Report n.61

Generalized Monotonicity

Siegfried SCHAIBLE

Pisa, luglio 1992

1. Introduction

In the second half of this century, the following three classes of mathematical models have been given particular attention in the management and economics literature:

— mathematical programming problems
— complementarity problems
— variational inequality problems.

In the analysis of mathematical programming problems, convexity of the model is often assumed. Fortunately, it holds in many applications. For the other two classes of models, complementary problems and variational inequality problems, the traditional assumption is monotonicity which also can often be found in applications.

However, one encounters numerous problems in management and economics where these classical assumptions of convexity and monotonicity do not hold. We realize that they are just sufficient conditions to guarantee certain properties of these models important in the solution process. They are by no means necessary. One can say that they pose an artificial limitation on the usefulness of these three classes of models.

In case of mathematical programming, this was realized almost from the beginning. As the result, a theory of generalized convex functions has been developed. Many of the results are summarized in the first monograph on generalized convexity by Avriel, Diewert, Schaible and Zang [3].

However, the situation is very different for the other two classes of models, complementarity problems and variational inequality problems. Several isolated results have become known over the years. Among these is the existence result by Karamardian in 1976 derived for complementarity problems; see [12]. But a rigorous study of generalized monotonicity still remains to be done. The subject has received renewed attention during the last few years. In this report, we try to summarize some of the major developments. The presentation is mainly limited to those results which the author himself has derived in collaboration with others.

As it will be seen, most of these results are related to conceptual matters, i. e. the definition and characterization of various kinds of generalized monotonicity. The use of these new concepts is currently under investigation.

2. Three classes of models

The mathematical programming problem, pioneered by Dantzig in the 1940's, is given as follows

MP $$\min \{ f(x) | x \in C \}. \quad (2.1)$$

Here $f : C \to \Re$ for $C \subseteq \Re^n$. Often C is given by a system of inequalities

$$C = \{ x \in \Re^n | g_i(x) \le b_i \quad i = 1, \cdots, m \}. \quad (2.2)$$

Application of linear and nonlinear programming problems are found in such diverse fields as management, economics, applied mathematics, statistics, the natural sciences and engineering.

The limitations of the use of MP's in economic equilibrium planning led to a more rapid development of the other two models, complementary problems and variational inequality problems [11].

The complementary problem, first pioneered by Cottle and Karamardian in the 1960's, is given in its simplest form as

CP $$x \geq 0, \quad F(x) \geq 0, \quad x^T F(x) = 0. \tag{2.3}$$

Here $F : \Re^n_+ \to \Re^n$ where $\Re^n_+$ denotes the nonnegative orthant of $\Re^n$. The CP is called a <u>linear complementary problem</u> if F is an affine map $F(x) = Mx + q$ where M is an $n \times n$ real matrix and $q \in \Re^n$.

The <u>generalized complementarity problem</u> is defined with respect to a closed convex cone $C \subseteq \Re^n$ as

GCP $$x \in C, \quad F(x) \in C^*, \quad x^T F(x) = 0. \tag{2.4}$$

Here $C^* = \left\{ y \in \Re^n \,\middle|\, y^T x \geq 0 \text{ for all } x \in C \right\}$ is the dual cone. It has also been studied in infinite-dimensional topological linear spaces.

Regarding applications, we first mention that every differentiable MP gives rise to a CP through the first-order optimality condition by Karush-Kuhn-Tucker. In this case

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad F(x) = \begin{pmatrix} \nabla f(x_1) + \left(\nabla g(x_1)\right)^T x_2 \\ -g(x_1) + b \end{pmatrix}$$

where x_1 is the vector of variables in the MP $\min\left\{ f(x_1) \mid g(x_1) \leq b \right\}$ and x_2 is the vector of Lagrange multipliers. A linear or quadratic program leads to a linear complementarity problem.

There are also other problems in economics and management that can be formulated as a CP such as Nash-equilibrium problems of non-cooperative games, bi-matrix games or certain economic equilibrium problems. We mention in passing that some equilibrium problems in mechanics give rise to complementarity problems as well.

Generalized Monotonicity+

Siegfried Schaible++

Abstract

Recently, several kinds of generalized monotone maps were introduced by Karamardian and the author which play a role in complementarity problems and variational inequality problems. Following a presentation of seven kinds of (generalized) monotone maps, various characterizations of differentiable and affine generalized monotone maps are reported which can simplify the identification of such properties. Also invariance properties of generalized monotone maps are presented.

+ The author gratefully acknowledges the research support he received as Visiting Professor of the Dipartimento di Statistica e Matematica Applicata All 'Economica, Universita' di Pisa, Spring 1992.

++ Graduate School of Management, University of California, Riverside, CA 92521, U.S.A.

gradient map if and only if M is symmetric; then $f(x) = \frac{1}{2}x^T M x + q^T x$. Thus most affine maps are not gradient maps.

In summary, we can say that the three classes of models are related to each other as follows:

$$\begin{array}{ccccc} \text{MP} & \rightarrow & \text{GCP} & \rightarrow & \text{VI} \\ & \not\leftarrow & & \not\leftarrow & \\ & & & \text{unless} & \\ & & & C \text{ is a cone} & \end{array} \tag{2.6}$$

This shows that the most general, and thus the most flexible model is the variational inequality problem.

3. Classical regularity assumptions

For the above models, theoretical and algorithmic results hold only if certain regularity assumptions are made. We will contrast here the most special with the most general model, namely MP's with VI's, in terms of such assumptions.

For MP's (2.1), the classical assumption, apart from convexity of C, is convexity of the objective function f. Then the following properties hold:

— the set of optimal solutions is convex
— a local is a global minimum
— a solution of the Karush-Kuhn-Tucker conditions is a minimum
— a minimum (if it exists) is unique, if f is strictly convex.

Furthermore, most algorithms converge to a minimum under convexity of f.

For VI's (2.5), the classical assumption, apart from convexity of C, is monotonicity of the map F. This is not surprising in case of a gradient map $F = \nabla f$, since a VI can be understood as the necessary optimality conditions of a MP with f as the objective function, and monotonicity of F is equivalent to convexity of f.

For monotone VI's, the following properties hold, regardless of whether F is a gradient map or not:

— the set of solutions is convex (though possibly empty)

— a solution (if it exists) is unique if F is strictly monotone

— a solution exists and is unique if F is strongly monotone.

Moreover, many algorithms converge to a solution if F is monotone.

4. Weakened regularity assumptions

Convexity in MP's and monotonicity in VI's are sufficient conditions for the above properties to be true. The question arises to what extend these assumptions can be relaxed such that the same properties still hold. This problem has extensively been studied for MP's where a rather elaborate theory of generalized convexity has been developed; see the monograph [3] and the conference proceedings [21], [22], [5]. One important type of such a generalized convex function is the pseudoconvex function [3].

On the other hand, for VI's only very few results are known so far in answer to the question to what extend properties of monotone VI's still hold in the nonmonotone case. One of these results is an existence theorem by Karamardian [12] for so-called pseudomonotone GCP's (2.4). It was recently extended to more general GCP's and to VI's ; see [9], [11]. A central assumption for the existence of a solution is pseudomonotonicity of F. It can easily be shown that for such VI's the set of solutions is still convex [11]. The proof shows that pseudomonotonicity is a very suitable concept in connection with VI's.

As Karamardian showed in [12], a gradient map $F = \nabla f$ is pseudomonotone if and only if f is pseudoconvex. This extends the corresponding result for monotone maps and convex functions. Pseudoconvex functions are central in the theory of MP's. It is conjectured that pseudomonotone maps will play an important role in VI's.

The comparison of pseudomonotone VI's and pseudoconvex MP's opens up a number of interesting questions:

— What other existence and uniqueness results can be established?
— Which algorithms do still converge to a solution?
— What kind of applications give rise to such VI's ?

The author is confident that the experience with generalized convex MP's in the last few decades will be a helpful guide in answering some of the above questions regarding generalized monotone VI's. Since it proved to be necessary to work with a variety of generalizations of convexity [3] rather than just one, it is expected that a variety of generalized monotone maps is needed as well. Some steps have been taken in this direction as we will see below. In [13] the authors introduce and discuss seven kinds of monotone and generalized monotone maps which are related to each other as follows:

$$\begin{array}{ccccc} \text{monotone} & \rightarrow & \text{pseudomonotone} & \rightarrow & \text{quasimonotone} \\ \uparrow & & \uparrow & & \\ \text{strictly monotone} & \rightarrow & \text{strictly pseudomonotone} & & \\ \uparrow & & \uparrow & & \\ \text{strongly monotone} & \rightarrow & \text{strongly pseudomonotone} & & \end{array} \qquad (4.1)$$

In case of gradient maps, they correspond to the following convex and generalized convex functions:

convex	→	pseudoconvex	→	quasiconvex
↑		↑		
strictly convex	→	strictly pseudoconvex		(4.2)
↑		↑		
strongly convex	→	strongly pseudoconvex		

We report on these results in the following section.

5. Seven kinds of monotone and generalized monotone maps

Throughout this section we assume that F denotes a map $F: C \to \Re^n$ where $C \subseteq \Re^n$. In the special case of a gradient map $F = \nabla f, f$ denotes a differentiable function $f: C \to \Re$ where C is open and convex.

5.1 Monotone, strictly monotone and strongly monotone maps

The notion of a monotone map F from $\Re^n$ into $\Re^n$ is a natural generalization of an increasing (non-decreasing) real-valued function of one variable.

Definition 5.1

F is monotone on C if for every pair of distinct points $x, y \in C$ we have

$$(y-x)^T (F(y) - F(x)) \geq 0. \tag{5.1}$$

F is strictly monotone on C if for every pair of distinct points $x, y \in C$ we have

$$(y-x)^T (F(y) - F(x)) > 0. \tag{5.2}$$

F is strongly monotone on C if there exists $\beta > 0$ such that for every pair of distinct points $x, y \in C$ we have

$$(y-x)^T (F(y) - F(x)) \geq \beta \|y-x\|^2. \tag{5.3}$$

Convexity of a function and monotonicity of its gradient are equivalent [3].

Proposition 5.1

f is convex (strictly convex, strongly convex) on C if and only if ∇f is monotone (strictly monotone, strongly monotone) on C.

We now present different generalizations of monotone maps. In case the map is the gradient of a function, such generalized monotonicity concepts can be related to some generalized convexity property of the underlying function.

5.2 Pseudomonotone maps

In [12] the concept of a pseudomonotone map was introduced.

Definition 5.2

F is pseudomonotone on C if for every pair of distinct points $x, y \in C$ we have

$$(y-x)^T F(x) \geq 0 \quad \text{implies} \quad (y-x)^T F(y) \geq 0. \tag{5.4}$$

Obviously, a monotone map is pseudomonotone as a comparison of (5.1) and (5.4) shows. But the converse is not true. See, for example,

$$F(x) = 1/(1+x), \quad C = \{ x \in \Re \,|\, x \geq 0 \}. \tag{5.5}$$

We recall the following definition [3]:

Definition 5.3

A function f is <u>pseudoconvex</u> on C if for every pair of distinct points $x, y \in C$ we have

$$(y-x)^T \nabla f(x) \geq 0 \quad \text{implies} \quad f(y) \geq f(x). \tag{5.6}$$

The following proposition was shown in [12].

Proposition 5.2

f is pseudoconvex on C if and only if ∇f is pseudomonotone on C.

Before we introduce new kinds of generalized monotonicity, we will show that in (5.4) both inequalities can be replaced by strict inequalities.

Proposition 5.3

F is pseudomonotone on C if and only if for every pair of distinct points $x, y \in C$ we have

$$(y-x)^T F(x) > 0 \quad \text{implies} \quad (y-x)^T F(y) > 0. \tag{5.7}$$

<u>Proof</u>

In view of (5.4), pseudomonotonicity is equivalent to

$$(y-x)^T F(y) < 0 \quad \text{implies} \quad (y-x)^T F(x) < 0. \tag{5.8}$$

Thus,

$$(x-y)^T F(y) > 0 \quad \text{implies} \quad (x-y)^T F(x) > 0. \tag{5.9}$$

❑

As we see from Proposition 5.3, replacing both inequalities in (5.4) by strict inequalities as in (5.7) will not give rise to a new type of generalized monotone map.

In the following two sections, we replace only one of the two inequalities by a strict inequality, and in this way we shall generate two new types of generalized monotone maps. As it turns out, they characterize two well-known types of generalized convex functions.

5.3 Strictly pseudomonotone maps

Let us introduce the following definition:

Definition 5.4
F is strictly pseudomonotone on C if for every pair of distinct points $x, y \in C$ we have

$$(y-x)^T F(x) \geq 0 \quad \text{implies} \quad (y-x)^T F(y) > 0. \tag{5.10}$$

Obviously, a strictly pseudomonotone map is pseudomonotone as a comparison of (5.4) and (5.10) shows. But the converse is not true. See, for example,

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases}, \qquad C = \Re. \tag{5.11}$$

Furthermore, every strictly monotone map is strictly pseudomonotone, as can be seen by comparing (5.2) and (5.10). The converse is not true as example (5.5) shows.

We now prove the equivalent of Proposition 5.2 for strictly pseudoconvex functions. Before, we recall the following definition [3]:

Definition 5.5
f is strictly pseudoconvex on C if for every pair of distinct points $x, y \in C$ we have

$$(y-x)^T \nabla f(x) \geq 0 \quad \text{implies} \quad f(y) > f(x). \tag{5.12}$$

Then we can prove the following proposition:

Proposition 5.4

f is strictly pseudoconvex on C if and only if ∇f is pseudomonotone on C.

Proof

Suppose that f is strictly pseudoconvex on C.
Let $x, y \in C$, $x \neq y$, be such that

$$(y-x)^T \nabla f(x) \geq 0. \tag{5.13}$$

We want to show that

$$(y-x)^T \nabla f(y) > 0. \tag{5.14}$$

Assume to the contrary that

$$(y-x)^T \nabla f(y) \leq 0. \tag{5.15}$$

Given (5.13), strict pseudoconvexity of f implies that

$$f(y) > f(x). \tag{5.16}$$

On the other hand, (5.15) can be written as

$$(x-y)^T \nabla f(y) \geq 0. \tag{5.17}$$

From strict pseudoconvexity of f, it follows that

$$f(x) > f(y),$$

which contradicts (5.16).

Conversely, suppose that ∇f is strictly pseudomonotone on C.
Let $x, y \in C$, $x \neq y$,be such that

$$(y-x)^T \nabla f(x) \geq 0. \tag{5.18}$$

We want to show that

$$f(y) > f(x).$$

Assume to the contrary that

$$f(y) \leq f(x). \tag{5.19}$$

From the mean-value theorem, we have

$$f(y) - f(x) = (y-x)^T \nabla f(\bar{x}) \tag{5.20}$$

where

$$\bar{x} = \bar{\lambda} x + \left(1 - \bar{\lambda}\right) y, \tag{5.21}$$

for some $0 < \bar{\lambda} < 1$. Now from (5.19), (5.20) and (5.21) we have

$$(x - \bar{x})^T \nabla f(\bar{x}) \geq 0. \tag{5.22}$$

Since ∇f is strictly pseudomonotone, we conclude that

$$(x - \bar{x})^T \nabla f(x) > 0. \tag{5.23}$$

Because of (5.21), this implies that

$$(x-y)^T \nabla f(x) > 0,$$

which contradicts (5.18). ❑

5.4 Quasimonotone maps

In view of Definition 5.2, Proposition 5.3 and Definition 5.4, there is still the case to be considered where the first inequality in (5.4) is a strict inequality.

Definition 5.6

F is quasimonotone on C if for every pair of distinct points $x, y \in C$ we have

$$(y-x)^T F(x) > 0 \quad \text{implies} \quad (y-x)^T F(y) \geq 0. \tag{5.24}$$

Every pseudomonotone map is quasimonotone as Proposition 5.3 demonstrates. But the converse is not true. See, for example,

$$F(x) = x^2, \quad C = \Re. \tag{5.25}$$

The term "quasimonotone" suggests a relationship to quasiconvex functions, which indeed exists. We recall the following definition [3]:

Definition 5.7

f is quasiconvex on C if for all $x, y \in C$, $\lambda \in [0,1]$,

$$f(y) \leq f(x) \quad \text{implies} \quad f(\lambda x + (1-\lambda)y) \leq f(x). \tag{5.26}$$

For differentiable functions, the following characterization of quasiconvex functions holds [3]:

Proposition 5.5

f is quasiconvex on C if and only if for every pair of distinct points $x, y \in C$ we have

$$f(y) \leq f(x) \quad \text{implies} \quad (y-x)^T \nabla f(x) \leq 0. \tag{5.27}$$

We now show the following proposition:

Proposition 5.6

f is quasiconvex on C if and only if ∇f is quasimonotone on C.

Proof

Suppose that f is quasiconvex. Let $x, y \in C$ be such that

$$(y-x)^T \nabla f(x) > 0. \tag{5.28}$$

The inequality

$$f(y) \le f(x) \tag{5.29}$$

is not possible, since then

$$(y-x)^T \nabla f(x) \le 0,$$

according to (5.27), which contradicts (5.28). Hence, we have

$$f(y) > f(x). \tag{5.30}$$

According to (5.27), $f(x) < f(y)$ implies that

$$(x-y)^T \nabla f(y) \le 0,$$

i. e.

$$(y-x)^T \nabla f(y) \ge 0. \tag{5.31}$$

Since we have shown that (5.28) implies (5.31), ∇f is quasimonotone.
Conversely, suppose that ∇f is quasimonotone. Assume that f is not quasiconvex. Then, there exists $x, y \in C$ such that

$$f(y) \le f(x), \tag{5.32}$$

and $\bar{\lambda} \in (0,1)$ such that, for $\bar{x} = x + \bar{\lambda}(y-x)$

$$f(\bar{x}) > f(x) \geq f(y). \tag{5.33}$$

The mean-value theorem implies the existence of $\hat{x}$ and x^* such that

$$f(\bar{x}) - f(y) = (\bar{x} - y)^T \nabla f(\hat{x}), \tag{5.34}$$

$$f(\bar{x}) - f(x) = (\bar{x} - x)^T \nabla f(x^*) \tag{5.35}$$

where

$$\hat{x} = x + \hat{\lambda}(y - x), \quad x^* = x + \lambda^*(y - x), \quad 0 < \lambda^* < \bar{\lambda} < \hat{\lambda} < 1. \tag{5.36}$$

Then (5.33) implies that

$$(\bar{x} - y)^T \nabla f(\hat{x}) > 0, \tag{5.37}$$

$$(\bar{x} - x)^T \nabla f(x^*) > 0. \tag{5.38}$$

This yields

$$(x^* - \hat{x})^T \nabla f(\hat{x}) > 0, \tag{5.39}$$

$$(\hat{x} - x^*)^T \nabla f(x^*) > 0, \tag{5.40}$$

in view of (5.36). From (5.40), we obtain

$$(x^* - \hat{x})^T \nabla f(x^*) < 0 \tag{5.41}$$

which together with (5.39) contradicts the quasimonotonicity of ∇f. Thus (5.33) does not hold for any pair $x, y \in C$, i. e. f is quasiconvex on C.

❑

We mention that there are also the concepts of semistrictly and strictly quasiconvex functions [3]. But it seems to be difficult to characterize these functions with help of the gradient only. Hence, no attempt is made here to introduce corresponding maps. Instead, we turn to a subclass of strictly pseudomonotone maps.

5.5 Strongly pseudomonotone maps

We introduce the following definition:

Definition 5.8

F is strongly pseudomonotone on C if there exists $\beta > 0$ such that for every pair of distinct points $x, y \in C$ we have

$$(y-x)^T F(x) \geq 0 \text{ implies } (y-x)^T F(y) \geq \beta \|y-x\|^2. \tag{5.42}$$

Every strongly monotone map is strongly pseudomonotone, as a comparison of (5.42) and (5.3) shows. The converse is not true. See, for example,

$$F(x) = 1/(1+x), \quad C = [0,1]. \tag{5.43}$$

Every strongly pseudomonotone map is strictly pseudomonotone, as a comparison of (5.42) and (5.10) shows. But the converse is not true, as illustrated by the example in (5.25) with $C = \{ x \in \Re | x \geq 0 \}$.

We will now relate strongly pseudomonotone maps to strongly pseudoconvex functions. From [3] we recall:

Definition 5.9

f is strongly pseudoconvex on C if there exists $\alpha > 0$ such that for every pair of distinct points $x, y \in C$ we have

$$(y-x)^T \nabla f(x) \geq 0 \text{ implies } f(y) \geq f(x) + \alpha \|y-x\|^2. \tag{5.44}$$

We can prove the following result:

Proposition 5.7

f is strongly pseudoconvex on C if ∇f is strongly pseudomonotone on C, where $\alpha = \frac{1}{2}\beta$.

Proof

Suppose that ∇f is strongly pseudomonotone.

Let

$$(y-x)^T \nabla f(x) \geq 0. \tag{5.45}$$

Consider

$$\phi(\lambda) = f(x+\lambda(y-x)), \quad \lambda \in [0,1]. \tag{5.46}$$

Then

$$\phi'(\lambda) = (y-x)^T \nabla f(x+\lambda(y-x)).$$

Let

$$x(\lambda) = x+\lambda(y-x).$$

Because of (5.45),

$$(x(\lambda)-x)^T \nabla f(x) \geq 0 \text{ for } \lambda \in [0,1]. \tag{5.47}$$

Since ∇f is strongly pseudomonotone, this implies that

$$(x(\lambda)-x)^T \nabla f(x(\lambda)) \geq \beta \; \|x(\lambda)-x\|^2 \text{ for } \lambda \in [0,1]. \tag{5.48}$$

Hence,

$$\lambda(y-x)^T \nabla f(x(\lambda)) \geq \beta \lambda^2 \|y-x\|^2,$$

implying that

$$\phi'(\lambda) \geq \beta \lambda \|y-x\|^2 \text{ for } \lambda \in [0,1]. \tag{5.49}$$

Then

$$\phi(1)-\phi(0) = \int_0^1 \phi'(\lambda) d\lambda \geq \int_0^1 \beta \lambda \|y-x\|^2 d\lambda ,$$

i. e.

$$f(y)-f(x) \geq \tfrac{1}{2}\beta\|y-x\|^2. \tag{5.50}$$

❑

The reverse of Proposition 5.7 is currently under investigation.

Now we have arrived at the end of this section. We presented seven kinds of monotone and generalized monotone maps and their relationship to each other, as summarized in the diagram (4.1). Furthermore, in case of gradient maps we related (generalized) monotonicity of the gradient to (generalized) convexity of the underlying function. For nondifferentiable functions, similar relationships have recently been shown by Komlosi [16] who uses directional Dini derivatives.

We also point out that Castagnoli and Mazzoleni study generalized monotonicity from a geometrical point of view using order-preserving functions [6], [7], [8], [17]. It is somewhat similar to the analysis in the following section, a more detailed presentation of which including additional results appears in [14], [15].

6. The differentiable case

Before we present first-order characterizations of differentiable generalized monotone maps, we provide a geometrical characterization for the one-dimensional case. This is significant because of the relationship between maps and the one-dimensional restrictions of their projections.

6.1 One-dimensional generalized monotone maps

As in the previous section, let $F: C \to \Re^n$ where $C \subseteq \Re^n$. We use the abbreviations QM, PM and SPM instead of "quasimonotone", "pseudomonotone" and "strictly pseudomonotone", respectively.

For every $v \in \Re^n$ and $x \in C$ we define the one-dimensional restriction of the projection of F on v by $\psi : \mathrm{I}_{x,v} \to \Re$ where

$$\psi_{x,v}(t) = v^T F(x+tv) \text{ and } \mathrm{I}_{x,v} = \left\{ t \in \Re \,|\, x+tv \in C \right\}. \tag{6.1}$$

The following theorem, whose proof is straightforward, establishes the relationship between F and $\psi_{x,v}$.

Proposition 6.1

F is QM , PM and SPM on C if and only if for every $v \in \Re^n$, $x \in C$ the function $\psi_{x,v}$ is QM , PM and SPM on $\mathrm{I}_{x,v}$, respectively.

The next proposition, whose proof again is straightforward, establishes the relationship between the sets C and $\mathrm{I}_{x,v}$.

Proposition 6.2

We have

(i) $\mathrm{I}_{x,v}$ is open (closed) for all $x \in C$ and $v \in \Re^n$ if and only if C is open (closed);

(ii) $\mathrm{I}_{x,v}$ is convex (i. e. an interval) for all $x \in C$ and $v \in \Re^n$ if and only if C is convex.

(iii) $\mathrm{I}_{x,v} = \Re$ for all $x \in C$ and $v \in \Re^n$ if and only if $C = \Re^n$.

Before we give a geometrical characterization for one-dimensional generalized monotone maps (functions), we introduce the following sign-preserving notions:

Definition 6.1

Let $\mathrm{I} \subseteq \Re$ and $F : \mathrm{I} \to \Re$. F is said to have the sign-preserving (SP) property on I if for any $x \in \mathrm{I}$ we have

(SP) $$F(x) > 0 \text{ implies } F(y) \geq 0 \text{ for all } y \in \mathrm{I}, y > x. \tag{6.2}$$

F is said to have the strict sign-preserving (SSP) property on I if for any $x \in \mathrm{I}$ we have

$$F(x) > 0 \text{ implies } F(y) > 0 \text{ for all } y \in \mathrm{I}, y > x \tag{6.3}$$

(SSP) and

$$F(x) < 0 \text{ implies } F(y) < 0 \text{ for all } y \in \mathrm{I}, y < x. \tag{6.4}$$

It is easy to show that (6.2), (6.3) and (6.4) are equivalent to (6.5), (6.6) and (6.7), respectively, where for any $x \in \mathrm{I}$

$$F(x) < 0 \text{ implies } F(y) \leq 0 \text{ for all } y \in \mathrm{I}, y < x, \tag{6.5}$$

$$F(x) \leq 0 \text{ implies } F(y) \leq 0 \text{ for all } y \in \mathrm{I}, y < x, \tag{6.6}$$

$$F(x) \geq 0 \text{ implies } F(y) \geq 0 \text{ for all } y \in \mathrm{I}, y > x. \tag{6.7}$$

The next proposition provides geometrical characterizations of one-dimensional QM, PM and SPM maps.

Proposition 6.3

Let $\mathrm{I} \subseteq \Re$ and $F : \mathrm{I} \to \Re$.

(i) F is QM on I if and only if F has the SP property on I.

(ii) F is PM on I if and only if F has the SSP property on I.

(iii) F is SPM on I if and only if F has the SSP property on I and $F(x) = 0$ has at most one real root.

The proof of this proposition is straightforward and follows from the definitions.

Geometrically, Proposition 6.3 states that F is QM on $I \subseteq \Re$ if and only if it has the property that, once $F(x)$ is positive for some x, it can never become negative for any $y > x$; or equivalently, if $F(x)$ is negative for some x, it could not have been positive for some $y < x$. Similarly, F is PM on I if and only if it has the property that, if $F(x)$ is positive for some x, it will remain positive for all $y > x$, and if $F(x)$ is negative for some x, it must be negative for all $y < x$.

As already mentioned, geometrical properties of generalized monotone maps have been the starting point for research in different and more abstract directions by Castagnoli and Mazzoleni; see [6], [7], [8], [17].

6.2 Relationship between QM and PM maps

As seen in Section 5, every PM map is QM, but the converse is not true. For the sake of completeness, we mention in passing a characterization of those QM maps which are PM. The interested reader is referred to the proof in[15].

Proposition 6.4

Let $C \subseteq \Re^n$ be open and convex, and $F: C \rightarrow \Re^n$ be continuous on C. Then F is PM on C if and only if

(i) F is QM on C, and

(ii) for every $x \in C$ with $F(x) = 0$ there exists a neighborhood $N(x)$ of x such that $(y-x)^T F(y) \geq 0$ for all $y \in N(x) \cap C$.

From this result it follows immediately:

Proposition 6.5

Let $C \subseteq \Re^n$ be open and convex, and $F: C \rightarrow \Re^n$ be continuous and QM on C. If $F(x) \neq 0$ for all $x \in C$, then F is PM on C.

6.3 Differentiable QM , PM and SPM maps

We now present first-order necessary conditions and sufficient conditions for a map to be QM, PM or SPM. Let $C \subseteq \Re^n$ be open and convex and the map $F: C \to \Re^n$ be differentiable with Jacobian matrix $J_F(x)$ evaluated at x. . We consider the following three conditions where $x \in C$ and $v \in \Re^n$:

(A)	$v^T F(x) = 0$	implies	$v^T J_F(x) v \geq 0$	(6.8)
(B)	$v^T F(x) = v^T J_F(x) v = 0$ $\hat{t} < 0, v^T F(x + \hat{t}v) > 0$	implies	there exists $\tilde{t} > 0$, $\tilde{t} \in \mathrm{I}_{x,v}$ such that $v^T F(x + tv) \geq 0$ for all $0 \leq t \leq \tilde{t}$	(6.9)
(C)	$v^T F(x) = v^T J_F(x) v = 0$	implies	there exists $\tilde{t} > 0$, $\tilde{t} \in \mathrm{I}_{x,v}$ such that $v^T F(x + tv) \geq 0$ for all $0 \leq t \leq \tilde{t}$.	(6.10)

By making use of the results on one-dimensional generalized monotone maps above, the following necessary and sufficient conditions for QM and PM maps can be established [15]:

Proposition 6.6

(i) F is QM on C if and only if (A) and (B) hold;

(ii) F is PM on C if and only if (A) and (C) hold.

We point out, that (A) is not sufficient for a map to be QM or PM. See, for example,

$$F(x) = -4x^3, \quad C = \Re. \tag{6.11}$$

Condition (A) holds, but F is neither QM nor PM since $F=\nabla f$ and $f(x)=-x^4$ is neither quasiconvex nor pseudoconvex on C.

The next proposition gives a somewhat different sufficient condition for F to be PM [14]:

Proposition 6.7

F is PM on C if in addition to (A) for every $x \in C$ and $v \in \Re^n$ $v^T F(x) = v^T J_F(x) v = 0$ implies there exists $\varepsilon > 0$ such that $v^T J_F(x+tv)v \geq 0$ for all $t \in \mathrm{I}_{x,v}$, $|t| \leq \varepsilon$.

This condition is not necessary for F to be PM. See, for example, $F=\nabla f$, $C=\Re$, where

$$f(x)=\begin{cases} -\int_0^x \xi^4\left(2+\sin\frac{1}{\xi}\right)d\xi & \text{if} \quad x<0 \\ 0 & \text{if} \quad x=0 \\ \int_0^x \xi^4\left(2+\sin\frac{1}{\xi}\right)d\xi & \text{if} \quad x>0. \end{cases} \tag{6.12}$$

The derivative of F changes its sign in every neighborhood of $x=0$, but nevertheless, F is PM.

In addition to proposition 6.7, we have [15]:

Proposition 6.8

F is SPM on C if for every $x \in C$ and $v \in \Re^n$

$$v^T F(x) = 0 \text{ implies } v^T J_F(x) v > 0. \tag{6.13}$$

In the next section, we turn to the special case of affine maps. Just as the criteria above extend second-order characterizations of generalized convex differentiable functions [3] (see also [1], [2], [4], [10]) to differentiable maps, the criteria below extend second-order characterizations of generalized convex quadratic functions ([3], see also [19], [20]) to affine maps.

7. The affine case

Let $F(x) = Mx + q$ where M is a real $n \times n$-matrix and $q \in \Re^n$. We consider F on an open and convex set $C \subseteq \Re^n$. As mentioned before, $F = \nabla f$ if and only if M is symmetric; then $f(x) = \frac{1}{2} x^T M x + q^T x$.

For affine maps, condition (A) becomes

$$\text{(A')} \qquad v^T(Mx+q) = 0 \text{ implies } v^T M v \geq 0, \qquad (7.1)$$

since $J_F(x) = M$ does not depend on x. Also, condition (B) and (C) are always satisfied by affine maps. Hence, Proposition 6.6 yields:

Proposition 7.1
$F(x) = Mx + q$ is QM on C if and only if F is PM on C if and only if (A') holds.

From this it follows easily:

Proposition 7.2
$F(x) = Mx + q$ is QM on the closure $\overline{C}$ of C if and only if F is PM on C.

Furthermore, Proposition 7.1 implies:

Proposition 7.3
Suppose there exists $x^o \in \Re^n$ such that $F(x^o) = Mx^o + q = 0$. Then F is monotone on $\Re^n$ (i.e. M is positive semidefinite) if and only if there exists an open neighborhood $N(x^o)$ of x^o where F is QM.

Thus, we conclude:

Proposition 7.4
If $F(x) = Mx + q$ is QM, but not monotone on C, then $F(x) \neq 0$ for all $x \in C$.

Finally, for the special case $C = \Re^n$ it can be shown:

Proposition 7.5
$F(x) = Mx + q$ is QM on $\Re^n$ if and only if F is monotone on $\Re^n$ (i. e. M is positive semidefinite).

We conclude this survey with some results on invariance properties of generalized monotone maps. The reader is referred to [18] for proofs and additional results.

8. Generalized monotonicity under variable transformations

Consider the variable transformation $z = Ax + b$ where A is an $m \times n$-matrix and $b \in \Re^m$. Let $D \subseteq \Re^m$ be a convex set and

$$C = \{x \in \Re^n \mid Ax + b \in D\}. \tag{8.1}$$

It can be shown:

Proposition 8.1

If $G(z)$ is QM (PM) on D, then $F(x) = A^T G(Ax+b)$ is QM (PM) on C. If $G(z)$ is SPM on D, $m = n$ and $\det A \neq 0$, then $F(x)$ is SPM on C.

In the special case of an orthogonal matrix A (i. e. $A^{-1} = A^T$) we have:

Proposition 8.2

If $G(z)$ is QM (PM, SPM) on D, then $F(x) = A^{-1}G(Ax+b)$ is QM (PM, SPM) on C.

We focus now on affine maps $G(z) = Mz + q$. From Proposition 8.1 we obtain:

Proposition 8.3

If $G(z) = Mz + q$ is QM (PM) on D, then $F(x) = (A^T MA)x + (A^T Mb + A^T q)$ is QM (PM) on C.

In the special case of $D = \Re_+^m$ and linear maps Mz we have:

Proposition 8.4

If $G(z) = Mz$ is QM (PM) on $\Re_+^m$, then for any nonnegative $m \times n$-matrix A $F(x) = (A^T MA)x$ is QM (PM) on $\Re_+^n$. $F(x) = (A^T MA)x$ is SPM on $\Re_+^n$ if in addition $m = n$ and $\det A \neq 0$.

Let us call M QM if Mz is QM on $\Re_+^m$. Proposition 8.4 can be used to generate QM matrices. Given a QM matrix M, then

$$N = A^T MA, \quad A \geq 0 \tag{8.2}$$

is again QM.

This result can be used to obtain numerical examples of QM matrices either of the same size $m \times m$ (if $n = m$) or of larger or smaller sizes $n \times n$ (if $n \neq m$).

To demonstrate this, we use the matrix

$$M = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}. \tag{8.3}$$

It is easy to see, either from the definition or Proposition 7.1, that M is QM.

Consider any nonnegative $2 \times n$ matrix A , i. e.

$$A^T = \left(a^1, a^2\right) \ , \ a^i \geq 0 \quad i = 1,2 \quad , \quad a^i \in \Re^n. \tag{8.4}$$

Then

$$N = A^T M A = \left(2a_i^1 a_j^2 - a_i^2 a_j^1\right) \tag{8.5}$$

is QM.

Additional results on invariance properties of generalized monotone affine maps have been derived in [18]. We currently study the question which properties of generalized monotone affine maps, known for the symmetric case of M , carry over to the nonsymmetric case.

References

[1] Arrow, K.J. and A.C. Enthoven, *Quasi-concave programming,* Econometrica 29, 1961, 779-800.

[2] Avriel, M., *r-convex functions*, Mathematical Programming 2, 1972, 309-323.

[3] Avriel, M., Diewert, W.E., Schaible, S. and I. Zang, Generalized concavity, Plenum Publishing Corporation, New York 1988.

[4] Avriel, M. and S. Schaible, *Second-order characterizations of pseudoconvex functions*, Mathematical Programming 14, 1978, 170-185.

[5] Cambini, A., Castagnoli, E., Martein, L., Mazzoleni, P. and S. Schaible (eds.), Generalized convexity and fractional programming with economic applications, Springer-Verlag, Berlin - Heidelberg - New York 1990.

[6] Castagnoli, E., *On order-preserving functions*, presented at the International Workshop on "Generalized Convexity and Fractional Programming", University of California, Riverside, October 1989.

[7] Castagnoli, E. and P. Mazzoleni, "*Order-preserving functions and generalized convexity*", presented at the International Workshop on "Generalized Convexity and Fractional Programming", University of California, Riverside, October 1989; J. of Optimization Theory and Applications (to appear in 1992).

[8] Castagnoli, E. and P. Mazzoleni, *Generalized monotonicity and poor vector order relations*" presented at the International Workshop on "Generalized Convexity and Fractional Programming", University of California, Riverside, October 1989.

[9] Cottle, R.W. and J.C. Yao, *Pseudomonotone complementarity problems in Hilbert space,* Department of Operations Research, Stanford University, Stanford 1990.

[10] Crouzeix, J.P. and J.A. Ferland, *Criteria for quasiconvexity and pseudoconvexity: relationships and comparisons"* Mathematical Programming 23, 1982, 193-205.

[11] Harker, P.T. and J.S. Pang, *Finite dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications"* in: Cottle, R.W. Kyparisis, J. and J. S. Pang (eds.), *Variational inequality problems,* Mathematical Programming 48, Series B, 1990, 161-220.

[12] Karamardian, S., *Complementarity over cones with monotone and pseudomonotone maps,* J. of Optimization Theory and Applications 18, 1976, 445-454.

[13] Karamardian, S. and S. Schaible, *Seven kinds of monotone maps,* J. of Optimization Theory and Applications 66, 1990, 37-46.

[14] Karamardian, S. and S. Schaible, *First-order characterizations of generalized monotone maps,* Working Paper 90 - 5, Graduate School of Management, University of California, Riverside, December 1989.

[15] Karamardian, S. Schaible, S. and J.P. Crouzeix, *Characterizations of generalized monotone maps,* J. of Optimization Theory and Applications (to appear in 1992).

[16] Komlosi, S., *Nonsmooth generalized monotone maps,* in: Giannessi, F. (ed.), Nonsmooth optimization methods and applications (to appear).

[17] Mazzoleni, P., Monotonicity *properties and generalized concavity*, presented at the International Workshop on "Generalized Convexity and Fractional Programming", University of California, Riverside, October 1989.

[18] Pini, R. and S. Schaible, *Invariance properties of generalized monotonicity,* Working Paper, University of Verona, May 1992.

[19] Schaible, S., *Second-order characterization of pseudoconvex quadratic functions*, J. of Optimization Theory and Applications 21, 1977,15-26.

[20] Schaible, S. *Quasiconvex, pseudoconvex and strictly pseudoconvex quadratic functions*, J. of Optimization Theory and Applications 35, 1981, 303-338.

[21] Schaible, S. and W.T. Ziemba (eds), Generalized Concavity in optimization and economics, Academic Press, New York 1981.

[22] Singh, C. and B.K. Dass (eds), Continuous-time, fractional and multiobjective programming, Analytic Publishing Company, New Delhi 1989.