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ABOUT AN INTERACTIVE MODEL FOR SEXUAL POPULATIONS

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Introduction

An as yet unsolved problem of modern mathematical demography is the so called "problem of the sexes", that arises every time we try to extend Lotka's notion of stable population (see Lotka and Sharpe (1911)) to sexual populations.

As we know, a stable population is an ideal object, achieved by assuming the existence of a one-sex world. At the moment that an attempt is made to remove this unrealistic feature from the model, by coupling two distinct Lotka's stable equations, one for each sex, a lot of unresolvable inconsistences arise. These inconsistences, initially observed at the empirical level, have also been studied from a theoretical point of view in important papers by Karmel (see Karmel (1947a), (1947b)).

The work of Karmel gave rise to a strong debate about the need to redefine the notion of stable population, particularly in relation to the problem of the interaction between individuals of the two sexes. One of the most important conclusions of this debate, was that the process of "internal metabolism" (see Ryder (1964)) of every population cannot be regarded as fully linear, as postulated by Lotka, because of the indisputable presence of nonlinearities in the mating process.

The need to clarify the nonlinear mechanism of the mating process has opened up many interesting and important research fields. Many of the first investigations in this area tried to develop simple explanations of the equilibrium existing in the ratio between the numbers of male and female individuals over time. The most elementary of these are, of course, the models that study the interaction between individuals of the two sexes by means of a couple of differential (or difference) equations, one describing the state of the male, the other the state of the female populations (see Kendall (1949), Goodman (1953) and Keyfitz (1965), (1968)). Though very rough (disregarding age structure as they do), these models nevertheless proved to be useful for the understanding of many qualitative features of the real world process.

In our paper we examine one mutualistic model of "random mating interaction", with a fertility function of the type known in the literature as the "harmonic mean fertility law". This model has been the object of attention, for instance, in Keyfitz (1968) and especially in Schoen (1983), although they limited their analysis to stable (in Lotka's sense) solutions.

What we point out here is that the role played by homogeneity of the formulation, coupled with its analytical simplicity, allows one to determine the general solution of the model in closed form. This solution is simple enough to be studied in a straightforward manner by means of the implicit functions theorem.

In this way it has been possible to highlight the behavior of the model during the adjustment process to the stable state, and also in all the non-stable cases.

In particular, we have shown that, among Lotka's stable patterns, the steady state equilibrium is structurally instable.

1. The Model

Following Schoen (1983), let us consider the model defined by the system of autonomous differential equations:

$$\begin{cases} \dot{x} = F(x,y) = -ax + \alpha\dfrac{xy}{x+y} \\ \dot{y} = G(x,y) = -by + \beta\dfrac{xy}{x+y} \end{cases} \tag{1.1}$$

with $(x,y) \in \mathbb{R}^2_+ - \{(0,0)\}$ and initial conditions $x(0) = x_0$ and $y(0) = y_0$. The system is undetermined in the origin but it is sufficiently reasonable to assume $F(0,0) = G(0,0) = 0$. The coefficients a, α, b, β are real and positive; furthermore, they will be supposed time-invariant.

The quantities x and y are functions of time describing the number of male and female individuals, respectively, existing in a given sexual population at time t.

System (1.1) defines a model of ecological interaction in which the growth rates $\dot{x}$ e $\dot{y}$ depend only on the number of males and females actually present at time t. In particular, the linear terms $-ax$ and $-by$ define the mortality of the two groups; the parameters a and b are to be intended as *istantaneous mortality rates* i.e. as the *forces of mortality* of the male and female population.

The nonlinear term $xy/(x+y)$ represents the istantaneous rate of mating between individuals of the two sexes, i.e. the rate at which couples are formed. The parameters α and β may be intended as *fertility coefficients*, defining respectively the average number of newborn male and female babies for each couple. Of course α and β are supposed to apply istantaneously. The so defined fertility law, known in the literature as the *harmonic mean fertility law*, is generally recognized (see Keyfitz (1968), Fredrickson (1971), Schoen (1981), (1983)) as the ideal formulation at least for the class of models we are considering.

Model (1.1) is defined as a "random mating" fertility model, because it is assumed that all individuals present at time t are capable of contributing to natality. Otherwise a third equation would be necessary describing the number of married couples, or simply, of mated couples.

Our model is a Lotka-Volterra system of the cooperative type (i.e. mutualistic or symbiotic), because the effect of the interaction is positive for both species (in this case, male and female subpopulations). It differs from the traditional Volterra model in that the interaction is not characterized in terms of the product

xy, that describes the number of possible meetings between individuals of the two sexes, but in terms of the *relative* number of meetings, given by the ratio of the quantity xy to the total population. That is to say that the effect of natality on the *relative growth rates* $\dot{x}/x$, $\dot{y}/y$, depends not on the absolute number of individuals of the other sex, but on their proportion of the total population.

For $x \neq 0$ and $y \neq 0$, the model may be rewritten in the form:

$$\begin{cases} \dfrac{\dot{x}}{x} = -a\dfrac{x}{x+y} + (\alpha - a)\dfrac{y}{x+y} \\ \dfrac{\dot{y}}{y} = -b\dfrac{y}{x+y} + (\beta - b)\dfrac{x}{x+y} \end{cases} \tag{1.2}$$

which shows that the relative growth rates of bothes sexes depend on two distinct quantities. One of these is always negative (and hence related to mortality), and dependent on the frequency of the individuals of the same sex; the other can be either positive or negative, depending on the frequency of the individuals of the other sex.

The parameters upon which, in the final analysis, the behaviour of the model therefore depends, are, apart from a and b , the differences $\sigma_x = \alpha - a$ and $\sigma_y = \beta - b$, which represent the *survival abilities* of the two sexes.

2. Equilibrium solutions

The functions F and G which define the system (1.1) are C^∞ in the *feasible region* $EP = \{(x,y) \in \mathbb{R}^2 \colon x \geq 0\, ,\ y \geq 0\}$ with the exception of the origin. Hence, the existence and unicity theorem guarantees us that for every point of $EP - \{(0,0)\}$ passes one and only one phase curve (see Arnold (1979)). In the origin only the existence of the solution is guaranteed.

Observe that, if $x = 0$, $\forall y > 0$, then $\dot{x} = 0$ and $\dot{y} = -by$. We may conclude that, in the absence of males, the female population will diminish exponentially at rate b. An analogous conclusion holds in the case of absence of females. Of course in the trivial case $x = y = 0$, by our assumption, the system will remain indefinitely in the origin.

Last observations permit us to say that EP and its boundary ∂EP are *invariant*, i.e. every solution starting here for $t = t_0$, will remain $\forall t > t_0$ (see Hofbauer and Sigmund (1988)). This is not true for the interior intEP.

Apart from the origin, the system will admit other equilibria for $x > 0$ and $y > 0$, if and only if the homogeneous system

$$\begin{cases} -ax + \sigma_x y = 0 \\ \sigma_y x - by = 0 \end{cases}$$

has a non trivial solution. This will obviously happen if the determinant

$$\Omega = \begin{vmatrix} -a & \sigma_x \\ \sigma_y & -b \end{vmatrix} = ab - \sigma_x \sigma_y \tag{2.1}$$

is zero. This condition is satisfied when $\sigma_x > 0$, $\sigma_y > 0$ and $\alpha = a\beta/\sigma_y$. It is easy to verify that all other combinations of values of the parameters are not compatible with the existence of equilibria.

For any given combination of the parameters such that $\Omega = 0$, every point of the line

$$y = \frac{\sigma_y}{b} x = \frac{a}{\sigma_x} x \tag{2.2}$$

is a rest point for the system. This is a direct consequence of the first degree homogeneity of the functions F and G.

The reader may notice the demographical meaning of the equilibrium condition: not only must both sexes exhibit a positive survival ability but mortality and fertility must also balance, keeping the products of survival abilities and forces of mortality equal.

Finally, with regard to the directions of motion in the phase space, we have:

$$\begin{array}{ccc} \dot{x} \gtrless 0 & \Leftrightarrow & y \gtrless \frac{a}{\sigma_x} x \\ \dot{y} \gtrless 0 & \Leftrightarrow & \frac{\sigma_y}{b} x \gtrless y \end{array} \tag{2.3}$$

3. System solutions

The functions F and G being homogeneous, the system (1.1) is invariant with respect to the one-parameter group of transformations $g^\lambda(x,y) = (e^\lambda x, e^\lambda y)$. This means that every phase curve of our system is transformed by the action of g^λ in another phase curve of the same system.

The existence of g^λ guarantees us that the solution of (1.1) may be obtained by quadratures (see Arnold (1989)) operating in polar coordinates or either in the ratio between the variables.

So, in order to find the solution in the phase space, we divide the two equations for $\dot{x} \neq 0$, obtaining the homogeneous differential equation:

$$\frac{dy}{dx} = \frac{\sigma_y xy - by^2}{\sigma_x xy - ax^2} . \tag{3.1}$$

Setting $y = zx$, with $y' = z'x + z$, we have:[1]

$$z'x = \frac{z(a\xi z - b\eta)}{-\sigma_x z + a} \tag{3.2}$$

where $\xi = (\sigma_x + b)/a$ and $\eta = (\sigma_y + a)/b$.

The analysis of the equilibria has shown the existence of two qualitatively different situations. We believe nevertheless that a more detailed inquiry of the behaviour of the model should be done by considering a somewhat finer classification based on the possible values of the survival abilities. Resolving (3.2) in correspondence of each one of these cases, it has been possible to draw the relative phase portraits of fig. 4.

Because of its greatest interest, in what follows we explicitly present a detailed analysis of the positive survival abilities case (case A in the sequel). Among the remaining cases, we have left in appendix those characterized by $\sigma_x < 0$, $\sigma_y > 0$ and $\sigma_x = 0$, $\sigma_y > 0$. We omit any treatment of the respectively symmetric cases $\sigma_x > 0$, $\sigma_y < 0$ and $\sigma_x > 0$, $\sigma_y = 0$. We also did not report any analitical consideration about the cases of non positive survival abilities, in virtue of their absolute simplicity.

With regard to case A, previous results about the sign of Ω suggest us to distinguish among three subcases.

Subcase A.1

Suppose $\Omega > 0$. In this case, the derivatives $\dot{x}$ and $\dot{y}$ may be either positive or negative as shown by (2.3). If $\dot{x} \neq 0$, since $\eta > 0$ and $\xi > 0$, we may write (3.2) as:

$$z'x = \frac{a\xi z(z - \theta)}{-\sigma_x z + a} \tag{3.3}$$

with $\theta = b\eta / a\xi > 0$. If $z \neq 0$ and $z \neq \theta$, the implicit form solution will be:

$$\Phi(x,y,k) = \frac{1}{a\xi}\left[-\frac{a}{\theta}\log\frac{y}{x} + \left(\frac{a}{\theta} - \sigma_x\right)\log\left|\frac{y}{x} - \theta\right|\right] - \log x + k = 0 \tag{3.4}$$

with $k \in \mathbb{R}$. Differentiating Φ with respect to x and y, we obtain:

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[1] System (1.1) may be solved also in the time domain. In effect, operating the substitution $y(t) = z(t)x(t)$, it may be converted into the following single separable equation in the ratio between the variables:

$$\dot{z} = -\frac{z}{1+z}[a\xi z - b\eta] \ .$$

$$\Phi_x = \frac{\sigma_y x - by}{a\xi x(y - \theta x)} \quad ; \quad \Phi_y = \frac{ax - \sigma_x y}{a\xi y(y - \theta x)} \ . \tag{3.5}$$

The derivatives Φ_x and Φ_y are positive or negative depending on the respective position in the phase space of the three lines:

$$y_1 = \frac{a}{\sigma_x}x \quad ; \quad y_2 = \frac{\sigma_y}{b}x \quad ; \quad y_3 = \theta x$$

Since, by assumption, $\alpha > a\beta/\sigma_y$, we have:

$$y_1 - y_2 = \frac{-\alpha\sigma_y + a\beta}{\sigma_x b}x < 0$$

Furthermore:

$$y_1 - y_3 = \frac{-\alpha\sigma_y + a\beta}{\sigma_x(\sigma_x + b)}x < 0$$

and

$$y_2 - y_3 = \frac{\alpha\sigma_y - a\beta}{b(\sigma_x - b)}x > 0 \ .$$

The line y_3 divides the positive quadrant in two regions that can not be "connected" by any phase curve. In the region above y_3 is defined, by the Implicit Function Theorem, a smooth function $y(x)$ that presents only one minimum point x^*. Let us observe that $(x^*, y(x^*))$ lies on the line y_2. It may be shown by means of simple calculations that $y(x)$ is convex. Analogously, in the region belove y_3 is defined a convex function $x(y)$ which unique minimum point is the solution of $x(y) = y_1$.

Keeping in mind that the sign of the derivatives $\dot{x}$ and $\dot{y}$, informs us on the direction of motion, we can draw the following phase portrait:

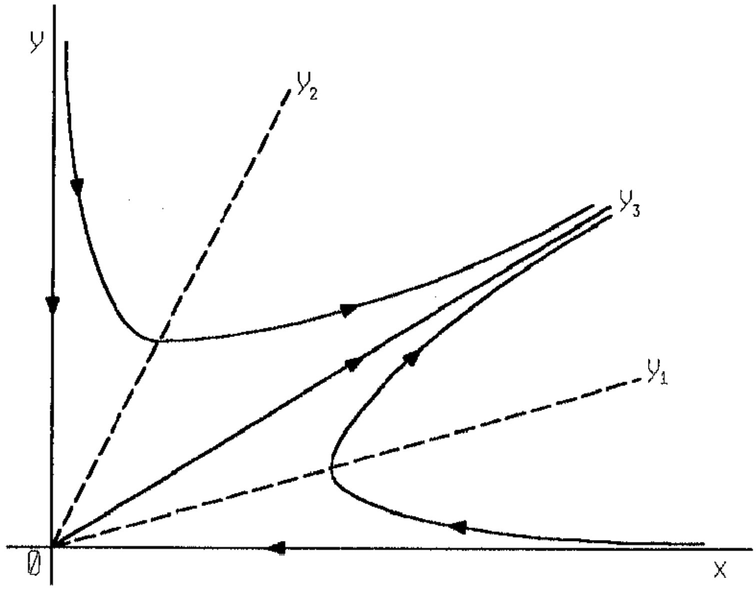

Figure 1. Phase portrait of subcase A.1.

Subcase A.2

Since $\Omega = 0$, the three lines y_1, y_2 and y_3 collapse into the single line y^*. From (2.3), it follows:

$$\dot{x} \gtrless 0 \quad \Leftrightarrow \quad y \gtrless y^* \quad \Leftrightarrow \quad 0 \gtrless \dot{y} \ .$$

So, we see that all the points of the line y^* (of course, in the admissible region) are rest points for the system.

The equation (3.1) simplifies into:

$$\frac{dy}{dx} = -\frac{\sigma_y}{a}\frac{y}{x} \ . \tag{3.6}$$

Integrating, for $y \neq 0$, we obtain:

$$y = kx^{-\sigma_y/a} \qquad k > 0 \ .$$

A simple study of (3.6) and the preceeding remarks give us the following phase portrait:

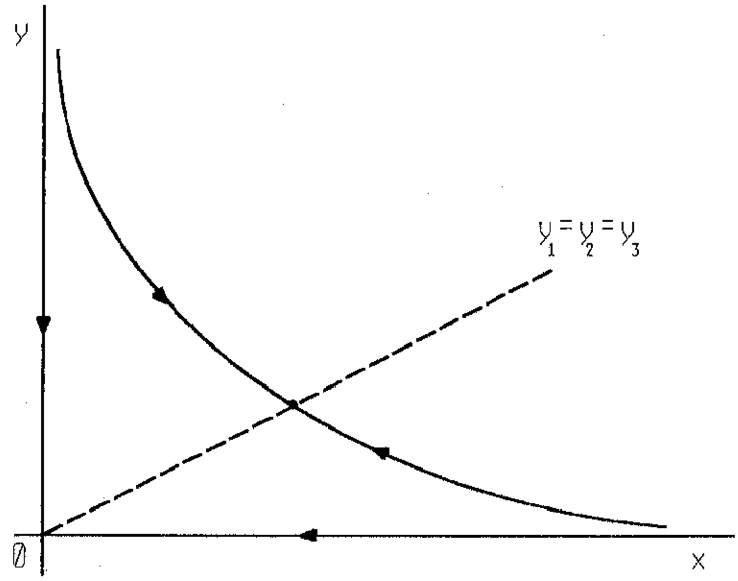

Figure 2. Phase portrait of subcase A.2.

Observe that every phase curve is the union of three different characteristic lines corresponding to distinct motions.

Subcase A.3

Assume $\Omega < 0$. Analogously to subcase A.1, the signs of $\dot{x}$ and $\dot{y}$ have to be determined from (2.3) and both ξ and η are positive. For $z \neq 0$ and $z \neq \theta$, the solution of (3.2) is given by (3.4). The partial derivatives (3.5) are positive or negative, as before, in relation to mutual position of the lines y_1, y_2 and y_3. In virtue of the assumptions made on the parameters, we find:

$$y_1 > y_2 \quad ; \quad y_1 > y_3 \quad ; \quad y_2 < y_3 \; .$$

The same reasoning as in subcase A.1, leads to the following diagram:

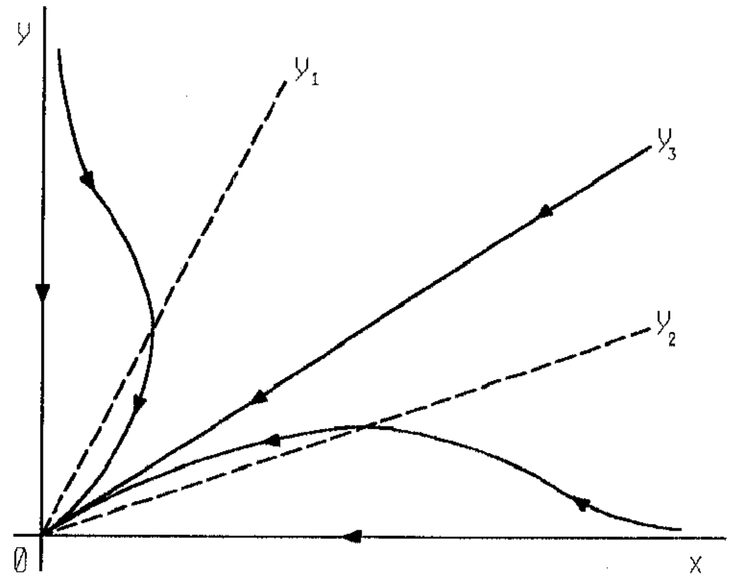

Figure 3. Phase portrait of subcase A.3.

For the sake of completeness we summarize in the next diagram the phase portraits relevant to all cases.

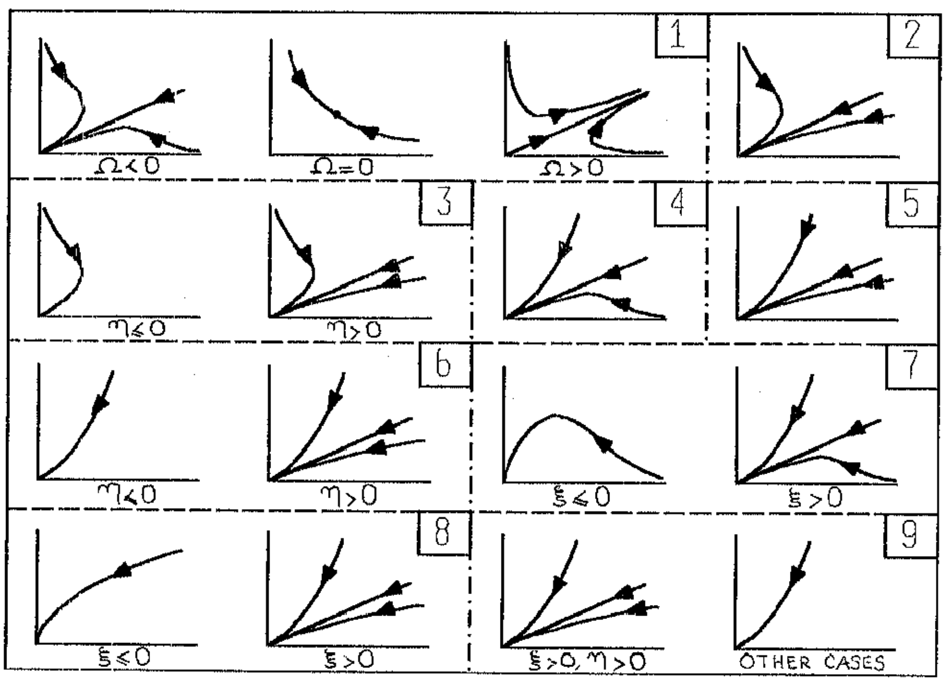

Figure 4. Global picture of the phase portraits relative to all possible values of the survival abilities:

1) $\sigma_x > 0$, $\sigma_y > 0$ 2) $\sigma_x > 0$, $\sigma_y = 0$ 3) $\sigma_x > 0$, $\sigma_y < 0$ 4) $\sigma_x = 0$, $\sigma_y > 0$ 5) $\sigma_x = 0$, $\sigma_y = 0$ 6) $\sigma_x = 0$, $\sigma_y < 0$ 7) $\sigma_x < 0$, $\sigma_y > 0$ 8) $\sigma_x < 0$, $\sigma_y = 0$ 9) $\sigma_x < 0$, $\sigma_y < 0$.

4. Behaviour of the model

Previous results permit us a straightforward study of the stability of the equilibria. The origin is always globally asymptotically stable, with the exception of cases A.1 and A.2. In the former case the zero equilibrium is unstable while in the latter is stable. Moreover, each one of the equilibrium points lying on y^* is stable.

It is interesting to notice that this last case is *structurally instable* i.e. a "small" perturbation in the values of the parameters modifies the topological image of the phase portrait of the system (Arnold (1989)). In fact, the set $\{\sigma_x > 0\ ,\ \sigma_y > 0\ ,\ \sigma_x\sigma_y = ab\}$ is a closed subset (the *bifurcation set*) of codimension one (Guillemin and Pollack (1974)) in the parameter space, that separates two open subsets associated to the cases A.1 and A.3. This means that if the system is in the case A.2, then a small perturbation would push it into one of the two contiguous cases in which, instead, the system will persist under further sufficiently small perturbations (for more details and a deeper treatment, see Hirsch and Smale (1974), Arnold (1989)).

It is possible to give a global view of the behaviour of the model with respect to the parameters fixing two of them, and drawing in the plane of the survival abilities the phase portraits relative to the cases considered. For instance, fixing a and b, we can show the way in which the phase portraits of the system evolve by varying α and β. The following picture shows what is going on when $a = b$. This situation is characterized by positiveness of ξ and η.

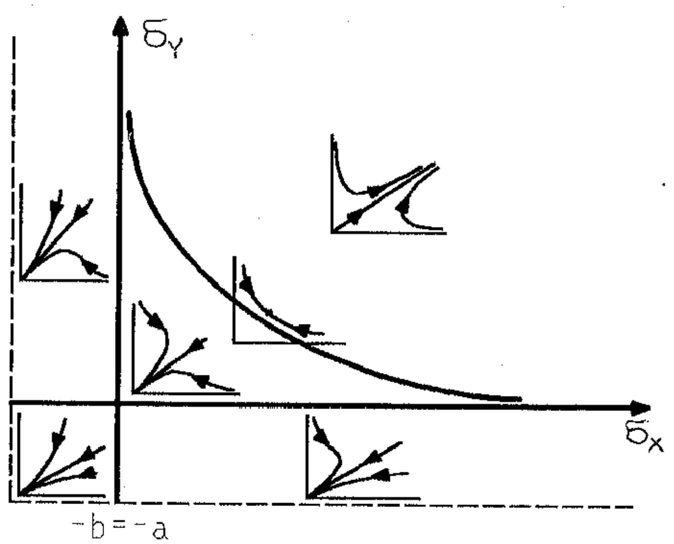

Figure 5. Phase portraits evolution depending on the survival abilities values for a and b fixed and $a = b$.

Inside each region the qualitative behaviour of the phase portraits does not change by varying the values of the parameters, but only a continuous deformation in the shape of the trajectories is observed. Instead, every transition from a region to another determines also a modification in the qualitative behaviour of the orbits.

For example, suppose $\sigma_y > 0$ and $\sigma_x < 0$. Every rightward mouvement along the σ_x axis gives rise to a compression towards the y axis of the characteristics lying above y_3, which will assume, when α exceeds a, the wave shape. Analogously, every downward deplacement along the σ_y axis generates a similar process for any orbit lying under y_3. Moreover, when α (β) increases the stable line becomes more (less) flat. Completely similar effects are observed by varying a and b .

When $a > b$ or $a < b$ the last picture will be slightly modified because the cases characterized by $\xi \leq 0$ or $\eta \leq 0$ will appear.

From a demographic point of view, quite important are the so-called *stable solutions* (in the Lotka sense). This are exponential solutions characterized by a time invariant sex composition.

To study their existence, we substitute the trial solutions $x = x_0 e^{\lambda t}$ and $y = y_0 e^{\lambda t}$ in (1.1), obtaining:

$$\begin{cases} \lambda x_0 = -\, a x_0 + \alpha \dfrac{x_0 y_0}{x_0 + y_0} \\ \lambda y_0 = -\, b y_0 + \beta \dfrac{x_0 y_0}{x_0 + y_0} \end{cases} \tag{4.1}$$

If $x_0 = 0$, the system is satisfied for $\lambda = -b$ that represents the rate of exponential decay of the female population, as previously pointed out .

If $x_0 \neq 0$, eliminating λ from both equations of (4.1), we obtain:

$$\frac{y_0}{x_0} a\xi = b\eta \tag{4.2}$$

Equation (4.2) admits one and only one demographically meaningful solution if and only if ξ and η are positive. In fact, a unique solution there exists if and only if $\xi \neq 0$. Furthermore, if $\xi < 0$, we have: $\sigma_y + a > \beta + \alpha > 0$ which implies $\eta > 0$.[2] So:

$$\lambda = -\frac{\Omega}{\alpha + \beta} \qquad \frac{y_0}{x_0} = \theta$$

where λ is the *stable rate* of growth or decay, while y_0/x_0 represents the *sex composition* in the stable state.

Stable solutions are so given, from a geometrical point of view, by the line y_3 in the phase space. As it may be realized from the phase portraits of the solutions of the system, whenever a stable solution exists, every other solution internal to *EP* tends to it as $t \to +\infty$. This means that the model exhibits the *ergodic property*: whatever the initial sex composition is, it will be forgotten during the adjustment process to stability.

In conclusion, the behaviour of the model is rather elementary. With the exception of case A, the population will unavoidably disappear as a consequence of the lack of survival ability of either sexes. The underlying demographical mechanism is obvious: the interdependence between the sexes generated by (1.1) is such that even if only one sex lacks survival ability, this will in the long run inhibit the growth possibilities of the other via the reduction in the number of partners on the marriage market.

Case A is the only one completely relevant, depending exactly on that both sexes have a positive survival ability. As we have shown in the subcases A.1-A.3, the model leads to all the possible cases of stable evolution: stable growth, stationarity, stable decay. On the basis of the previous considerations on the structural instability of case A.2, we conclude that a nonzero equilibrium can be considered as an "exceptional" event.

Case A shows very clearly the homeostatic role played by the sex composition. This fact may be easily noticed from (1.2): as we expect from the real world process, if one sex is "scarce" relatively to the other, then its numerical amount will experiment an upward pressure coupled with a downward pressure on the other (these forces operate trough the marriage market).

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2 Conversely, if $\eta < 0$, ξ is positive.

To directly realize the way in which the compositional forces do act, let us consider for instance the diagram relative to case A.1. Suppose the system starts, for $t = t_0$, in the region under y_1, in which the initial sex composition is strongly unbalanced. The very low frequency of women coupled with the too high frequency of men sets in motion forces which rise the number of women and diminish at the same time the number of men, generating in this way a balancing trend on sex composition. This process will continue until line y_1 is crossed: at this moment the sex composition will be sufficiently balanced so as to permit the growth of both sexes. Since in the region between y_1 and y_3 the percentage growth rate of the female population is still larger than the corresponding male rate, the equilibrating process of the sex composition will end only once the stable state has been reached. If the initial condition lies in the region above y_2, we will observe a similar process exchanging the role of women and men.

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Appendix

§1. If $\sigma_x < 0$ and $\sigma_y > 0$, inside *EP* $\dot{x}$ is always negative while $\dot{y} \gtrless 0$ if and only if $y_2 \gtrless y$. The origin is the only admissible equlibrium point. Beeing η always positive, we distinguish three subcases in relation to the possible values of ξ.

Subcase 1.

If $\xi = 0$, i.e. $b = \sigma_x$, equation (3.2) becomes:

$$z'x = -\frac{b\eta z}{zb + a} \quad .$$

Integrating, for $z \neq 0$, we obtain in implicit form:

$$\Phi(x,y,k) = \frac{y}{\eta x} + \frac{a}{b\eta} \ln \frac{y}{x} + \ln x + k = 0$$

where $k \in \mathbb{R}$. Partial derivatives with respect to x and y, are:

$$\Phi_x = -\frac{y}{\eta x^2} + \frac{\alpha + b}{b\eta x} \quad ; \quad \Phi_y = \frac{1}{\eta x} + \frac{a}{b\eta y} \quad .$$

where Φ_y is always positive and $\Phi_x \gtrless 0$ if and only if $y_2 \gtrless y$. Each phase curve is the diagram of a function $y(x)$ increasing for $y > y_2$. It is possible to show that $y(x)$ changes concavity on the right side of the maximum point.

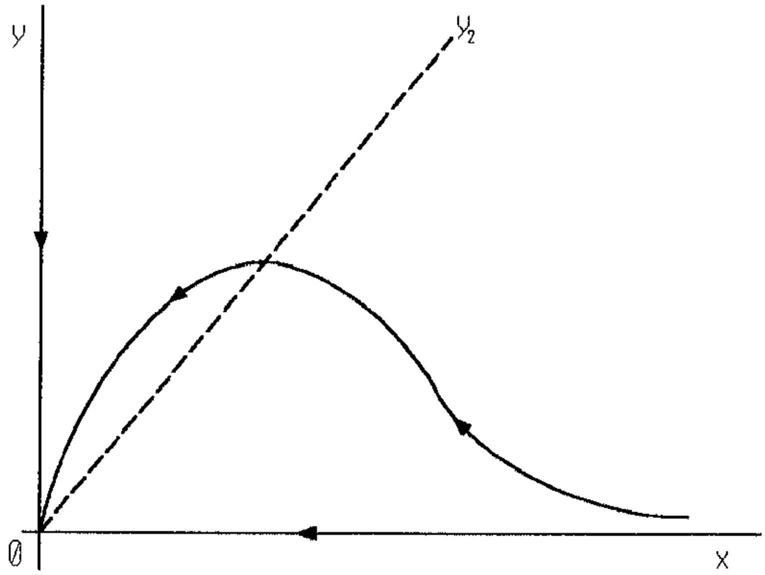

Figure 6. Phase portrait of the cases 1. and 3..

Subcase 2.

Supposing $\xi > 0$, i.e. $\sigma_x > -b$, the equation (3.3) has the solution (3.4) for $z \neq 0$ and $z \neq \theta$. Since $\sigma_x < 0$, taking partial derivatives (3.5), it follows that $\Phi_y \gtrless 0$ when $y \gtrless y_3$. Moreover, being:

$$y_2 - y_3 = \frac{x[\beta\sigma_x - b\alpha]}{b(\sigma_x + b)} < 0$$

Φ_x is positive in the region $\{(x,y)\colon\ y > y_2\} \cap \{(x,y)\colon\ y < y_3\}$ The behaviour of the solutions in the phase space is represented in the following diagram.

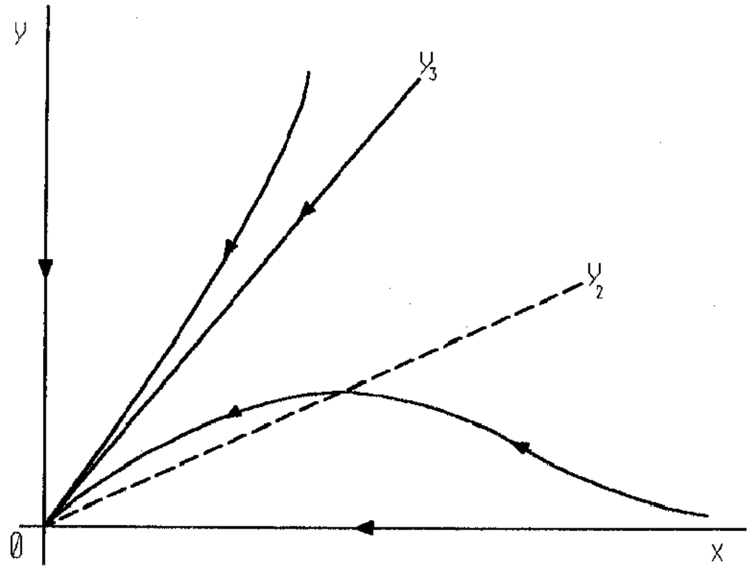

Figure 7. Phase portrait of subcase §1.2 and §2.

Notice that y_3 is a characteristic line.

Subcase 3.

In this case $\xi < 0$, i.e. $\sigma_x < -b$, and so θ is negative too. For $z \neq 0$, we can operate exactly as in subcase 2., obtaining an identical expression for Φ, where Φ_x is positive or negative depending on whether $y - y_2$ is positive or negative and Φ_y is always positive. The phase portrait of the solutions behaves in the same way as in subcase 1..

§2. If $\sigma_x = 0$ and $\sigma_y > 0$, $\dot{x}$ is always negative, while $\dot{y} > 0$ if and only if $y > y_2$. The equation (3.2) takes the form:

$$z'x = \frac{b}{a} z(z - \eta)$$

with $\eta > 0$.[3] So, $z = 0$ and $z = \eta$ are solutions. Integrating, for $z \neq 0$ and $z \neq \eta$, we obtain:

$$\Phi(x,y,k) = \frac{|y - \eta x|}{y} - kx^{b\eta/a} = 0 \qquad k > 0 \ . \tag{1}$$

that may be rewritten in closed form.

For $y > \eta x$, we have:

[3] If we assume $\alpha > \beta$ as is usual for human populations, we would have $\eta > 1$.

$$y = \Psi(x) = \frac{\eta x}{1 - kx^{b\eta/a}} .$$

The graph of Ψ is a subset of EP if and only if $x < k^{-a/b\eta} = x^*$. It is a simple task to verify that $\Psi(0) = 0$, $\lim_{x\to(x^*)^-} \Psi(x) = +\infty$ and:

$$\Psi'(x) = \frac{\eta}{(1 - kx^{b\eta/a})^2}\left(1 - k\left(1 - \frac{b\eta}{a}\right)x^{b\eta/a}\right) > 0 .$$

For $y < \eta x$, (1) becomes:

$$y = \Psi(x) = \frac{\eta x}{1 + kx^{b\eta/a}} \qquad k > 0$$

where $\lim_{x\to 0^+} \Psi(x) = 0^+$ and $\lim_{x\to+\infty} \Psi(x) = 0^+$. A simple study of the first derivative:

$$\Psi'(x) = \frac{\eta}{(1 + kx^{b\eta/a})^2}\left(1 - \frac{k\sigma_y}{a}x^{b\eta/a}\right)$$

shows the existence of a unique maximum point $x^{**} = (a/k\sigma_y)^{a/b\eta}$ at which $\Phi(x)$ crosses the line y_2. The corresponding phase portrait behaves as the one of subcase 2 of the case §1.

Abstract: In this paper the continuous random mating two-sex model with "harmonic mean" fertility function, partially investigated by Schoen (1983), is studied. Its closed form solution in the phase space is obtained in virtue of the homogeneity of the vector field considered. By introducing the notion of survival ability it has been clarified the balancing role played by the "marriage market".

Key words: Two-sex problem - Homogeneous equations - Stable population - Structural stability