Report n.49

Tangent cones in Optimization

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Pisa, december 1991
This research is partially supported by the Italian Ministry of Education

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Abstract

The aim of this paper is to investigate the relationships among some tangent cones, which has been shown useful in studying optimality conditions and regularity in scalar and vector optimization, and a new one.

+ This research is partially supported by the Italian Ministry of Education.
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The paper has been discussed jointly by the Authors ; Cambini has developed section 4 Martein has developed sections 1-2-3.

1. Introduction

The results stated in recent papers [3-9] have pointed out that the image space is a suitable framework in order to obtain a general approach for studying different fields of scalar and vector optimization; more precisely the image space would seem to be the natural framework within which the study of optimality conditions, duality, regularity, interior and exterior penalty could be carried on.

In particular, the introduction in the image space of some suitable cones [2,3,5,6,8], namely $\mathbf{T}_{\mathbf{E}}$, $\mathbf{T}_{\mathbf{K}}$, has been shown useful in studying optimality conditions and regularity.

The aim of this paper is to investigate the relationships among $\mathbf{T}_{\mathbf{E}}$, $\mathbf{T}_{\mathbf{K}}$ and a new cone $\mathbf{T}_{\|}$ which seems to be appropriate for obtaining results in the image space and in the decision space both.

The obtained results will allow us to deep some aspects studied in [4,7,8]; this will be the object of forthcoming papers.

2. Statement of the problem

As outlined in the introduction, the aim of this paper is the study of some tangent cones related to a scalar or vector optimization problem.

More precisely, consider the following vector extremum problem:

P: $\max \varphi(x)$, $x \in S=\{x \in X: g(x) \geq 0\}$

where $X \subset \mathbf{R}^n$ is an open set and $\varphi=(\varphi_1 ... \varphi_s) : X \to \mathbf{R}^s$, $g=(g_1 ... g_m) : X \to \mathbf{R}^m$ $s \geq 1$, $m \geq 1$ are continuous functions.

Let x^0 be a feasible point; since φ, g are continuous functions, we can suppose, without loss of generality, that x^0 is binding at all the constraints i.e. $g_i(x^0)=0$, $i=1 ... m$.

Set

$f(x)=\varphi(x)-\varphi(x^0)$, $F(x)=(f(x), g(x))$, $\mathbf{K}=F(X)$, $\mathbf{H}=U^0 \times V$ where

$U^0=\mathbf{R}_+^s \setminus \{0\}$, $V=\mathbf{R}_+^m$, $\mathbf{E}=\mathbf{K}-\mathrm{cl}\mathbf{H}$.

We will refer to $\mathbf{R}^n$ as the decision space and to $\mathbf{R}^{s+m}$ as the image space. Let $\mathbf{T_E}$, $\mathbf{T_K}$ be the tangent cones at the origin to $\mathbf{E}$ and $\mathbf{K}$, respectively; i.e.

$\mathbf{T_E} = \{ t : \exists \{\alpha_n\} \subset \mathbf{R}, (e_n) \subset \mathbf{E}, \alpha_n \to +\infty, e_n \to 0 \text{ with } \alpha_n e_n \to t \}$

$\mathbf{T_K} = \{ t : \exists \{\alpha_n\} \subset \mathbf{R}, (F(x_n)) \subset \mathbf{K}, \alpha_n \to +\infty, F(x_n) \to 0 \text{ with } \alpha_n F(x_n) \to t \}$

The cones $\mathbf{T_E}$, $\mathbf{T_K}$ has been introduced by several authors with different aims [1,11,12] : in particular they have been used in [2-9] for the study of optimality conditions and regularity.

On working in the image space, we must pay attention in establishing conditions which permit also to deduce some results in the decision space. From this point of view it seems to be appropriate the introduction of the following cone $\mathbf{T_1}$:

$\mathbf{T_1} = \{ t : \exists \{\alpha_n\} \subset \mathbf{R}, (x_n) \subset X, \alpha_n \to +\infty, x_n \to x^0 \text{ with } \alpha_n F(x_n) \to t \}$.

In the next section we will point out some relations among the cones $\mathbf{T_E}$, $\mathbf{T_K}$, $\mathbf{T_1}$ and, for this reason, we will assume, from now on, that X is a suitable neighbourhood of x^0.

3. The tangent cones $\mathbf{T_E}$, $\mathbf{T_K}$, $\mathbf{T_1}$

It is well known that the tangent cone to a set S at $z_0 \in \mathrm{cl}S$ is a closed cone, so that $\mathbf{T_E}$, $\mathbf{T_K}$ are closed cones. The following Theorem shows that $\mathbf{T_1}$ is a closed cone too.

Theorem 3.1 $\mathbf{T_1}$ is a closed cone.

<u>proof.</u> Consider a sequence $\{t_k\} \subset \mathbf{T_1}$ with $t_k \to t$, $t \in \mathrm{cl}\mathbf{T_1}$. We must show that $t \in \mathbf{T_1}$. Since $t_k \in \mathbf{T_1}$, there exist a sequence $\alpha_{n_k} \to +\infty$ and a sequence $x_{n_k} \to x^0$, such that $\alpha_{n_k} F(x_{n_k}) \to t_k$. Let $S(t, \frac{1}{m})$ be an open ball around t with radius $\frac{1}{m}$; then there exists h such that $t_h \in S(t, \frac{1}{m})$ and a neighbourhood I_{t_h} of t_h with $I_{t_h} \subset S(t, \frac{1}{m})$ such that

$\alpha_{n_h} F(x_{n_h}) \in I_{t_h} \subset S(t, \frac{1}{m})$.

Set $\beta_m = \alpha_{n_h}$, $nx_m = x_{n_h}$; we have $x_m \to x^o$, $\beta_m F(x_m) \to t$ and this implies $t \in \mathbf{T_1}$. □

The following Theorem states a relation among the cones $\mathbf{T_E}$, $\mathbf{T_K}$ and $\mathbf{T_1}$.

Theorem 3.2

i) $\mathbf{T_1} \subset \mathbf{T_K}$

ii) $\mathbf{T_1} - \mathrm{cl}\mathbf{H} \subset \mathbf{T_K} - \mathrm{cl}\mathbf{H} \subset \mathbf{T_E}$

proof.

i) it follows immediately from the given definitions of $\mathbf{T_K}$ and $\mathbf{T_1}$, taking into account the continuity of the function $F(x)$.

ii) Let $y \in \mathbf{T_K} - \mathrm{cl}\mathbf{H}$, that is $y = t - h$, $t \in \mathbf{T_K}$, $h \in \mathrm{cl}\mathbf{H}$. Since $t \in \mathbf{T_K}$, there exists $\alpha_n \to +\infty$, $(F(x_n)) \subset \mathbf{K}$, $F(x_n) \to 0$ with $\alpha_n F(x_n) \to t$.

Set $e_n = F(x_n) - \dfrac{h}{\alpha_n} \in \mathbf{E}$; we have $\alpha_n e_n = \alpha_n F(x_n) - h \to t - h \in \mathbf{T_E}$.

This completes the proof. □

The following example shows that the inclusion $\mathbf{T_K} - \mathrm{cl}\mathbf{H} \subset \mathbf{T_E}$ is proper.

Example 3.1

Consider problem P where $s=1$, $\varphi(x) = x^2$, $m=2$, $g_1(x) = x$, $g_2(x) = -x^4$, $x \in X \subset \mathbf{R}$, X being a neighbourhood of $x^o = 0$.

We have $\mathbf{K} = \{ F(x) = (x^2, x, -x^4), x \in X \}$ and

$$\frac{F(x)}{|F(x)|} = \frac{1}{|x|\sqrt{x^2+1+x^6}}(x^2, x, -x^4) \text{ so that}$$

$\mathbf{T_K} - \mathrm{cl}\mathbf{H} = \{ k(0,1,0), k \in \mathbf{R} \} - \mathrm{cl}\mathbf{H} = \{ (a, k, b), a,b \leq 0, k \in \mathbf{R} \}$.

Consider now the sequences $x_n = \dfrac{1}{n}$, $h_n = \left(0, \dfrac{1}{n} - \dfrac{1}{n^2}, 0\right)$.

We have $(F(x_n) - h_n) = \left(\dfrac{1}{n^2}, \dfrac{1}{n^2}, -\dfrac{1}{n^4}\right)$ so that $n^2(F(x_n) - h_n) \to (1,1,0)$ with $(1,1,0) \notin \mathbf{T_K} - \mathrm{cl}\mathbf{H}$. It is easy to show that $\mathbf{T_E} = \{(a,b,c), a,b \in \mathbf{R}, c \leq 0\}$.

In order to deep the relation among the cones, we analize, first of all, the simple case $\mathbf{T_1} \cap \mathrm{int}\mathbf{H} \neq \emptyset$. The following Theorem holds:

Theorem 3.3 Assume that $\mathbf{T_1} \cap \mathrm{int}\mathbf{H} \neq \emptyset$. Then:

$$\mathbf{T_E} = \mathbf{T_K} - \mathrm{cl}\mathbf{H} = \mathbf{T_1} - \mathrm{cl}\mathbf{H} = \mathbf{R}^{s+m}.$$

proof. It is sufficient to show that $\mathbf{T_1} - \mathrm{cl}\mathbf{H} = \mathbf{R}^{s+m}$. Since $\mathbf{T_1} \cap \mathrm{int}\mathbf{H} \neq \emptyset$, there exists $a=(a_1, ..., a_{s+m}) \in \mathbf{T_1}$ such that $a_i > 0$ $i=1, ..., s+m$.
Let $z=(z_1, ..., z_{s+m}) \in \mathbf{R}^{s+m}$ and set $I=\{i: z_i > 0\}$; obviously there exists $k>0$ such that $ka_i > z_i$ $\forall i \in I$. As a consequence $ka > z$ and so $z=ka-h$, $h \in \mathrm{cl}\mathbf{H}$.
This completes the proof. □

Taking into account the previous Theorem , the study will be carried on in the case $\mathbf{T_1} \cap \mathrm{int}\mathbf{H} = \emptyset$.
A sufficient condition in order to have $\mathbf{T_E} = \mathbf{T_K} - \mathrm{cl}\mathbf{H}$ is given in the following Theorem:

Theorem 3.4 Assume that $\mathbf{K} \cap \mathrm{cl}\mathbf{H} = \{0\}$ and $\mathbf{T_K} \cap \mathrm{cl}\mathbf{H} = \{0\}$. Then

$$\mathbf{T_E} = \mathbf{T_K} - \mathrm{cl}\mathbf{H} .$$

proof. Let $0 \neq t \in \mathbf{T_E}$, that is there exist $\{F(x_n)\} \subset \mathbf{K}$, $\{h_n\} \subset \mathrm{cl}\mathbf{H}$, $e_n= F(x_n)-h_n \to 0$, $\alpha_n \to +\infty$ with

$$\alpha_n (F(x_n-h_n) \to t . \tag{3.1}$$

First of all , let us note that $F(x_n) \to 0$; in fact $\{x_n\}$ is a bounded sequence so that[1] $x_n \to z$ and , for the continuity of $F(x)$, $F(x_n) \to F(z) \in \mathbf{K}$; on the other hand $F(x_n) - h_n \to 0$ implies $h_n \to z \in \mathrm{cl}\mathbf{H}$ and consequently $z=0$ since $\mathbf{K} \cap \mathrm{cl}\mathbf{H} = \{0\}$. □

[1] Since in a finite dimensional space any bounded sequence $\{z_n\}$ has a convergent subsequence, we will assume without loss of generality (substituting $\{z_n\}$ with a suitable subsequence, if necessary), that $z_n \to z$.

Now we prove that $\{\alpha_n |F(x_n|\}$ is a bounded sequence ; if not , from (3.1) we have $\alpha_n \frac{F(x_n)-h_n}{\alpha_n|F(x_n)|} \to 0$ that is $\lim \frac{F(x_n)}{|F(x_n)|} = z = \lim \frac{h_n}{|F(x_n)|}$ so that $z \in T_K \cap clH$, $z \neq 0$ since $\frac{F(x_n)}{|F(x_n)|}$ belongs to the unit ball, and this contradicts the assumption $T_K \cap clH = \{0\}$.

Since $\{\alpha_n |F(x_n|\}$ is a bounded sequence we have $\alpha_n F(x_n) \to t^* \in T_K$ and consequently $\alpha_n h_n \to t^* - t \in clH$. This completes the proof. □

Now we will study some relations between T_K and T_1.

The following example shows that $T_1 \neq T_K$

Example 3.2

Consider problem P where s=1, $\varphi(x_1,x_2) = x_1^2x_2$, m=2 , $g_1(x_1,x_2)= x_2$, $g_2(x_1,x_2) = -x_2$, $(x_1,x_2)\in X$, X being an open set containing $x^o=(0,0)$.

We have $K=\{ F(x_1,x_2)= (x_1^2x_2 , x_2 , -x_2), (x_1,x_2) \in X\}$ and

$$\frac{F(x)}{|F(x)|} = \frac{1}{|x_2|\sqrt{x_1^4+2}} (x_1^2x_2 , x_2 , -x_2)$$

Since in a finite dimensional space the elements of T_K are obtained by multiplying for a non negative real number the limit of $\frac{F(x_n)}{|F(x_n)|}$ for any sequence $\{ F(x_n)\}$ with $|F(x_n)| \to 0$, we have

$T_K = \{ \lambda (x_1^2, 1, -1) , \lambda \in \mathbf{R} , (x_1,x_2) \in X \}$

On the other hand, the elements of T_1 are obtained by multiplying for a non negative real number the limit of $\frac{F(x_n)}{|F(x_n)|}$ for any sequence $x_n \to x^o$.

As a consequence, we have $T_1 = \{ k (0, 1, -1) , k\in \mathbf{R} \}$ so that the elements of T_K with $x_1 \neq 0$ are not contained in T_1.

The following theorems characterize some classes of problems for which $T_1 = T_K$

Theorem 3.5 Consider problem P where φ and g are linear functions. Then $T_1 = T_K$.

proof. Obviously $T_1 \subset T_K$ so we must prove that $T_K \subset T_1$. Let $t \in T_K$; then there exist a sequence $\{F(x_n)\}$ with $F(x_n) \to F(x^0)$ and a sequence $\{\alpha_n\} \subset R$, $\alpha_n \to +\infty$, such that $\alpha_n F(x_n) \to t$.

Set $\bar{x}_n = x^0 + \frac{x_n - x^0}{n}$; obviously $\bar{x}_n \to x^0$ and furthermore, taking into account the linearity of F, we have $\alpha_n nF(\bar{x}_n) = \alpha_n F(x_n) \to t$, so that $t \in T_1$. □

Theorem 3.6 Consider problem P and assume that there does not exist a sequence $x_n \to x^0$ such that $F(x_n) = F(x^0)$ $\forall n$. Then $T_1 = T_K$.

proof. It is sufficient to prove that $F(x_n) \to F(x^0)$ implies $x_n \to x^0$.
The assumption implies the existence of a neighbourhood I of x^0 such that $F(x) \neq F(x^0)$ $\forall x \in I$, $x \neq x^0$.
Since $\{x_n\}$ is a bounded sequence, we can suppose[1] that $x_n \to \bar{x}$.
For the continuity of F, we have $F(x_n) \to F(\bar{x})$ so that $F(\bar{x}) = F(x^0)$ and, necessarily, $\bar{x} = x^0$. □

Corollary 3.1 We have $T_1 = T_K$ when one of the following conditions hold:
i) x^0 is a local optimal solution and there are not alternate solutions.
ii) x^0 is a local optimal solution, there exists a sequence $\{x_n\}$ of alternate solutions with $x_n \to x^0$ and there exists an index i such that $g_i(x_n) \neq g_i(x^0) = 0$ $\forall n$.

Taking into account the results given in Theorems 3.4, 3.6. and in Corollary 3.1 we have the following Theorem and Corollary:

Theorem 3.7 Assume that $\mathbf{K} \cap \mathrm{cl}\mathbf{H} = \{0\}$, $\mathbf{T_1} \cap \mathrm{cl}H = \{0\}$ and, furthermore, that there does not exist a sequence $x_n \to x^o$ such that $F(x_n) = F(x^o)$ $\forall n$.
Then $\mathbf{T_E} = \mathbf{T_K} - \mathrm{cl}\mathbf{H} = \mathbf{T_1} - \mathrm{cl}\mathbf{H}$.

Corollary 3.2 Assume that $\mathbf{K} \cap \mathrm{cl}\mathbf{H} = \{0\}$, $\mathbf{T_1} \cap \mathrm{cl}H = \{0\}$ and, furthermore, that one of the following conditions hold:
i) x^o is a local optimal solution and there are not alternate solutions.
ii) x^o is a local optimal solution, there exists a sequence $\{x_n\}$ of alternate solutions with $x_n \to x^o$ and there exists an index i such that $g_i(x_n) \neq g_i(x^o) = 0$ $\forall n$.
Then $\mathbf{T_E} = \mathbf{T_K} - \mathrm{cl}\mathbf{H} = \mathbf{T_1} - \mathrm{cl}\mathbf{H}$.

4. Characterization of T_1 in the differentiable case

In this section we will give a characterization of the tangent cone T_1 when P is a differentiable problem. Assume that in problem P, φ and g are differentiable functions at x^o.
We have

$$F(x) - F(x^o) = J(x - x^o) + \sigma(x, x^o) \qquad (4.1)$$

where J is the Jacobian matrix of F at x^o and $\dfrac{\sigma(x,x^o)}{|x-x^o|} \to 0$.

Set:

$$\mathbf{K_L} = \{ J(x-x^o), x \in \mathbf{R}^n \}$$

$$A = \{ t \in \mathbf{T_1}/\{0\} : \exists\, x_n \to x^o,\ \alpha_n \to +\infty \text{ with } \alpha_n F(x_n) \to t,\ \frac{x_n - x^o}{|x_n - x^o|} \to y \text{ and } J(y) = 0 \}$$

The following Theorem holds:

Theorem 4.1 Consider problem P where φ and g are differentiable functions at x^o. Then

$$\mathbf{T_1} = \mathbf{K_L} \cup A \qquad (4.2)$$

proof. First of all we prove that $\mathbf{T_1} \supset \mathbf{K_L}$. If $0 \neq z \in \mathbf{K_L}$, there exists $x^* \in \mathbf{R}^n$ such that $z = J(x^*-x^o)$. Consider the sequence $(x_n = x^o + \frac{x^*-x^o}{n})$.

We have $x_n \to x^o$ and $F(x_n) - F(x^o) = \frac{1}{n} J(x^*-x^o) + \sigma(x_n, x^o)$.

Taking into account that $n\,\sigma(x_n, x^o) = |x^*-x^o| \dfrac{\sigma(x_n, x^o)}{x_n-x^o} \to 0$ we have $nF(x_n) \to z \in \mathbf{T_1}$. Since $A \subset \mathbf{T_1}$ it results $\mathbf{T_1} \supset \mathbf{K_L} \cup A$.

Now we prove that $\mathbf{T_1} \subset \mathbf{K_L} \cup A$.

Let $0 \neq t \in \mathbf{T_1}$; then there exist a sequence $x_n \to x^o$ and a sequence $\alpha_n \to +\infty$ with $\alpha_n F(x_n) \to t$. Since the unit ball in $\mathbf{R}^n$ is a compact set, we can suppose[1] that $\dfrac{x_n-x^o}{|x_n-x^o|} \to y$. If $J(y)=0$ then $y \in A$, otherwise we have $\alpha_n F(x_n) = \alpha_n |x_n-x^o| (J(\dfrac{x_n-x^o}{|x_n-x^o|}) + \dfrac{\sigma(x_n, x^o)}{x_n-x^o})$

Since $J(y) \neq 0$, necessarily we have $\alpha_n |x_n-x^o| \to k \neq 0$ and $t=kJ(y)$, so that $t \in \mathbf{K_L}$.

The proof is complete. □

In the following Theorems we will give necessary and/or sufficient conditions in order to have $A = \emptyset$ in (4.2).

Theorem 4.2 If rank$J = n$ then $\mathbf{T_1} = \mathbf{K_L}$.

proof. The assumption implies that $J(y)=0$ is verified if and only if $y=0$. Consequently $A = \emptyset$. □

Theorem 4.3 Suppose that there does not exist a sequence $x_n \to x^o$ such that $F(x_n)=F(x^o)$ $\forall$ n. Then $\mathbf{T_1}=\mathbf{K_L}$ if and only if rankJ = n .

proof. Suppose that there exists $y \neq 0$ such that $J(y)=0$. The sequence $x_n = x^o + \frac{1}{n} y$ is contained in X , for n arbitrarily large ; we have $x_n \to x^o$, $F(x_n) \neq F(x^o)=0$, $\frac{F(x_n)}{|F(x_n)|} \to t \in \mathbf{T_1} \setminus \{0\}$, so that $A \neq \emptyset$ and this is a contradiction. □

Theorem 4.4 Consider problem P where φ and g are linear functions. Then $\mathbf{T_1}=\mathbf{K_L}$.

proof. Since $\mathbf{K} = F(X) = J(x-x^o)$, we have $\mathbf{T_K}= J(x-x^o)$. The thesis follows from Theorem 3.5. □

The following example shows that the condition rankJ=n is not necessary in order to have $\mathbf{T_1}=\mathbf{K_L}$.

Example 4.1
Consider problem P where s=1, $\varphi(x_1, x_2) = x_1+x_2$, m=2, $g_1(x_1,x_2) = x_1+x_2$, $g_2(x_1,x_2) = -x_1-x_2$, $x^o=0$.
Since the problem is linear, from Theorem 4.4 we have $\mathbf{T_1}=\mathbf{K_L}$ but rankJ $\neq$ 2. □

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