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Optimality conditions in Vector and Scalar Optimization: a unified approach

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Optimality conditions in Vector and Scalar optimization: a unified approach+

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Abstract

The aim of this paper is to carry on the study of optimality in the vector and in the scalar case jointly, by studying the disjunction of suitable sets in the image space.

A cone is introduced which allows us to find necessary and/or sufficient optimality conditions in the image space and in the decision space both.

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1. Introduction

In this paper we will consider the vector optimization problem

$$P: \max \varphi(x) = (\varphi_1(x), \ldots, \varphi_s(x)) \; ; \quad x \in S = \{x \in X \subset \mathbf{R}^n : g(x) = (g_1(x), \ldots, g_m(x) \geq 0\}$$

which reduces to a scalar problem when the objective function has only one component.

It is known (see for istance [2,5,7,10]) that a feasible point x^0 is a local optimal solution for P if and only if $\mathbf{K} \cap \mathbf{H} = \emptyset$, where $\mathbf{K}$ and $\mathbf{H}$ are suitable sets in the image space $\mathbf{R}^{s+m}$.

Since $\mathbf{K}$ does not have in general properties which are useful in the study of such a disjunction, some authors [5,7,10] have introduced suitable sets instead of $\mathbf{K}$ with different aims (for instance in studying regularity and proper efficiency).

The aim of this paper is to point out that the study of optimality in the vector and in the scalar case can be carried on jointly in the image space; more exactly any logical consequence of $\mathbf{K} \cap \mathbf{H} = \emptyset$ becomes a necessary optimality condition, while any condition which ensures $\mathbf{K} \cap \mathbf{H} = \emptyset$ becomes a sufficient optimality condition.

In this order of ideas, we will define a suitable tangent cone $\mathbf{T}_1$, which allows us to find necessary and/or sufficient optimality conditions in the image space.

The obtained results can be used to deduce necessary and/or sufficent optimality conditions in the decision space, whenever a characterization of $\mathbf{T}_1$ is established.

The characterization of $\mathbf{T}_1$ for differentiable problems given in [4], allows us to find, in a unified approach, new optimality conditions as well as F. John and Kuhn- Tucker conditions.

2. Statement of the problem

Consider the following vector extremum problem

P: $\max \varphi(x)$, $x \in S = \{x \in X: g(x) \geq 0\}$

where $X \subset \mathbb{R}^n$ is an open set and $\varphi = (\varphi_1 \ldots \varphi_s): X \to R^s$, $g = (g_1 \ldots g_m): X \to \mathbb{R}^m$ $s \geq 1$, $m \geq 1$ are continuous functions.
We recall that a point $x^0 \in S$ is said to be a (Pareto) optimal solution to the problem P if there is no $x \in S$ such that

$$\varphi_i(x) - \varphi_i(x^0) \geq 0 \quad i=1 \ldots s \tag{2.1}$$

where at least one inequality is strict.
We say that x^0 is a local Pareto optimal solution if (2.1) holds in a suitable neighbourhood of x^0.
Let us note that when $s=1$ problem P reduces to a scalar optimization problem and (2.1) collapses to the ordinary definition of a local maximum point.
Since we are interested to investigate local optimality conditions, for sake of simplicity, through the paper, X will play the role of a suitable neighbourhood of x^0.
Let x^0 be a feasible point ; since φ, g are continuous functions, we can suppose, without loss of generality, that x^0 is binding at all the constraints i.e. $g_i(x^0) = 0$, $i = 1 \ldots m$.
Set
$f(x) = \varphi(x) - \varphi(x^0)$, $F(x) = (f(x), g(x))$, $\mathbf{K} = F(X)$, $\mathbf{H} = U^0 \times V$ where
$U^0 = \mathbb{R}^m_+ \setminus \{0\}$, $V = \mathbb{R}^m_+$

Let us note that x^0 is either a local Pareto optimal solution or a local maximum point ($s=1$) if and only if

$$\mathbf{K} \cap \mathbf{H} = \emptyset \tag{2.2}$$

so that the study of optimality in the vector and in the scalar case can be carried on jointly by studying the disjunction between **K** and **H**.
More exactly any logical consequence of (2.2) becomes a necessary optimality condition, while any condition which ensures (2.2) becomes a sufficient optimality condition.
Taking into account the aforesaid considerations, from now on, we will refer to x^0 as a local optimal solution .
Since **K** does not have in general properties which are useful in the study of disjunction between **K** and **H**, some authors [2,3,5,7] have considered suitable sets instead of **K** with the aim to study such a disjunction.
As we will see in the next sections the cone $\mathbf{T_1}$, defined as

$\mathbf{T_1} = \{ t : \exists\ \alpha_n \to +\infty\ , x_n \to x^0 \text{ with } \alpha_n F(x_n) \to t \}$

seems to be appropriate in order to obtain optimality conditions in the image space and in the decision space both.
The properties of $\mathbf{T_1}$ and its relations with some other cones has been studied in [4].

3. Optimality conditions in the image space

In order to find necessary and/or sufficient conditions in the image space, we premize the following Lemma

Lemma 3.1 If $\mathbf{K} \cap \text{int}\mathbf{H} = \emptyset$ then $\mathbf{T_1} \cap \text{int}\mathbf{H} = \emptyset$

<u>proof.</u>
Assume $t^* \in \mathbf{T_1} \cap \text{int}\mathbf{H}$, that is $t^*>0$; then there exist a sequence $\{x_n\} \subset X$ with $F(x_n) \to F(x^0) = 0$ and a sequence $\alpha_n \to +\infty$, such that $\alpha_n F(x_n) \to t^*$. Hence $\exists\ m : \alpha_m F(x_m) > 0$ and this implies $F(x_m) > 0$, that is $\mathbf{K} \cap \text{int}\mathbf{H} \neq \emptyset$ and this is a contradiction. □

The following Theorem states a necessary optimality condition.

Theorem 3.1 Let x^0 be a local optimal solution for problem P. Then $T_1 \cap \text{int}H = \emptyset$

proof.
The thesis follows immediately from Lemma 3.1, taking into account that the assumption implies that $K \cap H = \emptyset$. □

The following example shows that $T_1 \cap \text{int}H = \emptyset$ is a necessary but not sufficient optimality condition.

Example 3.1
Consider problem P where $s=1$, $\varphi(x) = x^2$, $m=1$, $g(x) = x$, $x^0=0$.
It is easy to show that $T_1=\{\lambda(0,1), \lambda \in \mathbb{R}\}$ so that condition $T_1 \cap \text{int}H=\emptyset$ holds but $x^0=0$ is not an optimal solution for P.

The following Theorem gives a sufficient optimality condition.

Theorem 3.2 Consider problem P. If

$$T_1 \cap \text{cl}H = \{0\} \tag{3.1}$$

then x^0 is a local optimal solution for P.

proof.
If x^0 is not optimal for P there exists a sequence $x_n \to x^0$ such that $F(x_n) \in H$.
Since the unit ball S is a compact set, we can suppose[1] that the sequence $\frac{F(x_n)}{|F(x_n)|}$ converges at $t^* \neq 0$, $t^* \in T_1$. On the other hand $\frac{F(x_n)}{|F(x_n)|} \in H$ so that $t^* \in \text{cl}H$ and this is a contradiction. □

[1] Since in a finite dimensional space any bounded sequence $\{z_n\}$ has a convergent subsequence, we will assume without loss of generality (substituting $\{z_n\}$ with a suitable subsequence, if necessary), that $z_n \to z$.

The following example shows that (3.1) is not a necessary optimality condition.

Example 3.2
Consider problem P where $s=1$, $\varphi(x) = -x^2$, $m=1$, $g(x) = x$, $x^o=0$.
It is easy to verify that $\mathbf{T_1}=\{\lambda(0,1), \lambda \in \mathbf{R}\}$ so that $\mathbf{T_1} \cap \mathrm{cl}\mathbf{H} \neq \{0\}$ but $x^o=0$ is the optimal solution of problem P.

The following Theorem states a necessary and sufficient optimality condition.

Theorem 3.3 Consider problem P. The feasible point x^o is a local optimal solution for P if and only if condition I holds:

Condition I: Assume that $0 \neq t \in \mathbf{T_1} \cap \mathrm{cl}\mathbf{H}$. Then for any sequence $x_n \to x^o$ such that there exists $\alpha_n \to +\infty$ with $\alpha_n F(x_n) \to t$, we have $F(x_n) \notin \mathbf{H}$ $\forall n$.

proof.
if. The thesis follows immediately from (2.2).
only if. The proof is similar to the one given in Theorem 3.2. □

4. First order optimality conditions in the decision space

When P is a differentiable problem, it can be shown [4] that the tangent cone $\mathbf{T_1}$ can be characterized as $\mathbf{T_1}=\mathbf{K_L} \cup A$ where
$\mathbf{K_L}=\{J(x-x^o), x \in \mathbb{R}^n\}$, J is the Jacobian matrix of F at x^o,
$A=\{t \in \mathbf{T_1}/\{0\} : \exists\ x_n \to x^o,\ \alpha_n \to +\infty$ with $\alpha_n F(x_n) \to t$,

$$\frac{x_n-x^o}{|x_n-x^o|} \to y \text{ and } J(y)=0\}$$

This characterization will allow us to obtain in a very simple way new optimality conditions as well as the Fritz John optimality conditions either in the vector case or in the scalar case.

First of all we need to find a hyperplane Γ which separates $\mathbf{K_L} - \mathrm{cl}\mathbf{H}$ and $\mathrm{cl}\mathbf{H}$ such that $(\mathbf{K_L} - \mathrm{cl}\mathbf{H}) \cap \mathrm{cl}\mathbf{H} = \Gamma \cap \mathrm{cl}\mathbf{H}$.

In order to be able to find such a hyperplane, we must prove , first of all, that $\mathbf{K_L} - \mathrm{cl}\mathbf{H}$ is a closed set.

As regards to this last problem we will consider a linear subspace W of $\mathbf{R}^p$ and we will use the following notations:

if $I=\{i_1,....,i_s\}$ is a set of indices, we will denote with $z(I^*)$ the vector $z(I^*)=(z_{i_1}, ..., z_{i_s})$; if $z=(z_1, z_2,...., z_p)$ and (I^*,J^*) is a partition of $\{1,2,...,p\}$, without loss of generality, eventually by performing a rearrangement of the components of z , we set $z=(z(I^*), z(J^*))$.

Theorem 4.1. Let W be a linear subspace of $\mathbf{R}^p$. Then $W - \mathbf{R}^p_+$ is a closed convex cone.

<u>proof.</u>

If $W \cap \mathrm{int}\, \mathbf{R}^p_+ \neq \emptyset$, then $W - \mathbf{R}^p_+ = \mathbf{R}^p$ and the thesis is obvious.

It is easy to show that $W - \mathbf{R}^p_+$ is a convex cone. In order to prove that $W - \mathbf{R}^p_+$ is closed , let $\{w_k\}$ and $\{r_k\}$ be sequences such that $\{w_k\} \subset W$, $\{r_k\} \subset \mathbf{R}^p_+$ with $w_k - r_k \rightarrow z$. In order to show that $z \in W - \mathbf{R}^p_+$, we will find $w \in W$, $r \in \mathbf{R}^p_+$ such that $w - r = z$.

The proof is obvious when z=0 or when at least one of the sequences $\{w_k\}$, $\{r_k\}$ is convergent or , equivalently , when the intersection of the recession cones [10] $0^+(W-\mathbf{R}^p_+)$ and $0^+(\mathbf{R}^p_+) = \mathbf{R}^p_+$, is the singleton set$\{0\}$.

Consider the case $|w_k| \to +\infty$ and set $J = \{i: w_{k_i} < M\}$, so that $\{w_k(J)\}$ or one of its subsequences is convergent [1], i.e. $w_k(J) \to w(J)$.
Set $I_0 = \{ i: w_{k_i} \to +\infty \}$; taking into account that $r_k \geq 0$ and $w_k - r_k \to z$, necessarily we have $I_0 \neq \emptyset$ and $I_0 \cup J = \{1, ..., p\}$.
Now we will construct, by recurrence, a finite sequence of vectors $w^{(1)}, ..., w^{(s)}$, a finite sequence of sets of indices $I_1, ..., I_s$, and a finite sequence of sequences $\{w_k^{(1)}\}, ... ,\{w_k^{(s)}\}$, where s is the first index which verifies $I_s = \emptyset$.
Consider

$$w^{(1)} = \lim_{k \to +\infty} \frac{w_k}{|w_k(I_0)|} \quad ; \quad I_1 = \{ i \in I_0 : w_i^{(1)} = 0\}; \qquad (4.1.a)$$
$$w_k^{(1)} = w_k - |w_k(I_0)|\, w^{(1)}$$

and, if s >1

$$(4.1.b) \quad w^{(h)} = \lim_{k \to +\infty} \frac{w_k^{(h-1)}}{|w_k(I_{h-1})|} \quad ; \quad I_h = \{ i \in I_{h-1} : w_i^{(h)} = 0\}; \qquad (4.1.b)$$
$$w_k^{(h)} = w_k^{(h-1)} - |w_k(I_{h-1})|\, w^{(h)} \qquad h=2, ..., s.$$

We will prove , by induction, the following properties:

$$w^{(h)} \in W \qquad h=1, ..., s \qquad (4.2.a)$$

$$w^{(h)}(J) = 0 \qquad h=1, ..., s \qquad (4.2.b)$$

$$w^{(h)}(I_h) = 0 \qquad h=1, ..., s-1 \qquad (4.2.c)$$

$$w^{(h)}(I_{h-1} - I_h) > 0 \qquad h=1, ..., s \qquad (4.2.d)$$

$$w_k^{(h)}(J) = w_k(J) \qquad h=1, ..., s \qquad (4.2.e)$$

$$w_k^{(h)}(I_h) = w_k(I_h) \qquad h=1, ..., s-1 \qquad (4.2.f)$$

$$w_k^{(h)}(I_o-I_h) \to 0 \qquad h-1, ..., s \qquad (4.2.g)$$

$$w_k^{(s)} \to (0, w(J)) \in W. \qquad (4.2.h)$$

<u>Case h=1</u>. Consider the sequence:

$$\frac{w_k}{|w_k(I_0)|} = \left(\frac{w_k(I_0)}{|w_k(I_0)|}, \frac{w_k(J)}{|w_k(I_0)|}\right) \qquad (4.3)$$

Since $\frac{w_k}{|w_k(I_0)|} \to 0$ and $\frac{w_k}{|w_k(I_0)|}$ belongs to the unit ball, the sequence (4.3) or one of its subsequences[1] converges to $w^{(1)}$ such that $w^{(1)}(J) = 0$; furthermore $w^{(1)} \in W$ since W is a closed cone .

Properties (1.2c,e,f) follow from the given definition of I_1 and J ; since $w^{(1)} \geq 0$, $w^{(1)} \neq 0$, we have (4.2d) while (4.2g) follows by (4.2d) taking into account the limit in (4.1a).

Now assume that (4.2) holds for the index h ; we will show that (4.2) holds for the index h+1 too.

With this aim consider the sequence

$$\frac{w_k^{(h)}}{|w_k(I_h)|} = \left(\frac{w_k^{(h)}(I_o-I_h)}{|w_k(I_h)|}, \frac{w_k^{(h)}(I_h)}{|w_k(I_h)|}, \frac{w_k^{(h)}(J)}{|w_k(I_h)|}\right) \qquad (4.4)$$

Let us note that (4.2f) is equivalent to state that

$w_{k_i}^{(h)} = w_{k_i} \to +\infty \quad \forall\ i \in I_h$

From (4.2e,g) we have

$$\frac{w_k^{(h)}}{|w_k(I_h)|} \to 0 = w^{(h+1)}(J) \ , \quad \frac{w_k^{(h)}(I_o-I_h)}{|w_k(I_h)|} \to 0 = w^{(h+1)}(I_o-I_h) .$$

Furthermore $\frac{w_k^{(h)}}{|w_k(I_h)|}$ belongs to the unit ball so that it converges[1] to the non-negative element $w^{(h+1)}(I_h)$ and (4.4) converges to $w^{(h+1)}$ which belongs to W since W is a closed cone.

Taking into account that $w^{(h+1)}(I_{h+1})=0$, we have $w^{(h+1)}(I_h-I_{h+1}) > 0$ and consequently $w^{(h+1)}(I_h-I_{h+1}) \to 0$. This last result, together with the relation $w^{(h+1)}(I_0-I_h) = 0$, implies (4.2g) ; (4.2e,f) follows from the definitions of I_{h+1} and J.

At last , let us note that $w_k^{(h)} \in W$ $h=1,...,s$, since W is a vector space; on the other hand since $I_s = \emptyset$, from (4.2e,g) we have $w_k^{(s)}(I_0) \to 0$ and $w_k^{(s)}(J) = w_k(J) \to w(J)$; then (4.2h) holds since W is closed.

Now we are able to find $w \in W$, $r \in \mathbf{R}_+^p$ such that $w-r=z$, where

$z = \lim_{k\to+\infty} (w_k - r_k)$.

Since $w_k(J) \to w(J)$ and $w_k(J) - r_k(J) \to z(J)$, necessarily we have $r_k(J) \to r(J) = w(J) - z(J)$.

Consider now $w^* = \sum_{i=1}^{s} w^{(i)} \in W$; from (4.2d) we have $w^*(I_0) > 0$ and so there exists a scalar $k>0$ such that

$$kw^*(I_0) \geq z(I_0) . \tag{4.5}$$

We have, taking into account that from (4.2b) $w^*(J)=0$, that

$kw^* + (0,w(J)) = (kw^*(I_0) , w(J)) = w \in W$.

Set $r=(kw^*(I_0)-z(I_0) , r(J))$ where $r(J)= \lim_{k\to+\infty} r_k(J)= \lim_{k\to+\infty} w_k(J)=w \in W$.

Let us note that $r \in \mathbf{R}_+^p$ since (4.5) holds and $r_k(J) \geq 0$.

It is easy to verify that w-r=z and this completes the proof. □

Assume now that $W \cap \text{int}\, \mathbf{R}_+^p \neq \emptyset$; then $(W-\mathbf{R}_+^p) \cap \mathbf{R}_+^p \triangleq C$ is a face of $\mathbf{R}_+^p$ with dimC=k , $0 \leq k \leq p-1$. The following Lemma holds :

Lemma 4.1 The face C is contained in any hyperplane which separates $W-\mathbf{R}_+^p$ and $\mathbf{R}_+^p$.

proof.

Since $W\text{-}\mathbf{R}_+^p$ is a closed convex cone , it is also the intersection of its supporting half-space at the origin [8] . On the other hand , it is easy to show that any supporting hyperplane is also a hyperplane which separates $W\text{-}\mathbf{R}_+^p$ and $\mathbf{R}_+^p$. Consequently the face C is contained in the intersection of all hyperplanes separating $W\text{-}\mathbf{R}_+^p$ and $\mathbf{R}_+^p$. □

Theorem 4.2 There exists a hyperplane Γ separating $W\text{-}\mathbf{R}_+^p$ and $\mathbf{R}_+^p$, such that $\Gamma \cap \mathbf{R}_+^p = C$.

proof.

Let $e^1 , \dots , e^p$ be the edges of $\mathbf{R}_+^p$ where e^i is the vector whose i-th component is equal to 1 and the others are equal to 0 , and suppose , without loss of generality , that $e^1 , \dots , e^k$ $(0 \le k \le p-1)$ are those contained in C. From Lemma 4.1 , there exists a hyperplane $\Gamma_i = \{ z \in \mathbf{R}^p : (\alpha^i)^T z = 0 \}$, with $\alpha^i \ge 0$, $\alpha^i \ne 0$ and such that $e^i \not\subset \Gamma_i$ i.e. $\alpha_i^i > 0$ $i = k+1, \dots , p$.

Consider the hyperplane Γ whose equation is

$$\alpha^T z = 0 \quad , \quad \alpha = \sum_{i=k+1}^{p} \alpha^i$$

It is easy to verify that Γ separates $W\text{-}\mathbf{R}_+^p$ and $\mathbf{R}_+^p$, so that $C \subset \Gamma \cap \mathbf{R}_+^p$.

If $\Gamma \cap \mathbf{R}_+^p \ne C$, there exists $y = \sum_{i=k+1}^{p} \beta_i e^i$ such that $y \in \Gamma \cap \mathbf{R}_+^p$ and β_i , $i=k+1 , \dots , p$, are non-negative and at least one is positive.

On the other hand, since

$$\alpha^T y = \sum_{i=k+1}^{p} \beta_i(\alpha^T e^i) \quad \text{and} \quad \alpha^T e^i = (\sum_{j=k+1}^{p} \alpha^j)^T e^i = \sum_{j=k+1}^{p} \alpha^j_i \geq \alpha^i_i > 0$$

we have $\alpha^T y > 0$, so that $y \notin \Gamma$ and this is absurd. □

The optimality conditions given in the image space will allow us to deduce some results in the decision space.

Theorem 4.3 Consider problem P where φ and g are differentiable functions at the local optimal solution x^0. Then i) and ii) hold.

i) $\mathbf{K_L} \cap \text{int}\mathbf{H} = \emptyset$

ii) $\exists\ \alpha = (\alpha_1 \dots \alpha_{s+m}) \in \mathbf{R}_+^{s+m}$, $\alpha \neq 0$ such that

$$\alpha^T J = 0 \tag{4.6}$$

proof.

i) It follows immediately from ii) of Theorem 3.1 taking into account that $\mathbf{T_1} = \mathbf{K_L} \cup A$.

ii) Since $\mathbf{K_L}$ and cl$\mathbf{H}$ are convex sets and i) holds, there exists a hyperplane of the form $\alpha^T z = 0$ which separates $\mathbf{K_L}$ and $\mathbf{H}$, such that

$$\alpha^T z \geq 0 \quad \forall\ z \in \text{cl}\mathbf{H} \tag{4.7a}$$

$$\alpha^T z \leq 0 \quad \forall\ z \in \mathbf{K_L} \tag{4.7b}$$

Consequently $\alpha \in \mathbf{R}_+^{s+m}$, $\alpha \neq 0$.

Since $\alpha^T J(x - x^0) = 0 \ \forall\ x \in \mathbf{R}^n$, necessarily we have (4.6). □

Corollary 4.1 (Fritz-John optimality conditions)

Consider problem P where φ and g are differentiable functions at the local optimal solution x^0. Then there exists a vector $\alpha = (\alpha_1, \dots, \alpha_{s+m}) \in \mathbf{R}_+^{s+m}$,

$\alpha \neq 0$ such that

$$\sum_{i=1}^{s} \alpha_i \nabla \varphi_i(x^o) + \sum_{i=1}^{m} \alpha_{s+i} \nabla g_i(x^o) = 0 \tag{4.8}$$

Remark 4.1 (unconstrained problem)
Consider the vector optimization problem

$$\max (\varphi_1(x), \ldots\ldots \varphi_s(x)) \;,\; x \in X \;,\; s \geq 2$$

where X is an open set of R^n, $\varphi_i : X \to R$, i=1...s are differentiable functions at x^o.
It is obvious that all the previous results are valid setting $\mathbf{H} = U^o$.
As a consequence (4.8) becomes

$$\sum_{i=1}^{s} \alpha_i \nabla \varphi_i(x^o) = 0$$

which is a necessary condition for x^o to be an interior local optimal solution.

The following theorem states a necessary and sufficient condition in order to have $\alpha_i > 0$, i=1,...., s in (4.6) or in (4.8).

Theorem 4.4 We have $\alpha_i > 0$, i=1,...., s, in (4.8) if and only if $\mathbf{K_L} \cap \mathbf{H} = \emptyset$

proof.
if. If $z \in \mathbf{K_L} \cap \mathbf{H}$, then $z_i \geq 0$, i=1,...,m, and furthermore there exists j with with $1 \leq j \leq s$ such that $z_j > 0$. Consequently $\alpha^T z > 0$ and this contradicts (4.7 b).
only if. $\mathbf{K_L} \cap \mathbf{H} = \emptyset$ implies $\mathbf{K_L} \cap \mathrm{cl}\mathbf{H} \subset \{0 \times V\}$ and
$(\mathbf{K_L} - \mathrm{cl}\mathbf{H}) \cap \mathrm{cl}\mathbf{H} \subset \{0 \times V\}$. Set $C = (\mathbf{K_L} - \mathrm{cl}\mathbf{H}) \cap \mathrm{cl}\mathbf{H}$. From Thorem 4.2, there exists an hyperplane Γ such that $C - \Gamma \cap \mathrm{cl}\mathbf{H} \subset \{0 \times V\}$; this inclusion implies $\alpha_i > 0$, i=1,...., s, in (4.8). □

Corollario 4.2 Consider problem P where φ and g are differentiable functions at the local optimal solution x^0. Then (4.8) holds with $\alpha_i > 0$, $i=1,\dots,s$, if and only if $\mathbf{K_L} \cap \mathbf{H} = \emptyset$.

Remark 4.2 Consider the scalar case s=1 ; then $\mathbf{K_L} \cap \mathbf{H} = \emptyset$ becomes a necessary and sufficient condition in order to have the Kuhn-Tucker conditions. From this point of view any condition which ensure $\mathbf{K_L} \cap \mathbf{H}=\emptyset$ becomes a regularity condition. In a forthcoming paper we will deep this aspect.

In section 3 we have seen that $\mathbf{T_1} \cap \mathrm{cl}\mathbf{H} = \{0\}$ is a sufficient optimality condition; taking into account relation $\mathbf{T_1} = \mathbf{K_L} \cup A$, we obtain the following:

Theorem 4.5 If $A=\emptyset$ and $\mathbf{K_L} \cap \mathrm{cl}\mathbf{H} = \{0\}$, then x^0 is a local optimal solution for problem P.

Corollary 4.3 Assume that condition (4.8) holds with $\alpha_i > 0$, $i=1,\dots,s$. If $\mathrm{rank} J = n$ then x^0 is a local optimal solution for problem P.

<u>proof</u>.
The assumption $\mathrm{rank} J = n$ implies $\mathbf{T_1} = \mathbf{K_L}$ [4] , so that $A=\emptyset$; on the other hand the validity of (4.8) with $\alpha_i > 0$, $i=1,\dots,s$, implies $\mathbf{K_L} \cap \mathrm{cl}\mathbf{H} = \{0\}$. The thesis follows from Theorem 4.3. □

Consider the linearizing problem P_L :

$$P_L: \max J_f(x-x_0)$$
$$J_g(x-x_0) \geq 0 \ , \ x \in X$$

Let us note that $\mathbf{K_L} \cap \mathbf{H} = \emptyset$ becomes a necessary and sufficient optimality condition for problem P_L and consequently a necessary and sufficient optimality condition for the class of problems P where φ and g are linear functions.
Now, we will see that $\mathbf{K_L} \cap \mathbf{H} = \emptyset$ becomes a sufficient optimality condition for the class of generalized convex problems.

Theorem 4.6 Consider the differentiable problem P where φ_i , i=1, ...,s are pseudo-concave functions at x^o and g_i are quasi-concave functions at x^o. If $\mathbf{K_L} \cap \mathbf{H} = \emptyset$ then x^o is an optimal solution for P.

proof.
If x^o is not an optimal solution there exists x^* such that
$\varphi_i(x^*) \geq \varphi_i(x^o)$ i=1,...,s, where at least one inequality is strict (that is $\exists$ j , $1 \leq j \leq s$ such that $\varphi_j(x^*) > \varphi_j(x^o)$).
Since φ_j is pseudo-concave we have
$\nabla\varphi_j(x^o)(x^*-x^o) > 0$ and furthermore $\nabla\varphi_i(x^o)(x^*-x^o) \geq 0$ i=1,...,s, $i \neq j$
The assumption of quasi-concavity for g implies $\nabla g_i(x^o)(x^*-x^o) \geq 0$ i=1,...,m, so that $J(x^*-x^o) \in \mathbf{H}$ and this is a contradiction. □

Corollario 4.4 Consider the differentiable problem P where φ_i , i=1, ...,s are pseudo-concave functions and g_i are quasi-concave functions at x^o.
If (4.8) holds with $\alpha_i > 0$, i=1,...., s, then x^o is an optimal solution for P.

References

[1] **Bazaraa M. S., Shetty C. M.** : "Foundations of optimization" , Springer-Verlag, 1976.

[2] **Cambini A.** : " Non-linear separation theorems, duality and optimality conditions" , in "Optimization and Related Topics", Erice, Springer 1984.

[3] **Cambini A., Martein L.** : "Some optimality conditions in vector optimization". Journal of Informations and Optimization Sciences, vol.10, n.1, 1989, pp. 141-151.

[4] **Cambini A., Martein L.** : "Tangent cones in optimization", report n.49 , Dept. of Statistics and Applied Mathematics, University of Pisa, 1991.

[5] **Giannessi F.** : "Theorems of the alternative and optimality conditions", Journal of Optimization Theory and Applications, vol. 42, n. 3, 1984, pp. 331-365.

[6] **Mangasarian O. L.** : "Nonlinear Programming", McGraw-Hill, New York, 1969.

[7] **Martein L.** : "Stationary points and necessary conditions in vector extremum problems", Journal of Informations and Optimization Sciences, vol.10, n.1, 1989, pp. 105-128.

[8] **Rockafellar R. T.** : "Convex Analysis", Princeton, New Jersey, 1970.

[9] **Rockafellar R. T.** : "The theory of subgradients and its applications of optimization. Convex and non-convex functions", Eldermann, Verlag, Berlin, 1985.

[10] **Sawaragi Y., Nakayama H., Tanino T.** : "Theory of Multiobjective Optimization" Academic Press, 1985.

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