Report n.32

Solving a Quadratic Fractional Program by means of a Complementarity Approach

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Pisa, 1990

This research was supported in part by the Ministry of Public Education.

SOLVING A QUADRATIC FRACTIONAL PROGRAM BY MEANS OF A COMPLEMENTARITY APPROACH

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INTRODUCTION

In some recent literature regarding Portfolio Selection Theory, mathematical models expressed as Fractional Quadratic Problem or rather constrained problems in which a function subject to linear constraints is optimized and expressed as a ratio between a quadratic and a linear function have been suggested [11,13,14].

For a number of classes of these kinds of problems, interesting methods of solution applied to appropriate parametric quadratic problems have been proposed [4,5,6]; the strict convexity of the quadratic function plays an essential role in these methods, and as a result, they are not applicable in cases where the quadratic form is positive semi-definite.

All this has led to research which aims at the identification of an algorithm able to solve the fractional quadratic problem when the quadratic form is convex.

Taking into account that a convex quadratic programming problem can be transformed into a Linear Complementarity Problem as well as recent developments in the Linear Complementarity Problem field as regards both the theoretical and algorithmic- computational aspects, in this paper we will, first of all, prove how a Fractional Quadratic Problem can be expressed by means of a parametric complementarity linear problem and then how it is possible to generate a sequential method for solving this problem by means of some optimality conditions established in section 2 and a post-optimality analysis.

1. LINEAR COMPLEMENTARITY PROBLEM

As we have just outlined the aim of this paper is to solve a quadratic fractional programming arising in portfolio selection theory by means of a suitable complementarity approach.

In this section we will review one of the most popular methods for solving a linear complementarity problem, i.e. Lemke's algorithm.

Let us note that the sequential method that we are going to describe in section 3 does not depend on the algorithm used for solving a Linear Complementarity Problem, since it can be easily adjusted to any other complementarity method.

Before giving the main steps of Lemke's algorithm, we will recall the general formulation of a Linear Complementarity Problem [10]:

(1.1.a) $\quad w - M z = q$

(1.1.b) $\quad z, \; w \geq 0$

(1.1.c) $\quad z^T w = 0$

where M is a n x n matrix, $w, z, q \in IR^n$.

Lemke's Algorithm

Let us assume that one or more components of vector q are less than 0, otherwise $w = q$, $z = 0$ is a solution of (1.1). An artificial variable associated with the column vector $-e_n$ ($e_n \in IR^n$ is the column vector of all 1's) is introduced so that system (1.1) becomes:

$$
\begin{aligned}
&(1.2.a) \quad && w - Mz - e_n z_0 = q \\
&(1.2.b) && z,\ w \geq 0 \\
&(1.2.c) && z_0 \geq 0 \\
&(1.2.d) && z^T w = 0
\end{aligned}
$$

The Complementary Pivot Algorithm moves among feasible basic vectors for system (1.2.a,b,c) by means of simplex-like pivot operations.

We say that a feasible basic vector for (1.2.a,b,c) is a Complementary Feasible Basic Vector if there is exactly one basic variable for each complementary pair (w_j,z_j) and a feasible basic vector for (1.2.a,b,c) is an Almost Complementary Feasible Basic Vector if it satisfies the following properties :

i) There is at most one basic variable for each complementary pair of variables (w_j, z_j);

ii)It contains exactly one basic variable for each of (n-1) complementary pair of variables, and both the variables in the remaining complementary pair are non basic;

iii) z_0 is a basic variable in it.

Let us note that all the basic vectors obtained in the algorithm with the possible exception of the final basic vector are almost complementary feasible basic vectors and the algorithm generates a finite sequence of almost complementary feasible basic vectors.

The main property of the path generated by the algorithm is the following. Each Basic Feasible Solution obtained in the algorithm has two almost complementary edges containing it. We arrive at this solution along one of these edges and we leave it by the other edge. So the algorithm continues in a unique manner. It is also clear that a basic vector that was obtained in some stage of the algorithm can never reappear.

There are exactly two possible ways in which the algorithm can terminate:

1. At some stage of the algorithm, z_0 may drop out of the basic vector, or become equal to zero in the basic feasible solution of (1.2). If (w', z', $z_0' = 0$) is the basic feasible solution of (1.2) at that stage, then (w', z') is a solution of the Linear Complementarity Problem (1.1).
2. At some stage of the algorithm, z_0 may be strictly positive in the Basic Feasible Solution of (1.2), and the pivot column in that stage may turn out to be non-positive, and in this case the algorithm terminates with another almost complementary extreme half line (distinct from the initial almost complementary extreme half line). This is called ray termination.

Let us observe that when ray termination occurs, in the general case the algorithm is unable to solve the Linear Complementarity Problem, even if a solution to the Linear Complementarity Problem exists . However, when M satisfies some conditions (in particular when M is positive semi-definite matrix) it can be proved that ray termination in the algorithm will only occur when (1.1) has no solution.

2. SOME OPTIMALITY CONDITIONS FOR THE QUADRATIC FRACTIONAL PROBLEM

The quadratic fractional problem, that is the problem of minimizing the ratio between a quadratic and an affine function, is important since it naturally arises in many fields of economics and finance (such as Portfolio Selection Theory [11,13,14]).

We will consider the quadratic fractional problem in the following form:

$$\mathcal{P} : \quad \min F(x) = \frac{Q\,(x)}{D\,(x)} \quad , x \in R \equiv \{ x \in IR^n : Ax \geq b , x \geq 0\}$$

where: $Q\,(x) = 1/2\ x^T Q\, x + q^T x + q_0$, $D\,(x) = d^T x + d_0$, Q is a symmetric positive semi-definite n x n matrix, $q, d, x \in IR^n$, q_0, $d_0 \in IR$, A is a real m x n matrix and $b \in IR^m$.

In order to avoid trivial cases, we will suppose that $d^T x + d_0 > 0$ $\forall\ x \in R$.

Problem $\mathcal{P}$ has some interesting properties, which are summarized in the following Theorem [1,2,3,12] :

Theorem 1.1 : Let us consider the minimization problem $\mathcal{P}$. The following properties hold:

i) The objective function F(x) is semi-strictly quasi convex[1];

ii) The set of all optimal solutions of $\mathcal{P}$ is convex[2];

iii) A local minimum point for $\mathcal{P}$ is also global;

iv) A feasible point for which the Karush-Kuhn-Tucker conditions are verified is an optimal solution for $\mathcal{P}$.

The fractional quadratic problem has been studied by some authors [4,5,6,9] when Q (x) is a strictly convex quadratic function; for this particular case, a sequential method, which finds the optimal solution in a finite number of iterations, has been

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1 A real-valued function f defined on a convex set X is called semi-strictly quasi-convex if for all $x_1, x_2 \in X$ such that $f(x_1) \neq f(x_2)$ the inequality $f(x) < \max(f(x_1), f(x_2))$ holds for all x on the open line segment $]\, x_1, x_2\, [$.

2 We suppose without loss of generality that the empty set is convex.

suggested in [4,5,6]. In this method the non-singularity of matrix Q plays an important role, so that it cannot be extended to the convex case. It is for this reason that we will generalize some ideas given in [4,7,8,12] and suggest a new algorithm for solving $\mathcal{P}$ by means of complementarity approach.

With this aim in mind let us note that an optimal solution $\underline{x}$ of problem $\mathcal{P}$ is also optimal for the problem:

$$\begin{cases} \dfrac{1}{D(\underline{x})} \min\ Q(x) \\ D(x) = D(\underline{x}),\ x \in R \end{cases}$$

so that solving problem $\mathcal{P}$ is equivalent to finding the optimal value $\underline{\theta} = D(\underline{x})$ in the parametric problem:

$$P(\theta): \; Z(\theta) \equiv \{ \min \frac{1}{\theta} Q(x), \quad x \in R(\theta) \}$$

where $R(\theta) \equiv \{ x \in R : D(x) = \theta \}$.

We say that θ is a feasible level if $R(\theta) \neq \varnothing$.

Let us note that $P(\theta)$, for any θ, is a convex quadratic problem and thus, as is well known, its optimal solutions, if one exists, can be found by solving a suitable linear complementarity problem.

Now we will describe a procedure which, starting from a feasible level θ_0, allows us to verify, by means of sensitivity analysis applied to problem $P(\theta_0 + \theta)$ and by means of a suitable optimality criterium, if θ_0 is the optimal level; if not, to find a new level such that $Z' < Z_0$ where Z' and Z_0 are the minimum value of the objective function of problem $P(\theta')$ and $P(\theta_0)$, respectively.

To this aim consider the following linear complementarity problem obtained by applying Karush-Kuhn-Tucker conditions to problem $P(\theta_0 + \theta)$:

$$(2.1)\quad \begin{pmatrix} u \\ v \\ v_0 \end{pmatrix} - \begin{pmatrix} Q & -A^T & -d \\ A & 0 & 0 \\ d^T & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \lambda_0 \end{pmatrix} = \begin{pmatrix} q \\ -b \\ d_0 - \theta_0 \end{pmatrix} + \theta \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

where λ_0 is the Lagrange multiplier associated with the parametric constraint $D(x) = \theta$, y and u are the vectors of Lagrange multipliers associated with the constraints of the kind $Ax \geq b$ and with the non-negativity constraints, respectively, $v = b-Ax$ is the vector of slack variables associated with the constraint of the kind $Ax \geq b$ and $v_0= d_0-\theta_0- d^Tx_0$ is the slack variable associated with the parametric constraint.

At the beginning, we can choose θ_0 as the minimum value of the linear programming problem $\{\min D(x), x \in R\}$. Let us note that θ_0 exists since $D(x)$ is lower-bounded on R.

Solving (2.1) by means of Lemke's algorithm (or another similar method), we find a solution of this kind:

(2.3.a) $x(\theta) = x_0 + \alpha\theta$

(2.3.b) $y(\theta) = y_0 + \delta\theta$

(2.3.c) $\lambda_0(\theta) = \lambda_0 + \beta\theta$

where x_0 is the optimal solution of $P(\theta_0)$ and y_0, λ_0 are the Lagrange multipliers associated with. Obviously, $x(\theta)$ is optimal for $P(\theta)$ for every $\theta \in H(\theta) = \{\theta : x(\theta) \in R\} \cap \{\theta : y(\theta) \geq 0\}$. We will refer to such a solution as an optimal level solution.

Let us note that the multiplier λ_0 associated with the parametric constraint is not restricted in sign.

Set $\theta_1 = \sup_{x \in R} [D(x)]$ and observe that:

$$(2.4)\quad \inf_{x\in R} [F(x)] = \inf_{\theta} \inf_{x \in R(\theta)} [F(x)] = \inf_{\theta} [Z(\theta)]$$

The first idea of the sequential method that we are going to describe in section 3 is to generate all optimal level solutions of the problem P (θ), when the parameter θ assumes increasing values in the interval $[\theta_0,\theta_1]$, (or in the half-line) starting from θ_0, until the optimal level $\underline{\theta}$ is found.

Now we are able to establish some optimality conditions for problem $\mathcal{P}$. The following Lemma and Theorem hold.

LEMMA 2.1: It results:

i) $d^T \alpha = 1, \quad M \alpha = 0, \quad \alpha^T Q x_0 = - \alpha q^T + \lambda_0, \quad \alpha^T Q \alpha = \beta$

ii) $$Z(\theta) = \frac{1/2 \, \beta \, \theta^2 + \lambda_0 \theta + \theta_0 Z_0}{\theta_0 + \theta}$$

where $Z_0 = (1/2 \; x_0^T Q x_0 + q^T x_0 + q_0) / \theta_0$

Proof:

i) It follows from Karush-Kuhn-Tucker conditions for the problem P (θ), calculated in (2.3.a) (see [5]);

ii) It is obtained by substituting (2.3.a) in Z (θ), taking into account condition i).

THEOREM 2.1:

a) x_0 is optimal for problem $\mathcal{P}$ if one of the following conditions is verified:

i) $\beta = 0$ and $\lambda_0 \geq Z_0$;

ii) $\beta > 0$ and $\Delta \leq 0$;

iii) $\beta > 0$, $\lambda_0 \geq Z_0$ and $\Delta > 0$

where $\Delta = \beta^2 \theta_0^2 - 2 \beta \theta_0 (\lambda_0 - Z_0)$

b) x (θ^*) is optimal for problem $\mathcal{P}$ if the following condition is verified:

i) $\beta > 0$, $\lambda_0 < Z_0$, $\Delta > 0$ and $\theta^* \in H(\theta)$

where $\theta^* = - \theta_0 + \sqrt{\Delta} / \beta$

c) $Z(\theta)$ is a decreasing function in $H(\theta)$ if one of the following conditions is verified:

i) $\beta > 0$, $\lambda_0 < Z_0$, $\Delta > 0$ and $\theta^* \notin H(\theta)$

ii) $\beta = 0$ and $\lambda_0 < Z_0$; Furthermore if SUP $[H(\theta)] = +\infty$

then the problem has not optimal solutions and

INF $[Q(x)] = \lambda_0$, $x \in R$.

<u>**Proof**</u>: Taking into account (2.4), the thesis follows from inspection of the sign of function $Z'(\theta)$, where:

$$Z'(\theta) = \frac{1/2\, \beta\, \theta^2 + \beta\, \theta_0 \theta + \theta_0 (\lambda_0 - Z_0)}{(\theta_0 + \theta)^2}$$

Let us note that , from i) of Lemma 2.1, we have $\beta = \alpha^T Q \alpha$ and it results non-negative since Q is a positive semi-definite matrix; furthermore, in the case ii) of c) we have:

$$Z(\theta) = \frac{\lambda_0 \theta + \theta_0 z_0}{\theta_0 + \theta}$$

so that $\lim_{\theta \to +\infty} Z(\theta) = \lambda_0$

3. A SEQUENTIAL METHOD FOR SOLVING PROBLEM $\mathcal{P}$

The results given in the previous sections allow us to suggest a very simple algorithm for solving problem $\mathcal{P}$.

For sake of simplicity we will refer to a compact feasible region.

The idea of the sequential method that we are going to describe is to generate by means of a Complementary Method a feasible basic solution for the problem $P(\theta)$ and apply sensitivity analysis in (2.3). If one of the optimality conditions given in Theorem 2.1 is verified, then an optimal solution of problem $\mathcal{P}$ is found; otherwise we generate the next feasible basic solution.

Since the feasible basic solutions are finite, the method converges after a finite number of iterations.

Now we are able to describe the algorithm steps:

STEP 0: Solve the linear problem $\{\min D(x),\ x \in R\} = \theta_0$ and consider the linear complementarity problem:

$$\begin{matrix} u \\ v \\ v_0 \end{matrix} - \begin{matrix} Q & -A^T & -d \\ A & 0 & 0 \\ d^T & 0 & 0 \end{matrix} \begin{matrix} x \\ y \\ \lambda_0 \end{matrix} = \begin{matrix} q \\ -b \\ d_0 - \theta_0 \end{matrix} + \theta \begin{matrix} 0 \\ 0 \\ -1 \end{matrix}$$

set k=1 and go to STEP 1;

STEP 1 : Solve the linear complementarity problem with a complementarity method (such as Lemke's algorithm)[1].

A solution of this kind:

$$x^{(k)}(\theta) = x_0 + \alpha\,\theta$$

$$y^{(k)}(\theta) = y_0 + \delta\,\theta$$

$$\lambda_0^{(k)}(\theta) = \lambda_0 + \beta\,\theta$$

is found. Calculate $H^{(k)}(\theta) = \{\theta : x(\theta) \in R\} \cap \{\theta : y(\theta) \geq 0\}$ and go to STEP 2;

---

[1] At the first step, degeneracy occurs; in order to apply our method it is sufficient to have the multiplier λ_0 associated with the parametric constraint as a basic variable.

STEP 2 : If one of the following conditions is verified:

i) $\beta = 0$ and $\lambda_0 \geq Z_0$;

ii) $\beta > 0$ and $\Delta \leq 0$;

iii) $\beta > 0$, $\lambda_0 \geq Z_0$ and $\Delta > 0$, where:

$$\Delta = \beta^2 \theta_0{}^2 - 2 \beta \theta_0 (\lambda_0 - Z_0)$$

then x_0 is optimal for problem $\mathcal{P}$ and STOP;

If the following condition is verified:

$\beta > 0$, $\lambda_0 < Z_0$, $\Delta > 0$ and $\theta^* \in H(\theta)$,

where $\theta^* = - \theta_0 + \sqrt{\Delta} / \beta$

then $x^{(k)}(\theta^*)$ is optimal for problem $\mathcal{P}$ and STOP;

If one of the following conditions is verified:

i) $\beta > 0$, $\lambda_0 < Z_0$, $\Delta > 0$ and $\theta^* \notin H(\theta)$,

ii) $\beta = 0$ and $\lambda_0 < Z_0$

then set $\theta^{(k)} = \text{SUP} [H^{(k)}(\theta)]$ and update the Linear Complementary Problem setting:

$$x_0 = x^{(k)} (\theta^{(k)})$$

$$y_0 = y^{(k)} (\theta^{(k)})$$

$$\lambda_0 = \lambda_0{}^{(k)}(\theta^{(k)})$$

set k= k+1 and go to STEP 1.

5. A NUMERICAL EXAMPLE

Let us consider the quadratic fractional problem where :

$$\mathcal{P} : \min F (x_1 , x_2)= \frac{x_1{}^2}{x_2 + 1} , (x_1,x_2) \in \mathcal{R}$$

where $\mathcal{R} \equiv \{ (x_1,x_2) \in \mathbb{R}^2 : 4 x_1 - x_2 \geq 5 , 3/2 \leq x_1 \leq 3, x \geq 0\}$.

The parametric problem associated with this problem is :

$P(\theta) : z(\theta) \equiv \min x_1{}^2, (x_1,x_2)\in R(\theta)\equiv\{ (x_1,x_2)\in R : x_2 = \theta \}$.

At STEP 0, we solve the linear problem $\{\min (x_2 + 1), (x_1,x_2) \in R\} = \theta_0$ and obtain $\theta_0=1$. The linear complementarity problem associated with the parametric problem $P(\theta)$ is the following:

$$\begin{pmatrix} u \\ \\ v \\ \\ \\ v_0 \end{pmatrix} - \begin{pmatrix} 2 & 0 & -4 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 4 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ \\ y \\ \\ \\ \lambda_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 3 \\ -3/2 \\ 0 \end{pmatrix} + \theta \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

Set k =1 and go to Step 1.

At STEP 1, we solve the linear complementarity problem with a complementary method (such as Lemke'Algorithm) and we get the solution:

$$x_1^{(1)}(\theta) = 3/2$$
$$x_2^{(1)}(\theta) = \theta$$
$$v_1^{(1)}(\theta) = 1 - \theta$$
$$v_2^{(1)}(\theta) = 3/2$$
$$y_3^{(1)}(\theta) = 3/4$$
$$\lambda_0^{(1)}(\theta) = 3/4$$

so that $H^{(1)}(\theta) = [0, 1]$.

At STEP 2, since $\beta = 0$ and $\lambda_0 = 3/4 < Z_0 = 9/4$, the optimality conditions are not verified; set $\theta^{(1)} = \text{SUP}\,[H^{(1)}(\theta)] = 1$ and update the linear complementarity problem setting:

$$x_0 = x^{(1)}(\theta^{(1)})$$
$$y_0 = y^{(1)}(\theta^{(1)})$$
$$\lambda_0 = \lambda_0^{(1)}(\theta^{(1)})$$

Set k=k+1=2, go to STEP 1.

At STEP 1, we solve the updated linear complementarity problem, whose solution is:

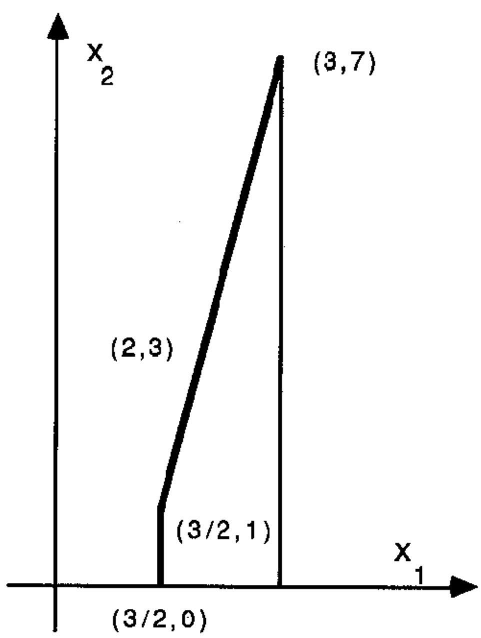
$$x_1^{(2)}(\theta) = 3/2 + \theta/4$$
$$x_2^{(2)}(\theta) = 1 + \theta$$
$$v_2^{(2)}(\theta) = 3/2 - \theta/4$$

$$v_3^{(2)}(\theta) = \quad + \theta/4$$
$$y_1^{(2)}(\theta) = 3/4 + \theta/8$$
$$\lambda_0^{(2)}(\theta) = 3/4 + \theta/8$$

so that $H^{(2)}(\theta) = [0, 6]$.

At STEP 2, since $\beta > 0$, $\lambda_0 = 3/4 < Z_0 = 9/8$, $\Delta = 1/4 > 0$ and $\theta^* = -\theta_0 + \sqrt{\Delta} / \beta = 2 \in H^{(2)}(\theta)$, then the optimality condition b) of Theorem 2.1 is verified and $x^{(2)}(\theta^*) = (2,3)$ is the optimal solution for problem $\mathcal{P}$ and the algorithm stops.

In the following picture, the feasible region for problem $\mathcal{P}$ is depicted; the optimal level solutions, generated with the proposed algorithm are represented by the bold line and the optimal solution of $\mathcal{P}$ is circled.



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