $N^{(k)}$ and $N^{(h)}$ be the columns of N associated with the indices k and h respectively. Then any feasible solution of the kind $x_B = B^{-1}b - B^{-1}N^{(k)}x_{N_k}, x_{N_k} \ge 0$ or of the kind $x_B = B^{-1}b - B^{-1}N^{(h)}x_{N_h}, x_{N_h} \ge 0$ is an optimal level solution since it is optimal for the linear problem

$$\begin{cases} \max(cx + c_0) \\ dx + dx_0 = \bar{d}_0 + \bar{d}_s x_{N_s} \\ x \in R \end{cases}$$

where $s = k$ or $s = h$. As a consequence, by performing a pivot operation corresponding to the selected index k or h, we obtain a new vertex which is a basic optimal level solution; if such a pivot operation is not possible, then any point of the ray $x_B = B^{-1}b - B^{-1}N^{(s)}x_{N_s}, x_{N_s} \ge 0$ is an optimal level solution; in such a case $s = k$ necessarily since the denominator does not turn out to be negative.

As we have just pointed out an optimal solution for P is also an optimal level solution; the following theorem states a necessary and sufficient condition for a basic optimal level solution to be optimal.
Set $\gamma \triangleq \bar{d}_0 \bar{c}_N - \bar{c}_0 \bar{d}_N$ and let γ_k and γ_h be the components of γ corresponding to the selected indices k and h respectively.

Theorem 2.3. Let x^0 be a basic optimal level solution for problem P. Then x^0 is optimal for P iff $\gamma_k \le 0$ and $\gamma_h \le 0$.

Proof: It is known (see Martos [12]) that x^0 is optimal for P iff $\gamma \le 0$. Obviously if $\gamma \le 0$, then $\gamma_k \le 0$ and $\gamma_h \le 0$; now we will prove that the nonpositivity of γ_k and γ_h implies $\gamma \le 0$.

Taking into account condition (2.2), it is sufficient to prove that $\gamma_k \le 0$ implies $\gamma_j \le 0 \quad \forall j \in J_1$ and $\gamma_h \le 0$ implies $\gamma_j \le 0 \quad \forall j \in J_2$.

For (2.2b) we have $\overline{c}_j / \overline{d}_j \le \overline{c}_k / \overline{d}_k \quad \forall j \in J_1$; on the other hand, $\gamma_k \le 0$ implies $\overline{c}_k / \overline{d}_k \le \overline{c}_0 / \overline{d}_0$, so that $\overline{c}_j / \overline{d}_j \le \overline{c}_0 / \overline{d}_0 \quad \forall j \in J_1$, i.e., $\gamma_j \le 0 \quad \forall j \in J_1$.

In a similar way, taking into account (2.2c), we can prove that $\gamma_j \le 0 \quad \forall j \in J_2$. This completes the proof.

The following theorem gives a sufficient condition for a ray $r \subset R$ to be an extremum ray.

Theorem 2.4. Let x^0 be a basic optimal level solution for problem P and suppose that $\gamma_k > 0$ and $B^{-1}N^{(k)} \le 0$. Then the ray r whose equation in the non-basic variables space is $x_B = B^{-1}b - B^{-1}N^{(k)}x_{N_k}, x_{N_k} \ge 0$, is an extremum ray for P that is $\sup_{x \in R} f(x) = \sup_{x \in r} f(x) = \overline{c}_k / \overline{d}_k$.

Proof: Let us note that $\gamma_k > 0$ implies, for ii) of theorem 2.2, $\gamma_j \le 0 \quad \forall j \in J_2$. Taking into account that any point of the ray $x_B = B^{-1}b - B^{-1}N^{(k)}x_{N_k}, x_{N_k} \ge 0$, is an optimal level solution (see remark 2.1), we have:

$$\sup_{x \in R} f(x) = \sup_{x_{N_k} \ge 0} \sup_{\substack{dx + d_0 = \overline{d}_0 + \overline{d}_k x_{N_k} \\ x \in R}} f(x) =$$

$$= \sup \frac{\overline{c}_k x_{N_k} + \overline{c}_0}{\overline{d}_k x_{N_k} + \overline{d}_0} = \lim_{x_{N_k} \to +\infty} \frac{\overline{c}_k x_{N_k} + \overline{c}_0}{\overline{d}_k x_{N_k} + \overline{d}_0} = \frac{\overline{c}_k}{\overline{d}_k}$$

This completes the proof.

Remark 2.2. Theorem 2.4 points out the main difference between Martos procedure and the algorithm that we are going to describe. In order to clarify this difference, consider the following problem:

$$\begin{cases} \sup f(x_1, x_2) = \frac{2x_1+3x_2}{x_1+2x_2+1} \\ -x_1 + x_2 \leq 2 \\ x_1 - 2x_2 \leq 1 \\ x_1, x_2 \geq 0 \end{cases}$$

We have $\sup\limits_{x \in R} f(x_1, x_2) = 7/4 = L$, while if we apply Martos algorithm we find the ray $r \subset R$ of equation $x_2 = 2 + t,\ t \geq 0$ such that $\sup\limits_{x \in r} f(x_1, x_2) = \lim\limits_{t \to +\infty} \frac{5t+2}{3t+5} = 5/3 < L$.

This happens since r does not have the property that any of its points is an optimal level solution.

Now we are able to suggest a sequential method for solving P for any feasible region.

The algorithm generates a finite sequence $x^i, i = 1, ..., \ell$ of optimal level solutions, the first of which is found in the following way:

solve the linear problem [1] $P_o : \min\limits_{x \in R}(dx + d_0)$ and let x^0 be an optimal solution. If x^0 is unique, then it is also an optimal level solution, otherwise solve the linear problem

$$P_1 \ : \ \begin{cases} \max(cx + c_0) \\ dx = dx^0 \\ x \in R \end{cases}$$

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[1] *Let us note that P_0 has optimal solutions, since the objective function is bounded below.*

If P_1 has no solutions, then we have $\sup_{x \in R} f(x) = +\infty$ for iii) of theorem 2.1; otherwise an optimal solution x^1 of P_1 is also an optimal level solution.

Let us note that, since we start solving P_0, we are interested in generating optimal level solutions corresponding to increasing levels of the denominator, so that, taking into account remark 2.1, in order to obtain x^{i+1} from x^i, we must perform a pivot operation corresponding to the index k given by the following rule:

$$\bar{c}_k / \bar{d}_k = \max_{j \in J_1} \bar{c}_j / \bar{d}_j \tag{2.3}$$

Cambini-Martein algorithm

Step. 0: Find the optimal level solution x^1; if such a solution does not exist, STOP: $\sup_{x \in R} f(x) = +\infty$, otherwise set $i = 1$ and go to step 1.

Step. 1: If $J_1 = \emptyset$, STOP: x^i is an optimal solution for P; otherwise let k be such that (2.3) holds and go to step 2.

Step. 2: The non-basic variable x_{N_k} enters the basis by means of a pivot operation, set $i + 1 = i$ and go to step 1. If such an operation is not possible, STOP: $\sup_{x \in R} f(x) = \bar{c}_k / \bar{d}_k$.

§ 3. On the equivalence

In [8] Charnes-Cooper have solved problem P applying the usual simplex algorithm to a suitable linear problem P_L; when the feasible region R is a compact set, it is easy to establish the relationship between an optimal solution of P_L and an optimal solution of P; when R is unbounded and the supremum of the linear fractional problem is finite but not maximum, Schaible has shown in [13] that infinite

suitable feasible rays in P are associated to an optimal solution of P_L. Let us note that several vertices of P_L can be generated by applying the simplex algorithm corresponding to feasible rays in P.
This remark and the fact that Martos algorithm does not process the problem are the main reasons for which the equivalence established by Wagner-Yuan in [14] cannot be extended to the non-compact case without any change.
In order to obtain such an extension we will suggest to solve P_L by means of a suitable choice of the variable entering the basis; this allows us to obtain a correspondence between an extremum ray of P and an optimal solution of P_L and viceversa. With this aim in mind, first of all we consider the following transformation suggested by Charnes-Cooper:

$$y = tx, \quad \text{where} \quad t = 1/(dx + d_0) \tag{3.1}$$

By means of (3.1) problem P is reduced to the following linear one:

$P_L \ : \quad \sup\ (cy + c_0 t), (y, t) \in R_L$

where $R_L \overset{\Delta}{=} \{(y, t) \in \mathbb{R}^{n+1} : Ay - bt = 0, dx + d_0 t = 1, y \geq 0, t \geq 0\}$.
It can be shown that if $(y^0, t_0), t_0 > 0$ is a (basic) optimal solution for P_L, then $x^0 = y^0/t_0$ is a (basic) optimal solution for P.
When R is bounded, the variable t is positive for any feasible point of R_L so that we can find an optimal solution for P solving the linear problem P_L.
Wagner-Yuan have shown in [14] that Martos and Charnes-Cooper algorithms are equivalent in the sense that they generate a sequence of vertices $x^i, i = 1, ..., \ell$ and a sequence of vertices $(y^i, t_i)\ i = 1, ..., \ell$ respectively, such that $x^i = y^i/t_i$.
Let us note that, when the feasible region of P is unbounded, the variable t turns out to be zero, so that the relation $x = y/t$ cannot be applied.

Nevertheless we will show that by solving P_L with a suitable choice of the variable entering the basis, we obtain an algorithm which is equivalent to the one suggested in section 2.
This last algorithm generates a finite sequence of vertices $x^i, i = 1, ..., \ell$, corresponding to increasing levels of the denominator $(dx+d_0)$; furthermore $x^i, i = 1, ..., \ell$ are optimal level solutions the last of which is optimal for the linear fractional problem or it is the origin of an extremum ray for P.
Let us note that by means of the Charnes-Cooper transformation the vertex x^i, with corresponding basis B, is transformed into the vertex (y^i, t_i) with $t_i = 1/(dx^i + d_0)$, $y^i = t_i x^i$.
On respect to this vertex, problem P_L can be rewritten as:

$$\begin{cases} \sup\ c_N^* y_N + c_0^* \\ y_B + N^* y_N = b^* \\ t + d_N^* y_N = t_0 \\ y_B \geq 0,\ y_N \geq 0,\ t \geq 0 \end{cases}$$

where:

(3.2)
$$c_N^* = \overline{c}_N - t_0 \overline{c}_0 \overline{d}_N;\ d_N^* = t_0 \overline{d}_N;\ b^* = t_0 \overline{b}$$
$$t_0 = 1/\ \overline{d}_0;\ N^* = \overline{N} + t_0 \overline{b}\ \overline{d}_N.$$

Starting from the basic solution (y^1, t_1) corresponding to x^1, we will solve problem P_L by means of the usual simplex algorithm with the following criterium for the variable entering the basis

(3.3)
$$\max_{d_j^* > 0} c_j^* / d_j^*$$

where c_j^* and d_j^* are the j-th component of c^* and d^* respectively.

In such a way we obtain an algorithm which is equivalent to Cambini-Martein algorithm, in the sense that it generates a sequence of vertices $(y^1, t_1), \dots, (y^s, t_s)$, satisfying the properties stated in the following theorem:

Theorem 3.1.

i) We have $x^i = y^i/t_i \quad i = 1, \dots, \ell$.

ii) x^ℓ is optimal for P iff $s = \ell$ and (y^ℓ, t_ℓ) is optimal for P_L.

iii) x^ℓ is the origin of an extremum ray for P iff $s = \ell+1$ and $(y^{\ell+1}, t = 0)$ is optimal for P_L.

Proof: Let us note that (2.2) hold substituting $\overline{d}_j, \overline{d}_s, \overline{c}_j, \overline{c}_s$ with $d_j^*, d_s^*, c_j^*, c_s^*$ respectively.

As a consequence we have:

$$\max_{d_j^* > 0} c_j^*/d_j^* = c_k^*/d_k^* \leq \min_{d_i^* < 0} c_i^*/d_i^*. \tag{3.4}$$

Furthermore, taking into account theorem 2.3, $\gamma_k \leq 0$ implies $c_N^* = t_0\gamma \leq 0$, so that ii) follows.

Relation (3.4) implies that the variables entering the basis, applying the two algorithms, are the corresponding variables x_{N_k}, y_{N_k}.

Let $w = (w_1, \dots, w_m)$ be the column of the matrix $\overline{N}$ associated to x_{N_k} and let $w^* = (w_1^*, \dots, w_m^*, w_0^*)$ be the column of the matrix N^* associated to y_{N_k}, where $w_0^* = t_0\overline{d}_k$ and $w_i^* = w_i + t_0\overline{b}_i\overline{d}_k$, $i = 1, \dots, m$. Set $I = \{i : w_i > 0\}$ and $I^* = \{i : w_i^* > 0\}$. Let us note that $I^* \neq \emptyset$ since $w_0^* > 0$ and furthermore $I \subset I^*$. Taking into account that:

$$\max_{w_i^* > 0} \left(\frac{w_i^*}{t_0\overline{b}_i}, \frac{w_0^*}{t_0} \right) = \tag{3.5}$$

$$= \max \left\{ \overline{d}_k + \frac{1}{t_0} \max_{i \in I} \frac{w_i}{\overline{b}_i}, \overline{d}_k \right\},$$

we have the following cases:

- $I \neq \emptyset$.

We have $\max_{i \in I} w_i/\overline{b}_i \triangleq w_t/\overline{b}_t$ and (3.5) reaches its maximum value in $\overline{d}_k + w_t/\overline{b}_t$ so that the corresponding variables x_{N_t} and y_{N_t} leave the basis and this implies i).

- $I = \emptyset$.

In such a case $x_B = \overline{b} - x_{N_k} w,\ x_{N_k} \geq 0$, is the equation (in the non-basic variables space) of an extremum ray for P and the supremum of the problem is $\overline{c}_k/\overline{d}_k$. On the other hand, in (3.5) the maximum is reached in $\overline{d}_k$ so that the variable t leaves the basis by performing a pivot operation on $\overline{d}_k^*$. The new basic solution is $(y^{\ell+1}, 0)$ and it is optimal for P_L with optimal value $\overline{c}_k/\overline{d}_k$, because of (3.4); this implies iii) and completes the proof.

Remark 3.1. By iii) of theorem 3.1 and taking into account (3.2), we can find the relationship between the equation of the extremum ray given by Cambini-Martein algorithm and the optimal solution of problem P_L obtained by using criterium (3.3).
In the first algorithm we obtain: the vertex $x^\ell = (\overline{x}_B = \overline{b},\ \overline{x}_N = 0)$, the extremum ray of equation $x_B = \overline{b} - x_{N_k} w,\ x_{N_k} \geq 0$, where w is the column of the matrix $\overline{N}$ associated to the variable x_{N_k} and the value $\overline{c}_k/\overline{d}_k$ for the supremum.
In the second algorithm, by performing a pivot operation on d_k^*, we obtain: the optimal solution $\left(y^{\ell+1} = (\overline{y}_B = -w/\overline{d}_k, \overline{y}_k = 1/\overline{d}_k, \overline{y}_N = 0), \overline{t} = 0\right)$, the optimal value $\overline{c}_k/\overline{d}_k$ (this is an indirect proof of the property that the supremum of the problems P and P_L are equals) and the column associated to the nonbasic variable t given by $u = (u_B, u_k)$ where

(3.6) $$u_{B_j} = -w_j/t_0\overline{d}_k - b_j, \quad j = 1, ..., m; \ u_k = 1/t_0\overline{d}_k.$$

It is easy to prove, setting $\mathcal{B}$ the set of indices corresponding to the bases B, that the value of t_0 and the index k are characterized by

$$t_0 = \min_{\substack{u_j>0 \\ j\in\mathcal{B}\cup\{k\}}} \overline{y}_j/u_j = \overline{y}_k/u_k. \tag{3.7}$$

Then we have the following correspondence:

– if $x_B = \overline{b} - x_{N_k}w, x_{N_k} \geq 0$ is the equation (in the non-basic variables space) of the extremum ray for P, then $(y^{\ell+1}, t = 0)$ is the optimal solution of P_L where

$$y^{\ell+1} = (\overline{y}_B = -w/\overline{d}_k, \overline{y}_k = 1/\overline{d}_k, \quad \overline{y}_N = 0)$$

and $\overline{d}_k$ is given by (2.3).

– Let $(y^{\ell+1}, t = 0)$ be a basic optimal solution of P_L where $y^{\ell+1} = (\overline{y}_{B+1}, \overline{y}_N = 0)$ and let u be the column associated to the non-basic variable t; furthermore let $\overline{y}_B, u_B$ be the vectors obtained deleting the k-th component in $\overline{y}_{B+1}$ and u respectively, where k is the index corresponding to (3.7).

Then $x_B = \overline{b} - x_{N_k}w, x_{N_k} \geq 0$, is the equation of an extremum ray for problem P where

$$\overline{b} = -u_B + \overline{y}_B/t_0, \quad w = -\overline{y}_B/\overline{y}_k.$$

Remark 3.2. As we have pointed out in the proof of the previous theorem, when the variable t leaves the basis we find the optimal solution of P_L; this is not true if we do not apply criterium (3.3).

4. Concluding Remarks

In this paper we have suggested two algorithms for solving a linear fractional problem which can be interpreted as a modified version of

Martos and Charnes-Cooper algorithms; furthermore we have shown that these algorithms are equivalent in the sense given by theorem 3.1. This last result can be viewed as an extension of the one given by Wagner-Yuan for a compact feasible region.

As last remark, we point out that in studying post-optimality in linear fractional programming, the equivalence between the two algorithms allows us to choose the one which is more appropriate. More exactly, the Cambini-Martein algorithm seems to be more appropriate when the parameter appears in the right-hand side of the constraints, while the second algorithm seems to be more appropriate when the parameter appears in the coefficients of the numerator of the objective function.

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