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**PARAMETRIC LINEAR FRACTIONAL  
PROGRAMMING FOR AN UNBOUNDED  
FEASIBLE REGION**

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## Parametric linear fractional programming for an unbounded feasible region

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### Abstract

Parametric analysis in linear fractional programming is significantly more complicated in case of an unbounded feasible region. We propose procedures which are based on a modified version of Martos' algorithm or a modification of Charnes-Cooper's algorithm, applying each to problems where either the objective function or the right-hand side is parametrized.

### 1. Introduction

Linear fractional programs, though not convex, have the same local-global properties as linear (convex) programs. An extensive literature exists for linear fractional programs with fixed values for the parameters in the objective function and in the constraints [8],[36]. In applications however, parameters are sometimes subject to changes. Since the 1960's a good number of articles have appeared in parametric fractional programming [1] - [7],[9],[10],[14] - [20],[22],[24] - [28],[30] - [33],[35] - [45]. Most of these contributions assume a bounded feasible region. In this case the analysis is comparatively easy since the problem is equivalent to a parametric linear program which has thoroughly been researched.

In the unbounded case difficulties arise since a supremum may not be attained at a vertex but along an extreme ray. In this case a vertex following procedure cannot be used to solve the problem.

In [10] an algorithm is suggested for a linear fractional program with an unbounded feasible region where the right-hand side is parametrized. It is based on Dinkelbach's algorithm in fractional programming where numerator and denominator are separated by a parameter.

More recently, Cambini and Sodini [14],[38] proposed a procedure for fractional programs with a parameter in the numerator of the objective function which is based on a modified version of Martos' algorithm by Cambini and Martein [11].

In the present paper we follow this approach. We study the same parametric problem as in [38] by applying a modified Charnes-Cooper algorithm [21] instead. In the second part we present procedures for fractional programs parametrized on the right-

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hand side by using either a modified Martos algorithm or a modified Charnes-Cooper algorithm. It turns out that for problems with a parameter in the objective function the algorithm based on Charnes-Cooper's transformation is more suitable whereas for problems with a parameter on the right-hand side the use of the modified Martos algorithm is preferable.

## 2. The linear fractional program

We consider the following linear fractional program:

$$P \quad \sup_{x \in X} f(x) = (c^T x + c_0) / (d^T x + d_0) \quad (1)$$

where  $X = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ ,  $A$  is a  $m \times n$  matrix,  $c, d \in \mathbb{R}^n, b \in \mathbb{R}^m, c_0, d_0 \in \mathbb{R}$ . We assume  $X \neq \emptyset$  and  $d^T x + d_0 > 0 \forall x \in X$ .

For problem  $P$  the following three cases can occur:

- 1) there exists an  $x' \in X$  such that  $\max \{f(x) : x \in X\} = f(x')$ ; in this case a vertex of  $X$  is an optimal solution;
- 2)  $\sup \{f(x) : x \in X\} = L < +\infty$ , but the supremum is not attained; in this case there exists a vertex  $x'$  and an extreme ray  $r$  such that  $\lim_{t \rightarrow +\infty} f(x' + tr) = L$ ;
- 3)  $\sup \{f(x) : x \in X\} = +\infty$ ; in this case there exists a vertex  $x'$  and an extreme ray  $r$  such that  $\lim_{t \rightarrow +\infty} f(x' + tr) = +\infty$ .

Clearly, if  $X$  is bounded, only case 1) occurs.

## 3. Algorithms

In this section we briefly describe two types of algorithms of fractional programming used in the parametric analysis below. These are the algorithm of Martos [29] and its modification by Cambini - Martein [11] and the algorithm by Charnes - Cooper [21].

Given a vertex  $x'$  of  $X$  and the corresponding base  $A_B$ , let

$$\begin{aligned} c'_N{}^T &= c_N{}^T - c_B{}^T A_B^{-1} A_N; & d'_N{}^T &= d_N{}^T - d_B{}^T A_B^{-1} A_N, \\ d'_0 &= d_B{}^T A_B^{-1} b + d_0, & c'_0 &= c_B{}^T A_B^{-1} b + c_0, \\ \gamma &= d'_0 c'_N - c'_0 d'_N. \end{aligned} \quad (2)$$

### The algorithm of Martos

**Step 0** Compute a basic feasible solution  $x'$ ; go to Step 1.

**Step 1** Compute  $c'_N, d'_N, \gamma$ . If  $\gamma \leq 0$ , stop;  $x'$  is optimal. Otherwise, select  $k$  such that  $\gamma_k = \max \{\gamma_j\}$ ; go to Step 2.

**Step 2** Compute  $y^k = A_B^{-1} A_N^k$ . If  $y^k \leq 0$ , then *stop*;  $c'_{Nk}/d'_{Nk} = \sup \{f(x) : x \in X\}$ . Otherwise, do a simplex iteration with  $x_{Nk}$  as entering variable. Let  $x'$  be the new basic solution; *go to Step 1*.

The algorithm of Martos guaranties the solution of the problem only if the feasible region is bounded [13],[38].

In order to overcome the limitations of Martos' algorithm in the unbounded case, Cambini and Martein have proposed a modification of it. The modified version of Martos' algorithm MVM differs from Martos' algorithm in the choice of the entering variable for simplex iterations. Only optimal level vertices are used. This modification is sufficient to eliminate the disadvantage of the algorithm of Martos.

In addition to the notation in (2), let

$$\gamma' = c'_N - f(x') d'_N. \quad (3)$$

### The algorithm MVM of Cambini - Martein

**Step 0** Solve problem  $P_0 : \min \{d^T x + d_0 : x \in X\}$ . If the optimal vertex  $x'$  of  $P_0$  is unique, then *go to Step 1*. Otherwise, solve problem  $P_1 : \sup \{c^T x + c_0 : x \in X'\}$  where  $X' = X \cap \{x : d^T x = d^T x'\}$ . If  $\sup \{c^T x + c_0 : x \in X'\} = +\infty$ , then *stop*;  $\sup \{f(x) : x \in X\} = +\infty$ . Otherwise *go to Step 1* with  $x'$  being an optimal vertex of  $P_1$ .

**Step 1** Compute  $c'_N$ ,  $d'_N$ ,  $\gamma'$  and  $J = \{j : \gamma'_j > 0\}$ . If  $J = \emptyset$ , then *stop*;  $x'$  is an optimal solution. Otherwise, select  $k$  such that  $c'_{Nk}/d'_{Nk} = \max \{c'_{Nj}/d'_{Nj} : j \in J\}$ ; *go to Step 2*.

**Step 2** Compute  $y^k = A_B^{-1} A_N^k$ . If  $y^k \leq 0$ , then *stop*;  $c'_{Nk}/d'_{Nk} = \sup \{f(x) : x \in X\}$ . Otherwise, do a simplex iteration with  $x_{Nk}$  as entering variable. Let  $x'$  be the new basic solution; *go to Step 1*.

### The algorithm of Charnes - Cooper

By using the transformation of variables  $t=1/(d^T x + d_0)$ ,  $y=tx$  Charnes-Cooper have shown that problem P is equivalent to the following one:

$$\begin{array}{l}
 \sup \quad c^T y + c_0 t \\
 \text{P}' \quad Ay - bt = 0 \\
 \quad \quad d^T y + d_0 t = 1 \\
 \quad \quad y \geq 0 \\
 \quad \quad t > 0
 \end{array} \quad (4)$$

If the feasible region  $X$  is a bounded set, we have  $0 < p \leq d^T x + d_0 < +\infty, \forall x \in X$ , and  $t=0$ , if permitted, could not occur in the optimal solution of problem P'. Consequently,

the constraint  $t > 0$  is equivalent to the constraint  $t \geq 0$ , and thus problem  $P'$  turns out to be a linear program which can be solved by any algorithm for linear programs. On the other hand when the feasible region is unbounded,  $t=0$  can occur, and when it happens,  $P$  has a supremum obtained along an extreme ray [23],[34].

#### 4. The parametric linear fractional program

Sensitivity and parametric analysis for a linear fractional problem with a bounded feasible region is similar to the one for linear programs [38]. However, when the feasible region is unbounded, some difficulties arise in studying the supremum of the problem  $P$  as a function of a parameter because of case 2) in section 2 which cannot occur in linear programming.

##### 4.1 Parametrization of the objective function

In this section we consider the parametrization of the numerator in the objective function of a linear fractional program. We point out that a problem with a parameter in the denominator can be reduced to one in the numerator by maximizing  $-1/f(x)$  (assuming  $f(x) \neq 0$ ). Let us consider the following parametric problem:

$$\text{POBJ}(\theta) \quad z(\theta) = \sup_{x \in X} [(c+\theta u)^T x + c_0 + \theta u_0] / (d^T x + d_0) \quad (5)$$

for some  $(u, u_0) \in \mathbb{R}^{n+1}$ .

Now we describe a procedure for solving problem  $\text{POBJ}(\theta)$  which is based on the algorithm MVM. Following that, we will apply the Charnes-Cooper algorithm to the same problem. First, problem  $\text{POBJ}(0)$  is solved by means of algorithm MVM.

Let  $x' = (A_B^{-1}b, 0)$  be the solution obtained by the algorithm. Clearly,  $x'$  is an optimal solution of the linear subproblem  $\text{PL}(0)$  where

$$\text{PL}(\theta) \quad \sup_{x \in X, d^T x = d^T x'} (c+\theta u)^T x + c_0 + \theta u_0 \quad (6)$$

Let  $c^*_{N'}{}^T = c_{N'}{}^T - c_B{}^T A^*_{B'}{}^{-1} A^*_{N'} \leq 0$  be the vector of the reduced cost of  $x'$  for  $\text{PL}(0)$ . Two cases can occur:

- 1)  $\gamma \leq 0$ ; then  $x'$  is a maximum of  $\text{POBJ}(0)$ ;
- 2) there exists an index  $k$  such that  $\gamma'_k > 0$ ,  $c'_{Nk}/d'_{Nk} = \max \{c'_{Nj}/d'_{Nj}; \gamma'_j > 0\}$  and  $y^k = A_B^{-1} A_N^k \leq 0$ ; in this case  $\text{POBJ}(0)$  has a supremum equal to  $c'_{Nk}/d'_{Nk}$  on the extreme ray from  $x'$  along  $y^k$ .

Let "the stability set" of vertex  $x'$  denote the set of values of  $\theta$  such that  $x'$  is a maximum or the supremum is attained on an extreme ray starting from  $x'$ . In order to

find the stability set of  $x'$ ,  $c'_N, \gamma, c'_0, c^*_N$  are considered as a function of the parameter  $\theta$ . Such functions are denoted by  $c'_N(\theta), \gamma(\theta), c'_0(\theta), c^*_N(\theta)$ .

Let

$$H = \{ \theta : \gamma(\theta) \leq 0 \}, \quad H' = \{ \theta : c^*_N(\theta) \leq 0 \}. \quad (7)$$

Clearly,  $H' \supseteq H$ .

- If case 1) holds,  $H \neq \Phi$  and for  $\theta \in H$   $x'$  is a maximum of  $POBJ(\theta)$  with  $z(\theta) = c'_0(\theta)/d'_0$ .
- If case 2) holds, let us consider the set  $H'' = \{ \theta : c'_{N_k}(\theta)/d'_{N_k} = \max \{ c'_{N_j}(\theta)/d'_{N_j} : \gamma'_j > 0 \} \}$ ; for  $\theta \in H'(\theta) \cap H''(\theta)$  the supremum is attained along  $y^k$  with  $z(\theta) = c'_{N_k}(\theta)/d'_{N_k}$ .

Let  $H' = \{ \theta : \theta' \leq \theta \leq \theta'' \}$ . We will solve problem  $PL(\theta^*)$  where  $\theta^* = \theta' - \varepsilon$  ( $\theta^* = \theta'' + \varepsilon$ ),  $\varepsilon > 0$ . Starting from  $x'$ , by means of one simplex iteration the optimal solution  $x^*$  of  $POBJ(\theta^*)$  is obtained.  $x^*$  lies on an edge of  $X$  and is not, in general, a vertex of  $X$ . We will find the best vertex (in terms of the value of the objective function) which belongs to the edge containing  $x^*$ . Let  $x''$  be such a vertex. The stability set of vertex  $x''$  is adjacent to the stability set of vertex  $x'$ .

Clearly, the described procedure allows us to find the function  $z(\theta)$  through a vertex following examination. As regards to the properties of the parametric function  $z(\theta)$ , it is easy to prove that  $z(\theta)$  is a convex piecewise linear function.

As an alternative to the above approach, let us now apply the Charnes-Cooper transformation to the parametric problem  $POBJ(\theta)$ . Then the following parametric linear program is obtained:

$$\begin{aligned} f(\theta) &= \sup [(c + \theta u)^T y + (c_0 + \theta u_0)t] \\ \text{C-C}(\theta) \quad & Ay - bt = 0, \\ & d^T y + d_0 t = 1, \quad y \geq 0, t \geq 0 \end{aligned} \quad (8)$$

$\text{C-C}(\theta)$  is a standard parametric linear program and can be solved using the usual procedure for parametric linear programming. Clearly, if in the optimal solution  $(y', t')$ ,  $t'$  is greater than zero, then it is possible to transform the results into the space of variables  $x$ , namely  $x' = y'/t'$ . This is not possible when  $t' = 0$ . In this case the above transformation cannot be used. Let  $B'$  be the set of basic variables of the optimal solution of  $\text{C-C}(\theta)$  and

$$A' = \begin{bmatrix} A & -b \\ d^T & d_0 \end{bmatrix}, \quad y_{B'} = A'_{B'}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u_{B'} = A'_{B'}^{-1} \begin{bmatrix} -b \\ d_0 \end{bmatrix}.$$

Further, let  $t_0 = y_{B'_k}/u_{B'_k} = \min \{ y_{B'_i}/u_{B'_i} : u_{B'_i} > 0 \}$  and set  $B = B' \setminus \{k\}$ . It easy to

prove [12] that the supremum of  $z(\theta)$  is attained along the extreme ray starting from the vertex  $x_B = (y_B - t_0 u_B)/t_0$  in the direction  $d_B = -y_B/y_{B'k}$ . Clearly, these results allow us to determine the function  $z(\theta)$  by solving problem C-C( $\theta$ ).

Example 1:

Let us consider the following numerical example

$$\begin{aligned} z(\theta) = \sup \{ & [(-1 + \theta) x_1 + (5 - 2\theta) x_2] / (x_1 + 2) \} \\ & -x_1 + x_2 \leq 2, \quad x_1 - 2x_2 \leq 4, \\ & x_1, x_2 \geq 0. \end{aligned}$$

By using the Charnes-Cooper transformation we obtain the following parametric LP:

$$\begin{aligned} f(\theta) = \sup \{ & (-1 + \theta) y_1 + (5 - 2\theta) y_2 \} \\ & -y_1 + y_2 - 2t + y_3 = 0, \\ & y_1 - 2y_2 - 4t + y_4 = 0, \\ & y_1 + 2t = 1, \\ & y_1, y_2, y_3, y_4, t \geq 0. \end{aligned}$$

The optimal solution for  $\theta = 0$  is given by the following simplex tableau:

			$y_1$	$y_2$	$y_3$	$y_4$	$t$	
$\theta$		2		1	0	2	0	0
<hr/>								
-z		-5		-1	0	-5	0	0
<hr/>								
t		1/2		1/2	0	0	0	1
$y_2$		1		0	1	1	0	0
$y_4$		4		3	0	2	1	0

tab. 1

Stability set:  $-1 + \theta \leq 0, -5 + 2\theta \leq 0$  imply  $\theta \leq 1$ ;  $z(\theta) = 5 - 2\theta, \theta \leq 1$  with the optimal solution ( $y_1 = 0, y_2 = 1, t = 1/2$ ). In the space of the variables  $x$  we have ( $x_1 = 0, x_2 = 2$ ).

For  $\theta > 1$  the reduced cost of variable  $y_1$  becomes positive and  $y_1$  substitutes  $t$  in the basis. The new tableau is the following:

			y <sub>1</sub>	y <sub>2</sub>	y <sub>3</sub>	y <sub>4</sub>	t	
θ		1		0	0	2	0	-2
-z		-4		0	0	-5	0	2
y <sub>1</sub>		1		1	0	0	0	2
y <sub>2</sub>		1		0	1	1	0	0
y <sub>4</sub>		1		0	0	2	1	-6

tab. 2

Stability set:  $2 - 2\theta \leq 0$ ,  $-5 + 2\theta \leq 0$  imply  $1 \leq \theta \leq 5/2$ ;  $z(\theta) = 4 - \theta$ ,  $1 \leq \theta \leq 5/2$  with the optimal solution ( $y_1 = 1$ ,  $y_2 = 1$ ,  $t = 0$ ). In the space of the variables  $x$  we have  $t_0 = \min \{1/2\} = 1/2$ ,  $x_2 = 2$ ,  $x_4 = (1 + (1/2) 6) / 1/2 = 8$ ,  $d_B = (-1, -1)$ .

For  $\theta > 5/2$  the reduced cost of variable  $y_3$  becomes positive and  $y_3$  substitutes  $y_4$  in the basis. The new tableau is the following:

			y <sub>1</sub>	y <sub>2</sub>	y <sub>3</sub>	y <sub>4</sub>	t	
θ		0		0	0	0	-1	4
-z		-3/2		0	0	0	5/2	-13
y <sub>1</sub>		1		1	0	0	0	2
y <sub>2</sub>		1/2		0	1	0	-1/2	3
y <sub>3</sub>		1/2		0	0	1	1/2	-3

tab. 3

Stability set:  $5/2 - \theta \leq 0$ ,  $-13 + 4\theta \leq 0$  imply  $5/2 \leq \theta \leq 13/4$ ;  $z(\theta) = 3/2$ ,  $5/2 \leq \theta \leq 13/4$  with the optimal solution ( $y_1 = 1$ ,  $y_2 = 1/2$ ,  $t = 0$ ). In the space of the variables  $x$  we have  $t_0 = \min \{1/2, 1/6\} = 1/6$ ,  $x_1 = (1 - (1/6) 2) / 1/6 = 4$ ,  $x_3 = (1/2 + (1/6) 3) / 1/6 = 6$ ,  $d_B = (-2, -1)$ .

For  $\theta > 13/4$  the reduced cost of variable  $t$  becomes positive and  $t$  substitutes  $y_2$  in the basis. The new tableau is the following:

			y <sub>1</sub>	y <sub>2</sub>	y <sub>3</sub>	y <sub>4</sub>	t	
θ		-2/3		0	-4/3	0	-1/3	0
-z		2/3		0	13/3	0	1/3	0
y <sub>1</sub>		2/3		1	-2/3	0	1/3	0
t		1/6		0	1/3	0	-1/6	1
y <sub>3</sub>		1		0	1	1	0	0

tab. 4



Stability set:  $13/3 - 4/3 \theta \leq 0$ ,  $1/3 - 1/3 \theta \leq 0$  imply  $\theta \geq 13/4$ ;  $z(\theta) = -2/3 + 2/3 \theta$ ,  $\theta \geq 13/4$  with the optimal solution  $(y_1 = 2/3, y_3 = 1, t = 1/6)$ . In the space of the variables  $x$  we have  $x_1=4$ ,  $x_3=6$ .

The function  $z(\theta)$ , depicted in fig. 1, is the following:

$$z(\theta) = \begin{cases} 5 - 2\theta & \theta \leq 1, \\ 4 - \theta & 1 \leq \theta \leq 5/2, \\ 3/2 & 5/2 \leq \theta \leq 13/4, \\ -2/3 + 2/3 \theta & \theta \geq 13/4. \end{cases}$$

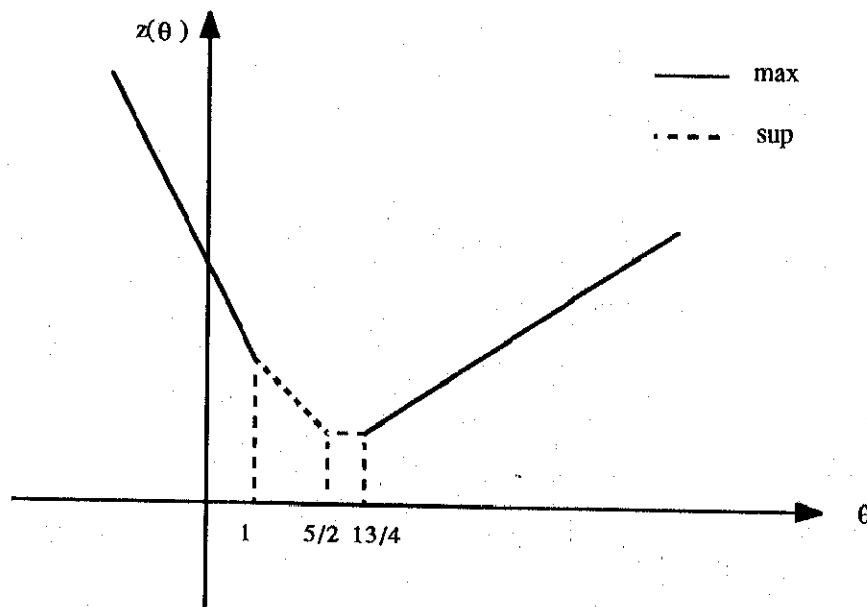


fig. 1

#### 4.2 Parametrization of the right-hand side

We now consider the following parametric linear fractional program:

$$\text{PRHS} \quad z(\theta) = \sup_{x \in X(\theta)} (c^T x + c_0) / (d^T x + d_0) \quad (9)$$

where  $X(\theta) = \{x \in \mathbb{R}^n : Ax = b + \theta b^*, x \geq 0\}$  for some  $b^* \in \mathbb{R}^m$ . We assume  $d^T x + d_0 > 0$   $\forall x \in X(\theta), \forall \theta$ .

If we apply the Charnes-Cooper transformation to problem PRHS we obtain

$$\begin{aligned} f(\theta) &= \sup [c^T y + c_0 t] \\ \text{C-C-RHS} \quad & Ay - (b + \theta b^*)t = 0, \\ & d^T y + d_0 t = 1, \quad y \geq 0, t \geq 0. \end{aligned} \quad (10)$$

C-C-RHS is a parametric linear program with the parameter in the column of the matrix corresponding to the variable  $t$ . To solve such a problem is not easy. For this reason we investigate the application of the algorithm MVM to problem PRHS instead.

Given a basis  $A_B$ , let us define:

$$\begin{aligned} c'_N{}^T &= c_N{}^T - c_B{}^T A_B^{-1} A_N, \quad c'_0(\theta) = c_0 + c_B{}^T A_B^{-1} (b + \theta b^*), \\ d'_N{}^T &= d_N{}^T - d_B{}^T A_B^{-1} A_N, \quad d'_0(\theta) = d_0 + d_B{}^T A_B^{-1} (b + \theta b^*), \\ x_B(\theta) &= A_B^{-1} b + \theta A_B^{-1} b^*, \quad \gamma(\theta) = d'_0(\theta) c'_N - c'_0(\theta) d'_N. \end{aligned}$$

The vertex associated with the basis  $A_B$  is optimal if the following conditions hold:

$$x_B(\theta) \geq 0 \quad (\text{feasibility}),$$

$$\gamma(\theta) \leq 0 \quad (\text{optimality}).$$

The supremum is attained on an extreme ray starting from the vertex  $x_B(\theta)$  if the following conditions hold:

$$x_B(\theta) \geq 0,$$

$$\gamma_k(\theta) > 0,$$

$$A_B^{-1} A_N^k \leq 0,$$

$$c'_{Nk}/d'_{Nk} = \max \{c'_{Nj}/d'_{Nj}; d'_{Nj} > 0\}.$$

Suppose that problem PRHS is solved by means of algorithm MVM for a given value of  $\theta$  and let  $A_B$  be the basis obtained.

If the vertex  $x_B(\theta)$  is optimal, then define

$$H' = \{\theta : x_B(\theta) \geq 0\}, \quad H'' = \{\theta : \gamma(\theta) \leq 0\}, \quad H^* = H' \cap H'' = \{\theta : \theta' \leq \theta \leq \theta''\}.$$

Clearly, for  $\theta \in H^*$  the vertex  $x_B(\theta)$  is optimal. For  $\theta \notin H^*$  different cases can occur:

- 1) feasibility is maintained while optimality is lost (i.e.  $x_B(\theta) \geq 0$  while  $\gamma(\theta) \not\leq 0$ );
- 2) feasibility is lost while optimality is maintained (i.e.  $x_B(\theta) \not\geq 0$  while  $\gamma(\theta) \leq 0$ );
- 3) feasibility and optimality are lost (i.e.  $x_B(\theta) \not\geq 0$  and  $\gamma(\theta) \not\leq 0$ ).

In case 1) there are two possibilities: i) the supremum is attained along an extreme ray from  $x_B(\theta)$  ( $\gamma_k(\theta) > 0$ ,  $A_B^{-1} A_N^k \leq 0$ ) and nothing must be done; ii) the new optimal solution is an adjacent vertex of  $x_B(\theta)$  which can be obtained by a primal simplex iteration. In case 2) the new optimal solution (if any) can be obtained by applying the dual simplex algorithm. In case 3) it is necessary to restart either feasibility or optimality and it is also necessary to maintain the optimality of level. This can be

done by simplex iterations (dual or primal) until case 1) or 2) is obtained.

If the supremum is attained along an extreme ray from the vertex  $x_B(\theta)$ , then define  $H' = \{\theta : x_B(\theta) \geq 0\}$ ,  $H'' = \{\theta : c'_{N_k}/d'_{N_k} = \max \{c'_{N_j}/d'_{N_j} : \gamma_j(\theta) > 0\}$ ,  $H^* = H' \cap H'' = \{\theta : \theta' \leq \theta \leq \theta''\}$ .

Clearly, for  $\theta \in H^*$  the extreme ray  $y^k$  from  $x_B(\theta)$  is optimal. For  $\theta \notin H^*$  different cases can occur:

- 4) feasibility is maintained and optimality is obtained (i.e.  $x_B(\theta) \geq 0$ ,  $\gamma(\theta) \leq 0$ );
- 5) feasibility is maintained while optimality is lost (i.e.  $x_B(\theta) \geq 0$  while  $\gamma(\theta) \not\leq 0$ );
- 6) feasibility and optimality are lost (i.e.  $x_B(\theta) \not\geq 0$  and  $\gamma(\theta) \not\leq 0$ ).

In case 4) clearly  $x_B(\theta)$  is an optimal vertex and nothing must be done. Case 5) is like case 1). Finally, case 6) is like case 3).

The proposed procedure is able to determine the function  $z(\theta)$  for all values of  $\theta$ .

#### Example 2:

Let us consider the following numerical example

$$z(\theta) = \sup [(-x_1 + 5x_2)/(x_1 + 2)]$$

$$-x_1 + x_2 \leq 2 + \theta, \quad x_1 - 2x_2 \leq 4 - 8\theta,$$

$$x_1, x_2 \geq 0.$$

First, we solve the problem for  $\theta = 0$ . We obtain:

		$x_1$	$x_2$	$x_3$	$x_4$
N	-10-5\theta	4	0	-5	0
D	-2	1	0	0	0
$x_2$	2+\theta	-1	1	1	0
$x_4$	8-6\theta	-1	0	2	1

tab. 5

Then  $\gamma(\theta) = (-2 - 5\theta, -10) \leq 0$ ,  $H'' = \{\theta : \theta \geq -2/5\}$ ,  $x_B(\theta) = (2+\theta, 8-6\theta) \geq 0$ ,  $H' = [-2, 4/3]$ , hence  $H^* = [-2/5, 4/3]$ . Then for  $\theta \in H^*$   $x^0 = (0, 2+\theta)$  is optimal and  $z(\theta) = (10+5\theta)/2$ .

For  $\theta > 4/3$  we obtain tab. 6:

		$x_1$	$x_2$	$x_3$	$x_4$
N	22- 29 $\theta$	0	0	3	4
D	6- 6 $\theta$	0	0	2	1
$x_2$	-6+ 7 $\theta$	0	1	-1	-1
$x_1$	-8+ 6 $\theta$	1	0	-2	-1

tab. 6

Then  $\gamma(\theta) = (26 - 40\theta, -2 - 5\theta) \leq 0$ ,  $H'' = \{\theta : \theta \geq 13/20\}$ ,  $x_B(\theta) = (-6 + 7\theta, -8 + 6\theta) \geq 0$ ,  $H' = \{\theta : \theta \geq 4/3\}$ , hence  $H^* = \{\theta : \theta \geq 4/3\}$ . Then for  $\theta \in H^*$   $x^1 = (-8 + 6\theta, -6 + 7\theta)$  is optimal and  $z(\theta) = (29\theta - 22)/(6\theta - 6)$ .

For  $\theta < -2/5$  it follows from tab. 5 that for  $-2 \leq \theta \leq -2/5$  the solution remains feasible while the reduced cost of variable  $x_1$  is positive and the relative column is nonpositive. This implies that the supremum is reached along the following extreme ray:

$$\begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 + \theta \\ 8 - 6\theta \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_1 \quad \text{with} \quad z(\theta) = 4.$$

For  $\theta < -2$  with a pivot transformation in tab. 5 we obtain tab. 7:

		$x_1$	$x_2$	$x_3$	$x_4$
N	-2- $\theta$	0	4	-1	0
D	$\theta$	0	1	1	0
$x_2$	-2- $\theta$	1	-1	-1	0
$x_1$	6- 7 $\theta$	0	-1	1	1

tab. 7

Then  $\gamma(\theta) = (-5\theta - 2, -2)$ ,  $\gamma_1(\theta) > 0$  for  $\theta < -2$ ,  $x_B(\theta) = (-2 - \theta, 6 - 7\theta) \geq 0$ ,  $H' = \{\theta : \theta \leq -2\}$ ,  $H^* = H'$ . Thus for  $\theta \in H^*$  the supremum is reached along the following extreme ray:

$$\begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 - \theta \\ 6 - 7\theta \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2 \quad \text{with} \quad z(\theta) = 4.$$

The function  $z(\theta)$  is now completely described. In fig. 2  $z(\theta)$  is depicted.

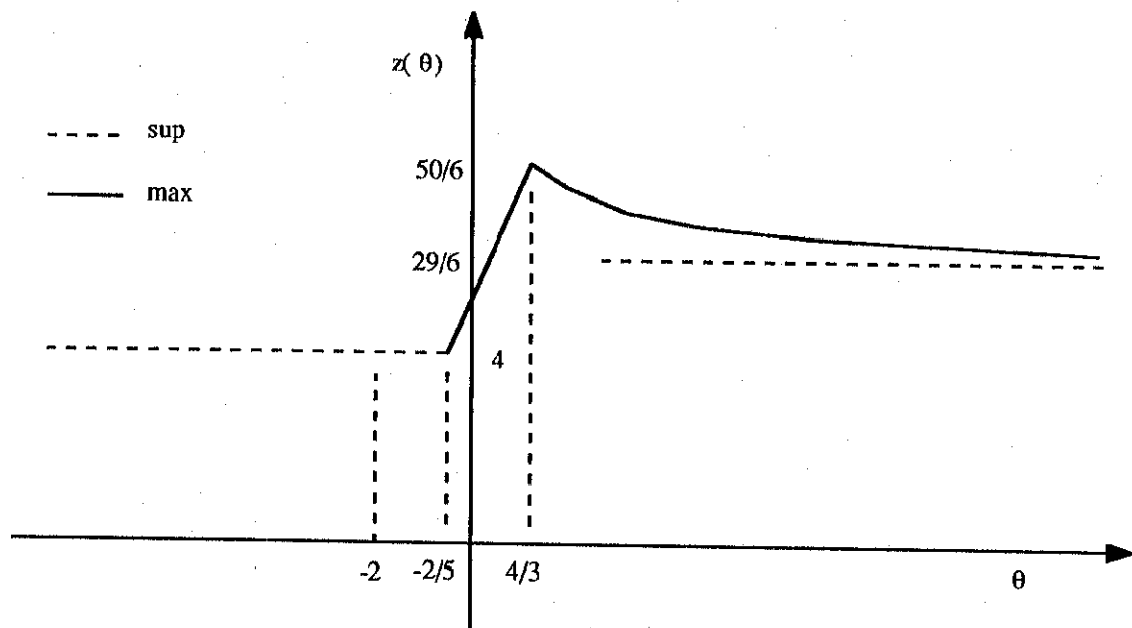


fig. 2

### 5. Conclusion.

Parametric linear fractional programming with an unbounded feasible region poses some difficulties not arising in the bounded case. Two types of algorithms have been applied to two kinds of parametric fractional programs. It turns out that a procedure based on the Charnes-Cooper transformation is more suitable in case of a parametrized objective function whereas a procedure based on the modified Martos algorithm is preferable in case of a parametrized right-hand side.

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