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A SEQUENTIAL METHOD FOR A BICRITERIA PROBLEM ARISING IN PORTFOLIO SELECTION THEORY

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INTRODUCTION

In recent years multiobjective mathematical programming has been widely studied because of its applications in different optimization fields; in fact, in many real problems, one is usually confronted with several objectives which are in mutual conflict.

For instance, a production manager, who is responsible for the operations in a plant, does not always want to maximize his profits only. For strategic reasons, he may also pursue the goal of minimizing the utilization of scarse resources in order to avoid their consumption.

In this paper, we will consider a bicriteria problem, i.e. the problem of maximizing, in the sense given by Pareto, a pair of functions f_1, f_2. Problems of this kind, naturally arise in various areas of Economic and Social Sciences, such as portfolio selection problem where we are interested in minimizing the risk of investment and, at the same time, in maximizing the profit.

It is well known that when f_1 and f_2 are convex functions, the set of all efficient points can be generated by solving a scalar convex parametric problem whose objective function is a non-negative linear combination of f_1 and f_2 [Markowitz [3], Pang [7], Sharpe [11]]. Unfortunately such an approach cannot be extended to a non convex bicriteria problem, not even to the class of generalized-convex bicriteria problems, since the quasi-convexity of f_1 and f_2 does not imply the quasi-convexity of their non-negative linear combination. For this reason and because of its wide applications, recently, some authors [Martein [4,5], Choo [2], Schaible [10]] have suggested a different approach for the bicriteria problem in order to study the connectedness of the set E of all efficient points and, at the same time, to give algorithms for generating E.

In such new methods one of the two objective functions plays the role of a parametric constraint; in this way E can be generated by means of a suitable post-optimality analysis and the properties of the objective functions are not lost.

Following this idea, in this paper, we will establish, first of all, some theoretical results in order to characterize the set E for a wide class of bicriteria problems that is the class of bicriteria problems where at least one of the objective functions has the property that a local minimum is also global. Then we will suggest a sequential method for generating E in the classic case where one of the objective functions is strictly convex quadratic function and the other one is linear.

1. STATEMENT OF THE PROBLEM

Let us consider the bicriteria problem in the following form:

$IP: (\min f_1(x), \ \max f_2(x)), x \in R$

where f_1, f_2 are real-valued continuous functions, defined on a compact subset R of IR^n.

A point $x^0 \in R$ is said to be efficient for the bicriteria problem if there does not exist a point $x \in R$ such that the following inequalities $f_1(x) \leq f_1(x^0)$, $f_2(x) \geq f_2(x^0)$ both hold, where at least one is strict. Let E be the set of all efficient points of IP.

Recently some authors [Martein [5], Schaible [10]] have pointed out that E can be expressed as a suitable union of sets of optimal solutions of a parametric problem where one of the objective functions plays the role of a parametric constraint. Without loss of generality we will consider the following parametric problem:

$IP(\theta): z(\theta) \equiv \min f_1(x), \ x \in R(\theta) \equiv \{ x \in R : f_2(x) \geq \theta \}$

and from now on we will refer to $f_2(x) \geq \theta$ as the parametric constraint.

In this section, we will establish some theoretical results in order to characterize the set E of all efficient points for a wide class of bicriteria problems. More exactly we will consider the class $\mathcal{C}$ of bicriteria problems where f_1, f_2 are real-valued continuous functions, defined on a compact subset R of IR^n and where at least one of the objective functions[1] has the property that a local optimum is also global.

The following propositions establish some properties of the parametric problem IP (θ), which are useful in the sequel.

Set $\theta_0 = \max\limits_{f_1(x)=\alpha_0,\, x \in R} f_2(x)$ and $\theta_1 = \max\limits_{x \in R} f_2(x)$

where $\alpha_0 = \min\limits_{x \in R} f_1(x)$

[1] We will suppose, without loss of generality, that f_1 has the property that a local minimum point is also global.

PROPOSITION 1.1 Let f_1 and f_2 be real-valued continuous functions and let x^0 be an optimal solution of the parametric problem IP(θ). If x^0 is not binding to the parametric constraint , then x^0 is a local minimum point for the problem { min $f_1(x)$, $x \in R$}.

Proof: Since we have $f_1(x) \geq f_1(x^0)$, $\forall x \in R(\theta)$ and $f_2(x^0) > \theta$, for the continuity of the functions f_1 and f_2 there exists a neighbourhood U of x^0 such that $f_2(x) > \theta$ and $f_1(x) \geq f_1(x^0)$, $\forall x \in U \cap R$.

As an immediate consequence of the previous result we have the following proposition:

PROPOSITION 1.2 If the function f_1 does not have a local minimum point on R, different from the global one, then any optimal solution of the parametric problem IP (θ) is binding to the parametric constraint $\forall \theta \in [\theta_0, \theta_1]$.

Now we can state the following theorem which gives a characterization of the set E of all efficient points for the aforesaid class of bicriteria problems.

THEOREM 1.1 Let us consider problem IP where f_1 and f_2 are continuous functions defined on the compact set R and f_1 has the property that a local minimum point is also global. Then,

$$E = \bigcup_{\theta \in [\theta_0, \theta_1]} S(\theta) \tag{1.1}$$

where S (θ) is the set of optimal solutions for the problem IP (θ).

Proof : We must show that x^0 is an optimal solution for IP (θ) with $\theta \in [\theta_0, \theta_1]$ if and only if x^0 is an efficient point for IP.

If x^0 is not an efficient point for IP, then there exists $x \in R$ such that:

$$f_2(x) > f_2(x^0) \quad \text{and} \quad f_1(x) \leq f_1(x^0) \tag{1.2}$$

or

$$f_2(x) \geq f_2(x^0) \quad \text{and} \quad f_1(x) < f_1(x^0) \tag{1.3}$$

(1.3) contradicts the optimality of x^0, while (1.2) implies the existence of an optimal solution x for IP (θ) which is not binding to the parametric constraint and this contradicts Proposition 1.2. Let x^0 be an efficient point for problem IP; if x^0 is not an optimal solution of the parametric problem IP(θ) with $f_2(x^0) = \theta$, then there exists $x \in R$ such that $f_1(x) < f_1(x^0)$, $f_2(x) \geq f_2(x^0)$ and this is absurd taking into account (1.3). This completes the proof.

Taking into account that the class of convex quadratic functions and the class of semi-strictly quasi convex functions[2] belong to $\mathcal{C}$, we can obtain, as a corollary of the previous theorem, the following result stated in Martein [5] and Schaible [10]:

COROLLARY 1.1 Consider the bicriteria problem where f_1 is a semi-strictly quasi-convex function, f_2 is a continuous function and R is a compact set. Then:

$$E = \bigcup_{\theta \in [\theta_0, \theta_1]} S(\theta)$$

2. SOME THEORETICAL RESULTS

In this section we will give some theoretical results in order to suggest a sequential method for generating the set E of all efficient points for the bicriteria problem where, now, f_1 is a strictly convex quadratic function and f_2 is a linear function defined on a compact set R, that is, we will consider the following problem:

$$P: [\min (1/2\, x^T Q x + q^T x + q_0),\ \max c^T x],\quad x \in R \equiv \{x \in \mathbb{R}^n : Ax \geq b\}$$

where the symmetric matrix Q of order n is positive definite, $q \in \mathbb{R}^n$, $q_0 \in \mathbb{R}$, $c \in \mathbb{R}^n$, A is a real m x n matrix and $b \in \mathbb{R}^m$.

[2] A real-valued function f defined on a convex set X is called semi-strictly quasi-convex if for all $x_1, x_2 \in X$ such that $f(x_1) \neq f(x_2)$ the inequality $f(x) < \max (f(x_1), f(x_2))$ holds for all x on the open line segment $]x_1, x_2[$.

Set $M = \max_{x \in R} c^T x$, the parametric problem can be rewritten in the following way:

$P(\theta)$: $z(\theta) \equiv \min \{1/2\, x^T Q x + q^T x + q_0\}$, $x \in R(\theta)$
where $R(\theta) \equiv \{ x \in R : c^T x \geq M - \theta\, , \theta \geq 0\}$

Since $f_1(x)$ is a strictly convex quadratic function,the problem $\min_{x \in R} f_1(x) = \alpha_0$ has a unique solution x^*; furthermore for every $\theta \in [0,\ M-\alpha_0]$, the corresponding problem $P(\theta)$ has a unique optimal solution since $R(\theta)$ is a compact set.

Taking into account Theorem 1.1, the set E of all efficient points of P becomes:

$$E = \bigcup_{\theta \in [0,\, M-\alpha_0]} S(\theta)$$

Let us note that, when $M = \alpha_0$, E reduces to the singleton set $\{ x^* \}$

The idea of the sequential method that we are going to describe is to generate all optimal solutions of problem $P(\theta)$ for every θ in the interval $[0,\ M-\alpha_0]$, by means of post-optimality analysis starting from $\theta = 0$.

Since $P(\theta)$ is a strictly convex quadratic problem the optimal solution $x(\theta)$, the Lagrange multipliers $\lambda(\theta)$ associated with the constraints and Lagrange multiplier λ_0 associated with the parametric constraint are linear functions with respect to θ .

As we will see, λ_0 turns out to be a linear non-negative decreasing function of θ and we will point out that λ_0 will play the role of a parameter instead of θ. This choice will allow us to obtain a simple sequential method for generating E.

With this aim in mind, let $x^{(K)}$ be the optimal solution of the problem $P(\theta^{(K)})$, the matrix A and the vector b can be partitioned as: $A = \begin{bmatrix} B \\ N \end{bmatrix}$ and $b = \begin{bmatrix} b_B \\ b_N \end{bmatrix}$

where B is the submatrix of A corresponding to the set of the l constraints binding at $x^{(K)}$ (i.e. $B x^{(K)} = b_B$) and N is the submatrix corresponding to the set of the constraints not binding at $x^{(K)}$, i.e. $N x^{(K)} > b_N$.

Taking into account Proposition 1.2, the Karush - Kuhn - Tucker conditions for problem P(θ) can be written in the following way:

(2.1.a) $Q x - B^T \lambda - \lambda_0 c = - q$

(2.1.b) $B x = b_B$

(2.1.c) $N x > b_N$

(2.1.d) $c^T x = M - \theta$

(2.1.e) $\lambda \geq 0 \quad \lambda_0 \geq 0$

and the multiplier λ_0 associated with the parametric constraint is positive for every $\theta \in [0, M-\alpha_0]$. Let us suppose that the matrix B has full rank $p < n$; the particular case of rank equal to n will be discussed in Remark 2.1.

Let us note that, setting:

$$D^* = \begin{bmatrix} Q & -D^T \\ D & 0 \end{bmatrix}$$

where $D^T = [B^T \vdots c]$, (2.1. a, b, d) can be rewritten as:

$$D^* \begin{bmatrix} x \\ \lambda \\ \lambda_0 \end{bmatrix} = \begin{bmatrix} -q \\ b_B \\ M \end{bmatrix} + \theta \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

D^* is not singular if and only if $D^T Q^{-1} D$ is not singular. Since D has full rank and Q is symmetric definite positive , then $D^T Q^{-1} D \equiv D^0$ is a symmetric definite positive matrix. This implies that system (2.2) has a unique solution and furthermore the Lagrange multiplier λ_0 is of the kind: $\lambda_0 = -a_1 \theta + a_2$ where a_1 is positive since it is the reciprocal of the (l +1, l +1) element of D^0. This remark points out that λ_0 is a piece-wise linear decreasing function of θ and thus λ_0 can play the role of the parameter instead of θ ; in other words post-optimality for problem P(θ) can be carried on by studying the variation of x and λ as functions of λ_0 .

With this aim in mind, from (2.1.a) and (2.1.b), we have:

$$H \begin{bmatrix} x \\ \lambda \end{bmatrix} = \lambda_0 \begin{bmatrix} c \\ 0 \end{bmatrix} + \begin{bmatrix} -q \\ b_B \end{bmatrix}$$

where:

$$H = \begin{bmatrix} Q & -B^T \\ & \\ B & 0 \end{bmatrix};$$

and, since H is not singular, we get :

$$(2.3.a) \quad x = \lambda_0 \; u_1 + v_1$$
$$(2.3.b) \quad \lambda = \lambda_0 \; u_2 + v_2$$
$$(2.3.c) \quad \theta = -c^T u_1 \lambda_0 - c^T v_1 + M$$

where $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = H^{-1} \begin{bmatrix} c \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = H^{-1} \begin{bmatrix} -q \\ b_B \end{bmatrix}$

It is important to note that for Proposition 1.2 when λ_0 becomes 0, the linear constraint is not yet binding so that the current solution is not an efficient point and the algorithm stops.

Starting from the optimal solution $x^{(K)}$, given by (2.3.a), with the value $\lambda_0^{(K)}$, we want to decrease the value of λ_0 and find a value $\lambda_0^{(K+1)} \in [0 , \lambda_0^{(K)}]$ which guarantees not only the non-negativity of the solution and of the Lagrange multipliers λ, but also the feasibility of the current solution with respect to the non- active constraints, i.e.:

$$(2.4.a) \quad x(\lambda_0) = \lambda_0 \; u_1 + v_1 \geq 0$$
$$(2.4.b) \quad \lambda(\lambda_0) = \lambda_0 \; u_2 + v_2 \geq 0$$
$$(2.4.c) \quad x(\lambda_0) \in R$$

This allows us to claim that all the solutions :

$$x(\lambda_0) = \lambda_0 \; u_1 + v_1$$

with $\lambda_0 \in [\lambda_0^{(K+1)}, \lambda_0^{(K)}]$ are optimal for the problem $P(\theta)$ with $\theta = c^T x(\lambda_0) + M$ and, at the same time, they are efficient points for the bicriteria problem P.

The following theorem gives us a condition in order to find $\lambda_0^{(K+1)}$, given $x^{(K)}$ and the corresponding $\lambda_0^{(K)}$. Set $v_3 = N v_1 - b_N$, $u_3 = N u_1$ and consider the following sets of indices:

$$J_1 = \{ j : v_{1j} < 0 \} \quad \text{with } j = 1,2,\dots,n$$

$$J_2 = \{ j : v_{2j} < 0 \} \quad \text{with } j = 1,2,\dots,1$$

$$J_3 = \{ j : v_{3j} < 0 \} \quad \text{with } j = 1,2,\dots,m-1$$

where v_{1j}, v_{2j}, v_{3j} denote the j-th component of the vectors v_1, v_2, v_3 respectively; we have:

THEOREM 2.1 The vector $x(\lambda_0)$ is optimal for the problem $P(\theta)$ with $\theta = c^T x(\lambda_0) + M$ for every $\lambda_0 \in [\lambda_0^{(K+1)}, \lambda_0^{(K)}]$ where $\lambda_0^{(K+1)} = \max\{\lambda_{01}, \lambda_{02}, \lambda_{03}\}$ and

$$\lambda_{01} = \begin{cases} \max\limits_{j \in J_1} - v_{1j} / u_{1j} & \text{if } J_1 \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_{02} = \begin{cases} \max\limits_{j \in J_2} - v_{2j} / u_{2j} & \text{if } J_2 \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_{03} = \begin{cases} \max\limits_{j \in J_3} - v_{3j} / u_{3j} & \text{if } J_3 \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

where u_{1j}, u_{2j}, u_{3j} denote the j-th component of the vectors u_1, u_2, u_3 respectively.

Proof: From the Karush - Kuhn - Tucker conditions applied to problem $P(\theta)$, taking into account (2.3), $x(\lambda_0)$ is optimal for $P(\theta)$ with $\theta = c^T x(\lambda_0) + M$ if the following conditions hold:

a) $x(\lambda_0) = \lambda_0 u_1 + v_1 \geq 0$
b) $\lambda(\lambda_0) = \lambda_0 u_2 + v_2 \geq 0$
c) $N x(\lambda_0) = \lambda_0 N u_1 + N v_1 \geq b_N$ that is : $\lambda_0 u_3 \geq v_3$.

Now, we will show that a) holds for any $\lambda_0 \in [\lambda_{01}, \lambda_0^{(K)}]$. In fact, if $v_{1j} \geq 0$ and $u_{1j} \geq 0$, then $x_j(\lambda_0) \geq 0$ for any $\lambda_0 \geq 0$; if $v_{1j} > 0$ and $u_{1j} < 0$ then $x_j(\lambda_0) \geq 0$ is true for any $\lambda_0 \leq -v_{1j}/u_{1j}$ and since $x^{(K)}$ is optimal for $P(\theta^{(K)})$ with $\theta^{(K)} = c^T x^{(K)} + M$, then condition a) is satisfied for $\lambda_0 = \lambda_0^{(k)}$ so that $\lambda_0^{(k)} \leq -v_{1j}/u_{1j}$ and $x_j(\lambda_0) \geq 0$ for any $\lambda_0 \leq \lambda_0^{(k)}$. Consider now the case $v_{1j} \leq 0$. Since $x(\lambda_0^{(k)}) \geq 0$ we necessarily have $u_{1j} > 0$, so that $x(\lambda_0) \geq 0$ for any $\lambda_0^{(k)} \geq -v_{1j}/u_{1j}$. As a consequence, condition a) is satisfied for any $\lambda_0 \in [\lambda_{01}, \lambda_0^{(K)}]$. In a similar way we can prove that conditions b) and c) hold for any $\lambda_0 \in [\lambda_{02}, \lambda_0^{(K)}]$ and for any $\lambda_0 \in [\lambda_{03}, \lambda_0^{(K)}]$ respectively. Obviously, all the conditions a), b) and c) are satisfied for any $\lambda_0 \in [\lambda_0^{(K+1)}, \lambda_0^{(K)}]$ where $\lambda_0^{(K+1)} = \max\{\lambda_{01}, \lambda_{02}, \lambda_{03}\}$. This completes the proof.

Remark 2.1 : When B has full rank equal to n, (2.3) become:

$$x = B^{-1}b_B$$
$$\lambda = [-(B^T)^{-1}c]\lambda_0 + [(B^T)^{-1}q + (B^T)^{-1}QB^{-1}b_B]$$
$$\theta = -c^T B^{-1}b_B + M$$

Let us note that x and θ are non-negative and they are independent of λ_0. The non-negativity condition on Lagrange multipliers allows us to know the constraint which must be deleted; so that the new matrix B' corresponding to the binding constraints again has rank less than n and we are in the previous case.

Remark 2.2 : Taking into account Theorem 2.1, if we know $x^{(k)}$ we can obtain $x^{(k+1)}$. In order to find the starting point, we calculate the optimal solution x^0 of the linear problem $P_L = \{\max c^T x, \ x \in R\}$. Since we will suppose that P_L has a unique solution[3], then the matrix B has full rank equal to n and we are in the case of Remark 2.1 with $\theta = 0$.

Let us note that $x^{(0)}$ is optimal solution for problem P(0), since the feasible region R(0) reduces to the singleton set $\{x^{(0)}\}$.

[3] The case of alternate optimal solutions will be considered in section 5.1.

3. THE ALGORITHM

In the previous sections, we established some theorical results, which now allow us to propose a simple algorithm to solve bicriteria problem P. The algorithm steps are stated with the aim of clarifing the computational aspects[4].

We will suppose, for the sake of simplicity, that degeneracy does not occur and that the linear problem P_L has a unique solution. Some special cases will be discussed in the following section.

STEP 0: Solve the linear problem $P_L = \{ \max c^T x, x \in R\}$. Let $x^{(0)}$ be the optimal solution, B the matrix of coefficients of constraints binding at $x^{(0)}$, N the matrix of the other constraints, b_B and b_N the corresponding right-hand side. Set K=0 and go to STEP 1.

STEP 1: Build the matrix:

$$H = \begin{bmatrix} Q & -B^T \\ B & 0 \end{bmatrix}$$

and the vectors: $h_1 = \begin{bmatrix} c \\ 0 \end{bmatrix}$ $h_2 = \begin{bmatrix} -q \\ b_B \end{bmatrix}$

Calculate the inverse of matrix H and the vectors:

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = H^{-1} \begin{bmatrix} c \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = H^{-1} \begin{bmatrix} -q \\ b_B \end{bmatrix},$$

$v_3 = N v_1 - b_N$ and $u_3 = N u_1$. Go to STEP 2.

STEP 2: Calculate $\lambda_0^{(K+1)} = \max \{ \lambda_{01}, \lambda_{02}, \lambda_{03} \}$ where:

$$\lambda_{01} = \begin{cases} \max\limits_{j \in J_1} - v_{1j} / u_{1j} & \text{if } J_1 \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_{02} = \begin{cases} \max\limits_{j \in J_2} - v_{2j} / u_{2j} & \text{if } J_2 \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

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[4] A Fortran implementation of this algorithm is running on IBM 30/90-VM computer of CNUCE Institute of Pisa.

$$\lambda_{03} = \begin{cases} \max\limits_{j \in J_3} - v_{3j} / u_{3j} & \text{if } J_3 \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

$v_3 = N v_1 - b_N$, $u_3 = N u_1$, $J_1 = \{ j : v_{1j} < 0 \}$, $j = 1,2,\ldots,n$;
$J_2 = \{ j : v_{2j} < 0 \}$, $j = 1,2,\ldots,l$,
$J_3 = \{ j : v_{3j} < 0 \}$, $j = 1,2,\ldots,m-l$. The new solution is $x^{(K+1)} = \lambda_0^{(K+1)} u_1 + v_1$. Go to STEP 3.

STEP 3: If $x^{(K+1)}$ is equal to $x^{(K)}$, set $S^{(k+1)} = \{ x^{(K+1)} \}$ and go to STEP 4; otherwise, set $S^{(k+1)} = \{ x : x = [\lambda_0 u_1 + v_1], \lambda_0^{(K+1)} \leq \lambda_0 < \lambda_0^{(K)} \}$ and go to STEP 4.

STEP 4: If $\lambda_0^{(K+1)} = 0$, the points of the set $E = \{ \bigcup_i S^{(i)}, i = 1,\ldots,K+1 \}$ are efficient for the problem P, STOP ; otherwise go to STEP 5.

STEP 5: If $\lambda_0^{(K+1)}$ is equal to λ_{02}, delete the constraint corresponding to $\lambda_0^{(K+1)}$ in matrix B and add it in matrix N, set $k=k+1$ and go to STEP 1; otherwise, add the constraint corresponding to $\lambda_0^{(K+1)}$ in matrix B and delete it in matrix N, set $k=k+1$ and go to STEP 1.

4. A NUMERICAL EXAMPLE

Let us consider the problem:
P_1: $(\max\ x_1 + 2x_2,\ \min\ x_1^2 + x_2^2 - 8x_1 - 8x_2)$,
with $x \in R = \{ x \in IR^2 : 2 \leq x_1 \leq 10,\ 3 \leq x_2 \leq 12 \}$.

Applying the algorithm, the following sequence of steps is obtained:

At **STEP 0**, the linear problem:
P_L: $\max\ x_1 + 2x_2$, $x \in R = \{ x \in IR^2 : 2 \leq x_1 \leq 10,\ 3 \leq x_2 \leq 12 \}$
has the unique solution $x^{(0)} = (10,12)$, thus we have:

$$B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad b_B = \begin{bmatrix} -10 \\ -12 \end{bmatrix}$$

$$N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad b_N = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Set K=0 and go to STEP 1.

At **STEP 1**, we built:

$$H = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \qquad h_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \quad h_2 = \begin{bmatrix} 8 \\ 8 \\ -10 \\ -12 \end{bmatrix}$$

The inverse of H is:

$$H^{-1} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

so that we have:

$$u = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \quad v = \begin{bmatrix} 10 \\ 12 \\ -12 \\ -16 \end{bmatrix} \quad u_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$$

At **STEP 2**, $\lambda_0^{(1)} = \max(\lambda_{01}, \lambda_{02}, \lambda_{03}) = 12$, since $\lambda_{01} = 0$, $\lambda_{02} = \max(12, 8) = 12$, $\lambda_{03} = 0$. The new solution is:

$$x^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \lambda_0^{(1)} + \begin{bmatrix} 10 \\ 12 \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \end{bmatrix}$$

At **STEP 3**, since $x^{(1)}$ is equal to $x^{(0)}$, set $S^{(1)} = \{ (10,12) \}$ and go to STEP 4.

At **STEP 4**, since $\lambda_0^{(1)}$ is not equal 0, go to Step 5.

At **STEP 5**, since $\lambda_0^{(1)} = \lambda_{02}$ we must delete the row of B corresponding to $x_1 \leq 8$ and add it in matrix N, i.e.:

$B = [0 \;\; -1]$ $\quad b_B = [-12]$ $\quad N = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad b_N = \begin{bmatrix} 2 \\ -10 \\ 3 \end{bmatrix}$

set k=1 and go to STEP 1.

At **STEP 1**, we have :

$$H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix} \qquad h_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad h_2 = \begin{bmatrix} 8 \\ 8 \\ -12 \end{bmatrix}$$

The inverse of H is:

$$H^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

so that we have:

$$u = \begin{bmatrix} 1/2 \\ 0 \\ 2 \end{bmatrix} \quad v = \begin{bmatrix} 4 \\ 12 \\ -16 \end{bmatrix} \quad u_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 2 \\ 6 \\ 9 \end{bmatrix}$$

At **STEP 2**, $\lambda_0^{(2)} = \max\{\lambda_{01}, \lambda_{02}, \lambda_{03}\} = 8$, since $\lambda_{01} = 0$, $\lambda_{02} = \max\{8\} = 8$, $\lambda_{03} = 0$. The new solution is:

$$x^{(2)} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \lambda_0^{(2)} + \begin{bmatrix} 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$$

At **STEP 3**, since $x^{(2)} \neq x^{(1)}$, set

$$S^{(2)} = \{ x : x = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \lambda_0 + \begin{bmatrix} 4 \\ 12 \end{bmatrix}, \; 8 \le \lambda_0 < 12 \}.$$

At **STEP 4**, since $\lambda_0^{(1)}$ is not equal 0, go to Step 5.

At **STEP 5**, since $\lambda_0^{(2)} = 8 = \lambda_{02}$, we must delete the row of B corresponding to the constraint $-x_2 \geq -12$ and add it in matrix N, i.e.:

$$N = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad b_N = \begin{bmatrix} 2 \\ -10 \\ 3 \\ -12 \end{bmatrix}$$

Set K=2 and go to STEP 1.

At **STEP 1**, we have:

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad h_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad h_2 = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$$

The inverse of H is:

$$H^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

so that we have:

$$u = \begin{bmatrix} 12 \\ 1 \end{bmatrix} \qquad v = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \quad u_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \\ -1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 2 \\ 6 \\ 1 \\ 8 \end{bmatrix}$$

At **STEP 2**, $\lambda_0^{(3)} = \max(\lambda_{01}, \lambda_{02}, \lambda_{03}) = 0$, since $\lambda_{01} = 0$, $\lambda_{02} = 0$, $\lambda_{03} = 0$. The new solution is:

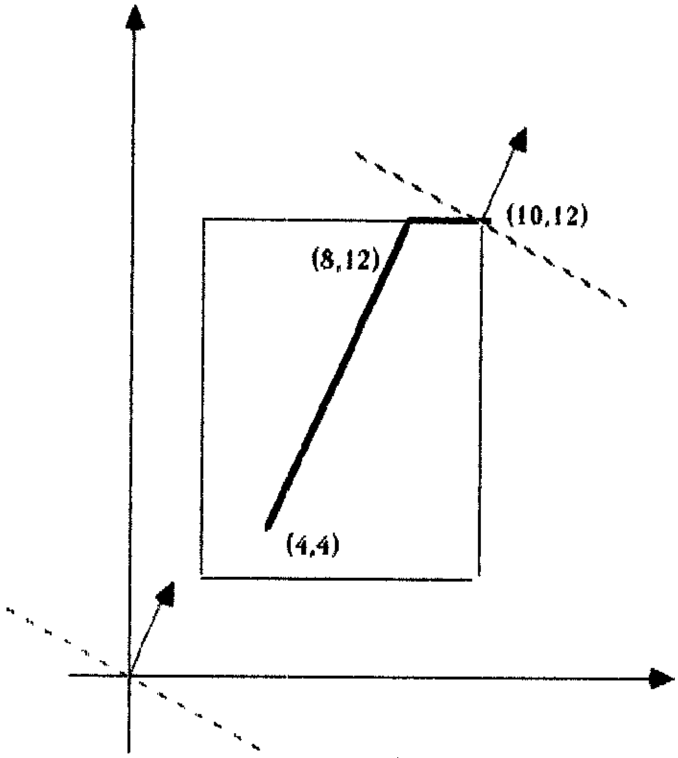
$$x^{(3)} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \lambda_0^{(3)} + \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

At **STEP 3**, $x^{(3)} \neq x^{(2)}$, then set:

$$S^{(3)} = \{ x : x = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \lambda_0 + \begin{bmatrix} 4 \\ 4 \end{bmatrix}, 0 \leq \lambda_0 < 8 \}$$

At **STEP 4**, since $\lambda_0^{(3)} = 0$, the points of the set $E = \{ \bigcup_i S^{(i)}, i = 1,2,3 \}$ are efficient for the problem P_1, the algorithm stops.

In the following picture , the feasible region R is depicted. The efficient points of P_I generated with the proposed algorithm are represented by the bold lines.



5. SPECIAL CASES

In this section, we will consider the following special cases which are important because they can be found in many applications:

1) Problem P_L has alternate optimal solutions,
2) Some of the linear constraints are in equality form.

5.1 Alternate optimal solutions

As we have pointed out in Remark 2.2, the starting point of the proposed algorithm is the unique solution of the linear problem P_L, since it is also an efficient point for P. When P_L has alternate optimal solutions, set $S_0 = \{ x \in R : \max c^T x \}$, we must choose a point, belonging to S_0, which is also efficient for P; it is easy to verify that now the starting point is the minimum of the quadratic function over the set S_0.

Let us note that, taking into account Remarks 2.1 and 2.2 for every $x \in S_0$, we have:

$$x = B^{-1}b_B$$
$$\lambda = [-(B^T)^{-1}c]\lambda_0 + [(B^T)^{-1}q + (B^T)^{-1}QB^{-1}b_B]$$
$$\theta = 0$$

Let x_L be a solution of P_L and set $J_a = \{ j : u_{2j} = 0 \text{ and } v_{2j} < 0 \}$; if J_a is empty, taking into account Theorem 2.1, we can find λ_0 which verifies conditions (2.4) and x_L turns out to be an efficient point for P. The algorithm can be applied, setting $x^{(0)} = x_L$.

Otherwise, if J_a is not empty, there is one or more λ_j which assumes negative value indipendently by λ_0.

In order to find $x^{(0)}$, the idea (see [6]) is to perturbate the linear objective function in such a way that the corresponding linear problem has a unique solution. Consider the problem:

$$P_L' \ : \max (c + \varepsilon c')^T x$$

where c' [5] is such that x_L is the unique solution for P'_L.

The solution of the perturbated problem $P_\varepsilon(\theta)$ becomes:

$$x(\lambda_0) = \lambda_0 (\varepsilon u'_1) + v_1$$
$$\lambda(\lambda_0) = \lambda_0 (u_2 + \varepsilon u'_2) + v_2$$

where $\begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = H^{-1} \begin{bmatrix} c' \\ 0 \end{bmatrix}$

Theorem 2.1 can be reformulated in the following way. Set:

$J'_1 = \{ j : u'_{1j} > 0 , v_{1j} < 0 \}$ with $j = 1,2,\dots,n$

$J'_2 = \{ j : u'_{2j} > 0 , j \in J_a \}$ with $j = 1,2,\dots,l$

---

5 A suitable choice for c' may be $c' = \Sigma_j a^{(i)}$, which $a^{(i)}$ denotes the gradient of the i-th linear constraint binding at x_L.

$J'_3 = \{ j : u'_{3j} > 0 , v_{3j} < 0 \}$ with $j = 1,2, \ldots , m-1$

where $u'_3 = N u'_1$ and $v_{1j}, v_{2j}, v_{3j}, u'_{1j}, u'_{2j}, u'_{3j}$ denote th j-th component of the vectors $v_1, v_2, v_3, u'_1, u'_2, u'_3$ respectively.
We have:

THEOREM 5.1 The vector $x(\lambda_0)$ is optimal for the problem $P_\varepsilon(\theta)$ with $\theta = (c + \varepsilon c')^T x(\lambda_0) + M$ for every $\lambda_0 \in [\lambda_0', \lambda_0'']$ where $\lambda_0' = \max \{\lambda'_{01}, \lambda'_{02}, \lambda'_{03}\}$, λ_0'' is the previous value of λ_0 and

$$\lambda'_{01} = \begin{cases} \max\limits_{j \in J'_1} - v_{1j} / \varepsilon u'_{1j} & \text{if } J'_1 \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda'_{02} = \begin{cases} \max\limits_{j \in J'_2} - v_{2j} / \varepsilon u'_{2j} & \text{if } J'_2 \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda'_{03} = \begin{cases} \max\limits_{j \in J'_3} - v_{3j} / \varepsilon u'_{3j} & \text{if } J'_3 \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Proof: the proof is the same as Theorem 2.1. Taking into account the arbitrary of ε, it is sufficient to note the maximum is reached when $u_{2j} = 0$.

Corollary 5.1 : If λ'_0 is equal to 0, then $x(\lambda'_0)$ is the optimal solution for $P(\theta)$ with $\theta=0$.
Proof : Taking into account Theorems 5.1 and 2.1, the statement hold since λ'_0 minimizes the quadratic function for every $x \in S_0$.

Remark 5.1 : Taking into account Corollary 5.1, until $\lambda'_0 > 0$, we must add or delete a row to or from B, respectively if $\lambda'_0 = \lambda'_{03}$ or λ'_{02} and set $\lambda''_0 = \lambda'_0$. When λ'_0 becomes equal to 0, $x(\lambda'_0)$ is an efficient point for problem P , thus setting $\varepsilon=0$, $x^{(0)} = x(\lambda'_0)$. The algorithm, proposed in section 3, can be restored.

5.2 Linear equality constraints

Now, we will show how to solve problem P when R is defined by one or more linear equality constraints.

Let us note that if for every equality constraint we introduce the corresponding two inequalities, the matrix H becomes singular and the proposed algorithm cannot be applied.

Thus, we will suppose that the matrix A and the vector b can be partitioned as :

$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ where A_1 is the submatrix of A corresponding to the set of the m_e equality constraints (i.e. $A_1 x = b_1$) and A_2 is the submatrix of A corresponding to the set of the $m-m_e$ inequality constraints (i.e. $A_2 x \geq b_2$).

Since all the solutions of problem P must be binding at all linear equality constraints, at every iteration of the algorithm, the matrix B must contain the submatrix A_2, even if the Lagrange multipliers, corresponding to these equality constraints, become negative.

So that, at the Step 2, the set J_2 becomes:

$$J_2 = \{ j : v_{2j} < 0 \text{ and } j \notin J^* \}, \ j = 1,2,\ldots,l$$

where J^* is the set of the indices corresponding to the equality constraints.

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