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On the Relationships between Bicriteria Problems and Non-Linear Programming Problems

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Abstract

The aim of this paper is to suggest a unifying approach for studying general classes of scalar optimization problems related to a bicriteria problem. More exactly we will point out how the knowledge of the set E of all efficient points of a vector optimization problem having two objective functions f_1, f_2, can be used in finding the optimal solutions of the class of functions:

$$H(x) = h\{ F[f_1(x)], G[f_2(x)] \},\ x \in X \subset \mathbb{R}^n$$

where F, G are increasing functions and h is a suitable real-valued function. Recently, results given in [3,6,8,11] establish that E can be expressed as the union of suitable sets of optimal level solutions so that it is possible to generalize the approach suggested by Geoffrion in [4] for the class of concave bicriterion mathematical programs.

1. Bicriteria Problems: preliminary results

For what concerns multi-objective programming, particular importance has been attributed, the last few years, to the solution of bicriteria problems [1,3,5,6,7,8,9,11]. These researches have led to the characterization of the set of Pareto solutions, with special attention to both the study of the connection of the efficient boundary and the determination, for special classes of functions, of sequential methods able to generate it.

Let $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ be continuous functions and let $X \subset \mathbb{R}^n$ be a compact set; we can associate with said functions the following bicriteria problems:

P_{B1} : (max $f_1(x)$, max $f_2(x)$), $x \in X$ $\qquad$ P_{B3} : (min $f_1(x)$, min $f_2(x)$), $x \in X$

P_{B2} : (max $f_1(x)$, min $f_2(x)$), $x \in X$ $\qquad$ P_{B4} : (min $f_1(x)$, max $f_2(x)$), $x \in X$

Taking into account the obvious relationships, min f = - max (- f) and max f = - min (-f), problems P_{B3} and P_{B4} may be formulated as problems of the kind P_{B1} and P_{B2}, respectively, and "viceversa". For this reason, without loss of generality, we

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associate with the functions f_1 and f_2 only the bicriteria problems P_{B1} and P_{B2}.

As it is well known, characterization of efficient points via scalarization is not possible when the bicriteria problem is not concave, so that, in order to generalize Geoffrion's results, we will refer to the approach given in [3,6,8,11]; such an approach points out the existing relationship between the set E_1 of the Pareto solutions of problem P_{B1}[1] and the set $S_1(\theta)$ of the optimal solutions of the following scalar parametric problem:

$$P_1(\theta): Z_1(\theta) \equiv \max f_1(x), \; x \in R_1(\theta) \equiv \{ x \in X : f_2(x) \geq \theta \}$$

(where one of the two objective functions plays the role of parametric constraint).

Let us note that P_{B1} and P_{B2} have different properties; in fact, if P_{B1} is a concave or generalized concave problem, P_{B2} is not. In order to determine a relationship involving both these problems P_{B1} and P_{B2}, we denote by E_2 the set of Pareto solutions to problem P_{B2} and by $S_2(\theta)$ the optimal solutions of the scalar parametric problem $P_2(\theta)$, where:

$$P_2(\theta): Z_2(\theta) \equiv \max f_1(x), \; x \in R_2(\theta) \equiv \{ x \in X : f_2(x) \leq \theta \}.$$

Set $a_0 = \max_{x \in X} f_1(x)$, $\quad \theta_1 = \max_{f_1(x)=a_0,\, x \in X} f_2(x)$, $\quad \theta_0 = \max_{x \in X} f_2(x)$,

$\theta_2 = \min_{f_1(x)=a_0,\, x \in X} f_2(x)$ and $\theta_3 = \min_{x \in X} f_2(x)$.

The results given in [6,8] can be easily extended to the pair of problems P_{B1}, P_{B2}, in order to obtain the following theorem:

Theorem 1.1: If f_1 does not have local maxima different from global ones, then the following properties hold:

1a) $P_1(\theta)$ has optimal solutions for each $\theta \in [\theta_3, \theta_0]$;

1b) each optimal solution to $P_1(\theta)$ with $\theta \in [\theta_1, \theta_0]$ is binding to the parametric constraint $f_2(x) \geq \theta$;

1c) $$E_1 = \bigcup_{\theta \in [\theta_1, \theta_0]} S_1(\theta);$$

2a) $P_2(\theta)$ has optimal solutions for each $\theta \in [\theta_3, \theta_0]$;

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[1] Let us note that a bicriteria problem consists of optimizing, in the sense given by Pareto, a pair of functions and that a point $x^0 \in R$ is called a Pareto solution or efficient solution for the bicriteria problem P_{B1} [P_{B2}], if there does not exist a point $x \in X$ such that the following inequalities are simultaneously verified:

$f_1(x) \geq f_1(x^0)$, $f_2(x) \geq f_2(x^0)$ $\quad$ [$f_1(x) \geq f_1(x^0)$, $f_2(x) \leq f_2(x^0)$]

with at least one strictly verified.

2b) each optimal solution to $P_2(\theta)$ with $\theta \in [\theta_3, \theta_2]$ is binding to the parametric constraint $f_2(x) \leq \theta$;

2c) $$E_2 = \bigcup_{\theta \in [\theta_3, \theta_2]} S_2(\theta).$$

Taking into account the previous theorem, the following remark shows how P_{B1} and P_{B2} can be solved through a unique parametric scalar problem.

Remark 1.1: Let us consider the scalar parametric problem:

(1.1) $P(\theta) : Z(\theta) \equiv \max f_1(x), \ x \in R(\theta) \equiv \{ x \in X : f_2(x) = \theta \}$

According to the properties 1b) and 2b) of Theorem 1.1, if f_1 does not have local maxima different from global ones, each optimal solution of $P_1(\theta)$ is binding to the parametric constraint $f_2(x) \geq \theta$ when $\theta \in [\theta_1, \theta_0]$, while each optimal solution of $P_2(\theta)$ is binding to the parametric constraint $f_2(x) \leq \theta$ when $\theta \in [\theta_3, \theta_2]$. Therefore, $P_1(\theta)$ is equivalent to $P(\theta)$ for every $\theta \in [\theta_1, \theta_0]$ and $P_2(\theta)$ is equivalent to $P(\theta)$ for every $\theta \in [\theta_3, \theta_2]$.

Now we will establish a condition which ensures that sets E_1 and E_2 are connected. Set $S_M = \{x \in X : f_1(x) = a_0\}$, the following theorem holds:

Theorem 1.2:

i) $E_1 \cap E_2 = S_M \Leftrightarrow$ the restriction of f_2 on S_M is constant;

ii) $E_1 \cup E_2 = \bigcup_{\theta \in [\theta_3, \theta_0]} S(\theta) \Leftrightarrow$ the restriction of f_2 on S_M is constant,

where $S(\theta)$ denotes the set of optimal solutions to problem (1.1).

Proof: The restriction f_2 on S_M is constant if and only if $\theta_2 = \theta_1$. For properties 1c) and 2c) of theorem 1.1 and remark 1.1, we have ii) and $E_1 \cap E_2 \neq \emptyset$. In particular, $E_1 \cap E_2 = S_M$ from which i).

Under suitable assumptions of generalized concavity, we obtain some properties for the functions Z_1 and Z_2. The following theorem holds:

Theorem 1.3: If f_1 is a semi-strictly quasi-concave function and f_2 is continuous on the compact set X then $Z_1(\theta)$ is non-increasing on X and $Z_2(\theta)$ is non-decreasing on X. Specifically, $Z_1(\theta)$ is constant in the interval $[\theta_3, \theta_1]$ and $Z_2(\theta)$ is constant in the interval $[\theta_2, \theta_0]$.

Proof: For what concerns the monotonia of functions $Z_1(\theta)$ and $Z_2(\theta)$ it is sufficent to observe that if $\theta' < \theta''$ then $R_1(\theta') \supset R_1(\theta'')$ so that $Z_1(\theta') \geq Z_1(\theta'')$ while if $\theta' < \theta''$ then $R_2(\theta') \subset R_2(\theta'')$ so that $Z_2(\theta') \leq Z_2(\theta'')$. The fact that $Z_1(\theta)$ is monotonically decreasing in the interval $[\theta_1, \theta_0]$ (and the fact that $Z_2(\theta)$ is monotonically increasing in the interval $[\theta_3, \theta_2]$) is a direct conseguence of properties 1b) and 2b) of theorem 1.1. The rest of the proof is straightforward.

2. Non-linear programming problems related to bicriteria problems

In this section we will consider a wide class of non linear optimization problems involving functions f_1, f_2, whose optimal solutions are efficient points of the previous bicriteria problems . With this aim, let us consider these classes of scalar extremum problems:

P_{h1} : $\max H_1 = h_1 (F[f_1(x)],\ G[f_2(x)]), x \in X$

P_{h2} : $\max H_2 = h_2 (F[f_1(x)],\ G[f_2(x)]), x \in X$

where h_1: $\mathbb{R}^2 \to \mathbb{R}$ is an increasing function in each argument[2], h_2: $\mathbb{R}^2 \to \mathbb{R}$ is a function which is increasing in the first argument and decreasing in the second one, f_1 and f_2 are real-valued and continuous functions defined on the compact set $X \subset \mathbb{R}^n$, F and G are continuous functions defined on the subsets $X_1, X_2 \subset \mathbb{R}$ containing, respectively, the outcomes of the functions f_1 and f_2.

Let us note that the class P_{h1} reduces to the one studied by Geoffrion in [4], if f_1, f_2 are concave functions and F, G are increasing functions.

If S_{h1} and S_{h2} denote the sets of optimal solutions for the problems P_{h1} and P_{h2}, respectively, the following theorem points out the relationships between S_{h1} and E_1, S_{h2} and E_2.

Theorem 2.1:

i) If F and G are increasing functions then $S_{h1} \subset E_1$,

ii) If F and G are increasing functions then $S_{h2} \subset E_2$.

Proof: i) We must prove that, if x^0 is an optimal solution for P_{h1}, then x^0 is an efficient solution for P_{B1}. Let x^0 be optimal solution for P_{h1} then

$h_1 \{ F [f_1(x)], G [f_2(x)] \} \leq h_1 \{ F [f_1(x^0)], G [f_2(x^0)] \}$ for each $x \in X$.

Let us suppose ab absurdo that x^0 is not a Pareto solution for P_{B1}, then there exists $x^1 \in X$ such that:

$f_1(x^0) \leq f_1(x^1)$ and $f_2(x^0) < f_2(x^1)$

or

$f_1(x^0) < f_1(x^1)$ and $f_2(x^0) \leq f_2(x^1)$.

Since F and G are increasing functions, it follows that:

(2.1.a) $F [f_1(x^0)] \leq F [f_1(x^1)]$ and $G [f_2(x^0)] < G [f_2(x^1)]$

or

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2 Let us recall that $\phi(x,y)$: $D \subset \mathbb{R}^2 \to \mathbb{R}$, is an increasing function in each argument if $\phi(x_1,y) < \phi(x_2,y)$, for each $(x,y) \in D$ such that $x_1 < x_2$ and $\phi(x,y_1) < \phi(x,y_2)$ for each $(x,y) \in D$ such that $y_1 < y_2$.

(2.1.b) $F[f_1(x^0)] < F[f_1(x^1)]$ and $G[f_2(x^0)] \leq G[f_2(x^1)]$.

Since h_1 is an increasing function in each argument, taking into account (2.1.a), we have:

(2.2.a) $h_1\{F[f_1(x^0)], G[f_2(x)]\} \leq h_1\{F[f_1(x^1)], G[f_2(x)]\}, \forall x \in X$,

(2.2.b) $h_1\{F[f_1(x)], G[f_2(x^0)]\} < h_1\{F[f_1(x)], G[f_2(x^1)]\}, \forall x \in X$.

In particular, (2.2.a) holds for $x = x^0$ and (2.2.b) holds for $x = x^1$, so that we have:

$h_1\{F[f_1(x^0)], G[f_2(x^0)]\} \leq h_1\{F[f_1(x^1)], G[f_2(x^0)]\}$

and

$h_1\{F[f_1(x^1)], G[f_2(x^0)]\} < h_1\{F[f_1(x^1)], G[f_2(x^1)]\}$.

That contradicts the optimality of point x^0 for problem P_{h1}. In the same way we will prove that (2.1.b) contradicts the optimality of x^0.

ii) The proof is similar to the one given in i).

The previous theorem allow us to characterize particular classes of problems to which it is possible to apply the obtained results; consider, for instance, the following subclass of problems:

(2.3) $P^* = \max \{F[f_1(x)] * G[f_2(x)]\}, x \in X$

where F, G are increasing functions and * denotes an algebraic composition law (+, ·, - , :).

Let S_A, S_S, S_P and S_Q be the sets of optimal solutions of the type P* problems, when * is, respectively, + , -, · , : . Recalling that E_1 is the set of Pareto solutions for P_{B1} and E_2 for P_{B2}, the following corollary points out the relationship between S_A, S_S, S_P, S_Q and E_1, E_2.

Corollary 2.1:

i) $S_A \subset E_1$.

ii) If $f_1(x) > 0$ and $f_2(x) > 0$ for every $x \in X$, $F(0) \geq 0$ and $G(0) \geq 0$ then $S_P \subset E_1$.

iii) $S_S \subset E_2$.

iv) If $f_1(x) > 0$ and $f_2(x) > 0$ for every $x \in X$, $F(0) \geq 0$ and $G(0) \geq 0$ then $S_Q \subset E_2$.

In remark 1.1 we noted that bicriteria problems P_{B1}, P_{B2} may be related by means of the unique scalar parametric problem (1.1). Such a problem can be utilized even to solve the class of problems (2.3) how the following remark shows.

Remark 2.1: If x^0 is an optimal solution of the problem P* then it is also the optimal solution of the problem:

$$\max F[f_1(x)]$$
$$G[f_2(x)] = G[f_2(x_0)], \; x \in X$$

Therefore, solving P* is equivalent to find the optimal level $G[f_2(x_0)] = G[\theta_0]$ in the parametric problem (1.1).

3. Algorithmic aspects

The previous theoretical results point out that the optimal solutions of the scalar problems P_{h1} and P_{h2} are contained, respectively, in the sets E_1 and E_2 of all efficient points of the bicriteria problem P_{B1} and P_{B2}. In order to outline this important aspect, let us note that a problem belonging to the class:

$$P' = \max z'(x) = [\, f_1(x)\,]^{\alpha} * [\, f_2(x)]^{\beta}\ , x \in X\ , \alpha, \beta \in \mathbb{R}$$

where * is an algebraic law (+ , · , - , :), f_1, f_2 are fractional functions or f_1 is a linear function and f_2 is quadratic function, is not easy to solve since the objective function does not have particular properties (except trivial cases), so that we can have several local maximum not global; on the contrary it is possible to generate all efficient solutions of the related bicriteria problem[1,6,9] which can be utilized to solve every problem of the class P'. More precisely, from an algorithmic point of view, taking into account the previous results, it is possible to propose a unifying approach to solve problems of the kind P_{h1} and P_{h2}.

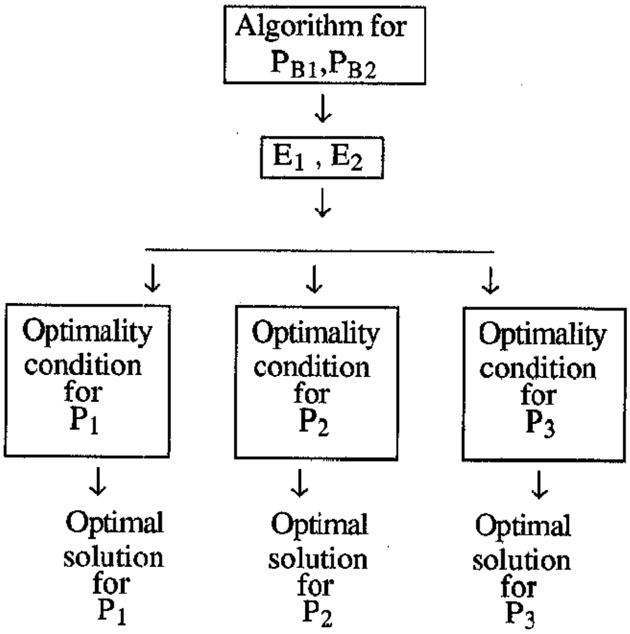
Let us consider, for instance, the following problems:

$$P_1 = \max\ \sqrt{f_2(x)} + k\, f_1(x),\ x \in X,\ k \in \mathbb{R}$$
$$P_2 = \max\ [f_1(x) \cdot f_2^{\alpha}],\ x \in X,\ \alpha \in \mathbb{R}$$
$$P_3 = \max\ [f_1^{\alpha}(x) - f_2^{\beta}],\ x \in X,\ \alpha, \beta \in \mathbb{R}$$

which are structurally different but they are related to the same bicriteria problem, that is P_{B1} or P_{B2}.

Algorithms, based on different theoretical results, have been proposed to solve, in particular case, problems P_1, P_2 [2,12,13]. With our approach we are able to solve P_1, P_2, P_3 and each problem of the type P_{h1} and P_{h2} with a unique algorithm, only applying different optimality conditions.

More exactly, the idea is the following one: starting from an algorithm which solves P_{B1} or P_{B2}, at each iteration we determine a subset of E_1, E_2 and we verify, by means of suitable optimality conditions, if such a subset contains the optimal solutions of problems P_1, P_2, P_3 or one of the type P_{h1} and P_{h2}. If it does not happen we perform another iteration and so on. That is synthetized in the following picture.



Furthermore, we want to outline that, when a sufficient optimality condition for a particular problem of the kind P_{h1} and P_{h2} is not easy to find but, at the same time, the set of all efficient points of the related bicriteria problem has a particular structure, for instance, when it is the union of segments, taking into account that we have:

$$\max_{x \in X} H_1(x) = \max_{x \in E_1} H_1(x) \qquad \max_{x \in X} H_2(x) = \max_{x \in E_2} H_2(x)$$

it is possible to solve the considered problem through one of the known basic descendent methods applied on the set of all efficient solutions, for instance, as it is suggested in [10].

4. An illustrative example

For sake of clearness, the following example shows an application of the previous results. Let us consider the bicriteria problems:

P_{B1} : ($\max f_1(x)$, $\max f_2(x)$), $x \in X$

P_{B2} : ($\max f_1(x)$, $\min f_2(x)$), $x \in X$

where $f_1(x) = 5 - \frac{5}{4} x_1 + x_2$, $f_2(x) = 2 - \frac{1}{2} x_1 + x_2$ and $X = \{x \in IR^n, x_1+x_2 \geq 1, x_1+x_2 \leq 2, x_1 \geq 0, x_2 \geq 0\}$.

The sets of all efficient points are, respectively:

$E_1 = \{A\}$ and $E_2 = \{seg[AD], seg[DC], seg[CB]\}$,

where A = (0,2), B = (2,0), C = (1,0), D = (0,1).

Taking into account the previous results, we know that the optimal solutions for the problems belonging to the classes:

P_{h1} : $\max H_1 = h_1 (F[f_1(x)], G[f_2(x)]), x \in X$

P_{h2} : $\max H_2 = h_2 (F[f_1(x)], G[f_2(x)]), x \in X$

are in the sets E_1, E_2, respectively.

In particular this happens for the following problems of the type P_{h1}:

$P_1 = \max \sqrt{f_2(x)} + k f_1(x), x \in X, k \in IR$

$P_2 = \max [f_1^{\alpha}(x) \cdot f_2^{\beta}], x \in X, \alpha, \beta \in IR^+$

$P_3 = \max [f_1^{\alpha}(x) \cdot \ln (f_2)], x \in X, \alpha \in IR^+$

and for the following problems of the type P_{h2}:

$P_4 = \max [f_1(x) - f_2(x)], x \in X$

$P_5 = \max [f_1(x) - f_2^2(x)], x \in X$

$P_6 = \max \frac{\sqrt{f_1(x)}}{f_2(x)}, x \in X.$

Since the set E_1 has a unique element, the optimal solution of P_1, P_2, P_3 and of each problem of the type P_{h1} is A, while the optimal solutions of P_1, P_2, P_3 are, respectively, seg [AD], E = (3/2,0), B and these elements are contained in the set E_2 because of theorem 2.1.

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