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# An algorithm for a non-differentiable non-linear fractional programming problem

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# AN ALGORITHM FOR A NON-DIFFERENTIABLE NON-LINEAR FRACTIONAL PROGRAMMING PROBLEM(*)

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## Abstract

The problem of minimizing a non-differentiable fractional function on a polytope is considered. The numerator is the sum of a linear function and the square root of a convex quadratic form while the denominator is a linear function. Patkan and Stancu-Minasian have studied the problem and, for the case of a bounded feasible region, they have proposed different approaches for solving it. In this paper we will be discussing a finite algorithm, which solves the problem by means of a parametric quadratic programming, even when the feasible region is unbounded.

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## 1. The problem

The problem is:

$$(1.1) \qquad \inf_{x\in S} f(x)= \frac{c^Tx+c_0+\sqrt{\frac{1}{2}x^TQx}}{d^Tx+d_0}$$

where Q is a symmetric positive definite nxn matrix, $c, d \in \mathbb{R}^n$, $c_0, d_0 \in \mathbb{R}$, $S=\{x\in \mathbb{R}^n: Ax \geq b\}$[1], A is an mxn matrix and $b \in \mathbb{R}^m$. We will suppose that $d^Tx+d_0>0, \forall x\in S$.

We will consider vectors $(c^T, c_0)$ and $(d^T, d_0)$ linearly independent, because, if $(c^T,c_0) = \lambda (d^T,d_0)$ with $\lambda \in \mathbb{R}$, problem (1.1) becomes:

$$\inf_{x\in S} f(x)= \frac{\lambda (d^Tx+d_0)+\sqrt{\frac{1}{2}x^TQx}}{d^Tx+d_0} = \lambda + \frac{\sqrt{\frac{1}{2}x^TQx}}{d^Tx+d_0}$$

and this problem has been studied and solved in [1].

## 2. Theoretical properties

In [2,3,4] Patkar and Stancu-Minasian proved that the objective function f(x) is "explicitly" quasi-convex. This property implies that any local minimum is also global.

The function f(x) is non-differentiable in x=0. This implies that "a priori" methods based on validation of Kuhn -Tucker (K-T) conditions cannot be applied. We will show that it is possible to define a procedure, based on validation of K-T conditions, which solves the problem in a finite number of steps.

For problem (1.1) the following three cases can occur:

a) there exists an $x^{ott}\in S$ such that $\inf \{f(x): x \in S\} = f(x^{ott})$;

b) $\inf \{f(x): x \in S\} = L > -\infty$, but the inf is not attained; then there exists a half-line $\{x = x_0 + k u , k \geq 0, x_0 \in S\}\subset S$ such that:
$$\lim_{k\to+\infty} f(x_0 + k u) = L;$$

c) $\inf\{f(x): x\in S\} = -\infty$ ; i.e., there exists a half-line $\{x = x_0 + k u$ $, k \geq 0, x_0\in S\}\subset S$ such that:
$$\lim_{k\to+\infty} f(x_0 + k u) = -\infty.$$

These three cases will be shown implicitly by the algorithm proposed in section 4.

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[1] We will suppose that S is non empty and the vertices of S are non degenerate.

Set $C_0=c^Tx_0+c_0$ and $D_0 = d^Tx_0+d_0$.

**Remark 2.1**: Let us note that if S is a bounded region only case a) occurs. If S is unbounded, let us consider a half-line $\{x = x_0 + k u, k \geq 0, x_0 \in S\} \subset S$ and denote with $\phi(k)$ the restriction of f(x) on this half-line, we have:

i) if $d^Tu > 0$ then $\lim_{k \to +\infty} \phi(k) = \dfrac{c^Tu + \sqrt{\frac{1}{2}u^TQu}}{d^Tu} = L_1$;

ii) if $d^Tu = 0$ and $c^Tu + \sqrt{\frac{1}{2}u^T Q u} = 0$ then $\lim_{k \to +\infty} \phi(k) = \dfrac{C_0}{D_0} + \dfrac{x_0^T Q u}{D_0 \sqrt{2\, u^TQu}} = L_2$;

iii) if $d^Tu = 0$ and $c^T u + \sqrt{\frac{1}{2}u^T Q u} < 0$ then $\lim_{k \to +\infty} \phi(k) = -\infty$;

iv) if $d^Tu = 0$ and $c^T u + \sqrt{\frac{1}{2}u^T Q u} > 0$ then $\lim_{k \to +\infty} \phi(k) = +\infty$.

By applying the Charnes-Cooper transformations $y = x\,t$ , $t = \dfrac{1}{d^Tx+d_0}$ to the problem (1.1), we obtain :

$$(2.1) \qquad \inf_{(y,t)\in R} \varphi(y,t) = c^Ty + c_0t + \sqrt{\tfrac{1}{2}y^T Q\, y}$$

where $R = \{(y, t) \in IR^{n+1}: Ay\text{-}bt \geq 0, d^Ty+d_0t=1, t>0\}$.

The following theorems state some conditions which will allow us to solve problem (1.1) through problem:

$$(2.2) \qquad \inf_{(y,t)\in R'} \varphi(y,t) = c^Ty + c_0t + \sqrt{\tfrac{1}{2}y^T Q\, y}$$

where $R' = \{(y, t) \in IR^{n+1}: Ay\text{-}bt \geq 0, d^Ty+d_0t =1, t \geq 0\}$.

**Theorem 2.1**: To each ray u which verifies condition i) of Remark 2.1 corresponds a $(y,0) \in R'$ such that $\varphi(y,0) = L_1$ and viceversa.

**Proof**: If u satisfies conditions i) of Remark 2.1, then

$$Au \geq 0,\ Ax_0 \geq b,\ d^Tu > 0 \text{ and } \lim_{k \to +\infty} \phi(k) = \frac{c^Tu + \sqrt{\frac{1}{2}u^TQu}}{d^Tu} = L_1;$$

it follows that in problem (2.2):

$(u / d^Tu,0) \in R'$ and $\varphi ( u / d^Tu, 0 ) = L_1$.

Viceversa, if $(y,0) \in R'$ then

$$Ay \geq 0,\ d^T y = 1,\ \varphi(y,0) = c^Ty + \sqrt{\tfrac{1}{2} y^TQy};$$

it follows that:

$$A(x_0+k\,y) \geq b,\ x_0 \in S,\ \forall k \geq 0 \text{ and } \lim_{k\to+\infty} \phi(k) = \frac{c^T y + \sqrt{\frac{1}{2} y^T Q y}}{d^T y} = \varphi\,(y,0)$$

being $d^T y=1$. This completes the proof.

Let us denote with $\psi(\theta)$ the restriction of $\varphi(y,t)$ on $(y,t) = (y_0, t_0)+\theta\,(\omega,\delta)$, $\theta\geq 0$:

**Theorem 2.2**: To each point $x_0$ and ray u which verify condition ii) of Remark 2.1 corresponds a half-line $\{(y,t) = (y_0,t_0)+\theta\,(\omega,\delta),\ \theta\geq 0,\ (y_0,t_0) \in R'\} \subset R'$ such that:

$$\lim_{k\to+\infty} \phi(k) = \lim_{\theta\to+\infty} \psi(k) = L_2$$

and viceversa.

**Proof:** If $x_0$ and u satisfy conditions ii) of Remark 2.1, then

$$Au \geq 0,\ Ax_0 \geq b,\ d^T u = 0,\ c^T u + \sqrt{\frac{1}{2} u^T Q u} = 0 \text{ and } \lim_{k\to+\infty} \phi(k) = L_2;$$

It follows that in problem (2.2) there exists the corresponding half-line with: $y_0 = x_0/D_0$, $t_0 = 1/D_0$, $\omega = u/D_0$ and $\delta = 0$ such that:

$$\lim_{\theta\to+\infty} \psi(\theta) = L_2.$$

Viceversa, we must prove that if:

$$\lim_{\theta\to+\infty} \psi(\theta) = L_2$$

then there exists a half-line in S such that condition ii) of remark 2.1 is verified. Let us consider a half-line $(y,t) = (y_0, t_0) + \theta\,(\,\omega,\delta\,)$, $\theta\geq 0$ such that:

$$Ay_0 - bt_0 \geq 0,\ A\omega - b\delta \geq 0,\ d^T y_0 + d_0 t_0 = 1,\ d^T\omega + d_0\delta = 0, \text{ then}$$

$$\text{if } c^T\omega + c^T\delta + \sqrt{\frac{1}{2}\omega^T Q\omega} = 0,\ \lim_{\theta\to+\infty} \psi(\theta) = c^T y_0 + c^T t_0 + \frac{y_0^T Q\,\omega}{\sqrt{2\,\omega^T Q\omega}}\,.$$

Let us note that $\delta$ must equal zero, in fact if $\delta\neq 0$ then:

$A\,\omega/\delta \geq b$, $\omega/\delta \in S$, $d^T\omega/\delta + d_0 = 0$, contradicting the hypothesis $d^T x+d_0 > 0$, $A\,x \in S$. It follows that $y_0/t_0 \in S$, $\omega/t_0$ is a ray of S such that :

$$\lim_{k\to+\infty} \phi(k) = \lim_{\theta\to+\infty} \psi(k) = L_2$$

This completes the proof.

**Theorem 2.3**: To each point $x_0$ and ray u which verify condition iii) of Remark 2.1 corresponds a half-line $\{(y,t) = (y_0, t_0) + \theta\,(\omega,\delta),\ \theta\geq 0,\ (y,t) \in R'\} \subset R'$ such that:

$$\lim_{k\to+\infty} \phi(k) = \lim_{\theta\to+\infty} \psi(k) = -\infty$$

and viceversa.

**Proof:** Similar to proof of theorem 2.2.

**Remark 2.2**: As a consequence of the previous theorems we have that:

1) if (y',t') is a finite optimal solution of problem (2.2) then two cases can occur:
   - $t' > 0$; in this case, $x^{ott} = y'/t'$ is the optimal solution of problem (1.1);
   - $t' = 0$; in this case, problem (1.1) is lower bounded on S but the inf is not attained, in fact, $\inf\{f(x): x \in S\} = \varphi(y',0)$ on ray y';
2) if problem (2.2) is unbounded then problem (1.1) is unbounded;
3) if problem (2.2) is lower bounded, but the inf is obtained on the half-line $\{(y,t)=(y',t')+\theta(\omega,\delta), \theta \geq 0, (y,t) \in R'\}$ then also problem (1.1) is lower bounded and $\inf\{f(x): x \in S\} = \inf\{\varphi(y,t): (y,t) \in R'\} = c^T y' + c_0 t' + \sqrt{\frac{1}{2} y'^T Q y'}$ on half-line $\{x = y'/t' + k\, w'/t',\ k \geq 0\} \subset S$.

In [5] a procedure has been proposed to solve the following problem:

$$\text{(2.3)} \qquad \min\ g(z) = q^T z + k_0 \sqrt{\frac{1}{2} z^T H z}\ ,\ z \in S'$$

when H is a symmetric positive definite pxp matrix, $q \in \mathbb{R}^p$, $k_0 \in \mathbb{R}^+$, $S' = \{z \in \mathbb{R}^p: A'z \geq b'\}$, A' is an m'xp matrix and $b' \in \mathbb{R}^{m'}$. Let us note that it is not possible to directly apply this procedure to solve problem (2.2), since if we set :

$$z = \begin{bmatrix} y \\ t \end{bmatrix},\ q = \begin{bmatrix} c \\ c_0 \end{bmatrix},\ H = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix},\ k_0 = 1,$$

$$A' = \begin{bmatrix} A & -b \\ d^T & d_0 \\ 0 & 1 \end{bmatrix},\ b' = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } p = n+1$$

H is semi-definite positive. The procedure proposed in [5] solves problem (2.3) by means of parametric quadratic programming. If we apply this procedure, we may have the following results for problem (2.3):

1) there exists an optimal solution $z_{ott} \in S'$;
2) problem (2.3) is not lower bounded;
3) problem (2.3) is lower bound but the inf is not attained.

The following is a brief description of this procedure, called SO (see [5] for further information):

**Step 0**: Determine $H^{-1}$.

If $(2q^T H^{-1} q)^{1/2} < k_0$ and $b' \leq 0$, then set $z_{ott} = 0$ and STOP.

If $(2\, q^T H^{-1} q)^{1/2} = k_0$ and $S'(\theta) = \{\theta: A' z(\theta) \geq b', \theta \leq 0\} \neq \emptyset$ where

$z(\theta)=\dfrac{\theta}{q^T H^{-1} q} H^{-1}q$ then set $z_{ott} = z(\theta)$ for $\theta \in S'(\theta)$ and STOP.

If $(2\, q^T H^{-1} q)^{1/2} > k_0$ and $\inf S'(\theta) = -\infty$ then problem (2.2) is unbounded and STOP.

If $(2\, q^T H^{-1} q)^{1/2} > k_0$ and $\inf S'(\theta) = \theta' \neq 0 > -\infty$ then set $k=1$, $z^{(k)} = z(\theta')$, $\theta^0 = q^T z^{(k)}$ and go to Step 2.

Otherwise, go to Step 1.

**Step 1:** Set $k=1$, let $z^{(k)}$ be the optimal solution of the linear problem min $q^T z$, $z \in S'$. If $z^{(k)} = 0$ then set $z_{ott} = 0$ and STOP.

If $z^{(k)} \neq 0$ and $S' \cap \{ z : q^T z = q^T z^{(k)} \} = \{ z^{(k)} \}$ then set $\theta^0 = q^T z^{(k)}$ and go to Step 2.

If $z^{(k)} \neq 0$ and $S' \cap \{ z : q^T z = q^T z^{(k)} \} \neq \{ z^{(k)} \}$ then solve $P'(\theta) = \{\min z^T H z,\ z \in S',\ q^T z = \theta \}$ with $\theta = q^T z^{(k)}$; let z' be the optimal solution of this problem, set $z^{(k)} = z'$ and $\theta^0 = q^T z^{(k)}$ and go to Step 2.

If $\{ \min q^T z,\ z \in S' \} = -\infty$, determine a solution $z' \in S'$ such that $q^T z' < 0$ and let $z^{(k)}$ be the optimal solution of $P'(q^T z')$, set $\theta^0 = q^T z^{(k)}$ and go to Step 2.

**Step 2:** Let $M' z^{(k)} = b'_M$ the equations of the constraints binding at $z^{(k)}$. If $z^{(k)}$ is not a vertex, select from the matrix $B' = \begin{bmatrix} M' \\ q^T \end{bmatrix}$ the submatrix $B = \begin{bmatrix} M \\ q^T \end{bmatrix}$ of maximum rank and go to Step 2. Otherwise, select a feasible basis[3] $B = \begin{bmatrix} M \\ q^T \end{bmatrix}$ of the matrix B'and go to Step2.

**Step 3:** Solve the following system:

$$(2.4)\qquad \begin{aligned} H z - M^T \mu - q\, \mu_0 &= 0 \\ M z &= b_M \\ q^T z &= \theta_0 + \theta \end{aligned}$$

Let
$$z(\theta) = z^{(k)} + \alpha\, \theta$$
$$\mu(\theta) = \mu^{(k)} + \beta\, \theta$$
$$\mu_0(\theta) = \mu_0^{(k)} + \gamma\, \theta$$

be the solution of the system. If $z^{(k)}$ is a vertex of S', go to Step 5; otherwise

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3 $B = \begin{bmatrix} M \\ q^T \end{bmatrix}$ is a feasible basis if the basic solution of the system $M^T \mu + q\, \mu_0 = H z^{(k)}$ verifies conditions $\mu \geq 0$. In general, different bases correspond to the matrix $B' = \begin{bmatrix} M' \\ q^T \end{bmatrix}$.

go to Step 4.

**Step 4**: If $\mu_0^{(k)} = 2(\theta_0 - z_0)/k_0^2$ then set $z_{ott}=z^{(k)}$ and STOP.

If $\mu_0^{(k)} < 2(\theta_0 - z_0)/k_0^2$ then set $U(\theta) = H(\theta) \cap [0, +\infty)$ where $H(\theta) = \{\theta : z(\theta) \in R\} \cap \{\theta : \mu(\theta) \geq 0\}$; otherwise set $U(\theta) = H(\theta) \cap (-\infty, 0]$;

If $U(\theta) = \{0\}$ then set $z_{ott}=z^{(k)}$ and STOP. Otherwise set :

$$\theta_1 = \frac{-k_0^2\mu_0(2-k_0^2\beta) - \sqrt{2k_0^2(2-k_0^2\beta)(k_0^2\mu_0^2 - 2\beta(z_0-\theta_0)^2)}}{k_0^2\beta(2-k_0^2\beta)}$$

If $\theta_1 \in H(\theta)$ then set $z_{ott}= z(\theta_1)$ and STOP.

If $\inf U(\theta) = -\infty$ and $q^T\alpha = -k_0\sqrt{\frac{1}{2}\alpha^T Q\alpha}$ then $\inf\{g(z), z \in S'\}=$

$= q^T z^{(k)} + \dfrac{k_0 z^{(k)T} H\alpha}{\sqrt{\frac{1}{2}\alpha^T H\alpha}}$ on ray $\alpha$ and STOP;

If $\inf U(\theta) = -\infty$ and $q^T\alpha \neq -k_0\sqrt{\frac{1}{2}\alpha^T Q\alpha}$ then $\inf\{g(z), z \in S'\} = -\infty$ and STOP. Otherwise let $\theta^{(k+1)}$ be the end point of $U(\theta)$ different from zero, set:

$$z^{(k+1)} = z(\theta^{(k+1)})$$
$$\mu^{(k+1)} = \mu(\theta^{(k+1)})$$
$$\mu_0^{(k+1)} = \mu_0(\theta^{(k+1)})$$

If $z^{(k+1)} = 0$ then set $z_{ott}=0$ and STOP. Otherwise update the constraints binding at $z^{(k+1)}$ in system (2.4), deleting the constraint i such that $\mu_i^{(k+1)}= 0$ and adding the constraint j such that $a_j z^{(k)} > b_j$ and $a_j z^{(k+1)} = b_j$ ( $a_j$ denotes the j-th row of A), set $k = k+1$ and go to Step 3.

**Step 5**: If there are two different bases $B_1$ and $B_2$ such that $\mu_{0B_1}^{(k)} > 2(\theta_0 - z_0)/k_0^2$ $\mu_{0B_2}^{(k)} < 2(\theta_0 - z_0)/k_0^2$ or $\mu_{0B_1}^{(k)} < 2(\theta_0 - z_0)/k_0^2$, $\mu_{0B_2}^{(k)} > 2(\theta_0 - z_0)/k_0^2$ then set $z_{ott}=z^{(k)}$ and STOP.

If we have $U(\theta) = \{0\}$ for any feasible basis B, then set $z_{ott}= z^{(k)}$ and STOP. Otherwise, go to Step 3.

## 3. The algorithm

As we have just observed in section 2, procedure SO cannot be directly applied to problem (2.2), since the quadratic form of this problem is semi-definite positive, in fact:

1) Step 0 cannot be utilized, since H is singular,

2) at Step 3, a linear system, which contains H, must be solved. If H is definite

positive, certainly, the system matrix is non singular, but this is not, in general, true when H is semi-definite positive as in this case.

In this section, however, we will show how it is possible to utilize however this procedure to solve problem (1.1) in the cases $d_0 \neq 0$ or $d_0 = 0$ and $c_0 \neq 0$.

Let us observe that the matrix of system (2.4), in relation to problem (2.2) is:

$$MS = \begin{bmatrix} \begin{matrix} Q & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} -M^T & -d & -c \\ b_M^T & -d_0 & -c_0 \end{matrix} \\ \begin{matrix} M & -b_M \end{matrix} & \begin{matrix} 0 & 0 & 0 \end{matrix} \\ \begin{matrix} d^T & d_0 \\ c^T & c_0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \end{bmatrix}$$

By rearranging columns and rows, the matrix can be put in the following form:

$$MS = \begin{bmatrix} Q & -M^T & -d & -c & 0 \\ M & 0 & 0 & 0 & -b_M \\ d^T & 0 & 0 & 0 & d_0 \\ c^T & 0 & 0 & 0 & c_0 \\ 0 & b_M^T & -d_0 & -c_0 & 0 \end{bmatrix}$$

Since the sub-matrix $[-M^T,-d,-c]$ has full rank (i.e. the columns are linearly independent) it is easy to show that, in cases $d_0 \neq 0$ or $d_0 = 0$ and $c_0 \neq 0$, MS is not singular and system (2.4) has solution.

Therefore, in cases $d_0 \neq 0$ or $d_0 = 0$ and $c_0 \neq 0$, by utilizing procedure SO, we can solve problem (2.2), starting from Step 1 instead of Step 0. Taking into account remark 2.2, the results of this procedure can be interpreted to solve problem (1.1).

In the particular case $d_0 \neq 0$, problem (2.2) can be transformed into an equivalent problem where the quadratic form is definite positive. Procedure SO can be applied to this new problem without any modification; in fact, by setting:

$$t = \frac{1}{d_0}(1 - d^T y)$$

problem (2.2) becomes:

(3.1) $$\frac{c_0}{d_0} + \inf_{y \in T} f(y) = (c^T - \frac{c_0}{d_0} d^T)\, y + \sqrt{\frac{1}{2} y^T Q y}$$

$$\text{where } T = \left\{ y \in IR^n : \left[A + \frac{b\, d^T}{d_0}\right] y \geq \frac{b}{d_0},\ -\frac{d^T}{d_0} y \geq -\frac{1}{d_0} \right\}.$$

Problem (3.1) is equivalent to problem (2.3) by setting:

(3.2) $$z=y,\ p = n,\ k_0 = 1,\ q^T = c^T - \frac{c_0}{d_0} d^T,\quad H = Q,\ A' = \begin{bmatrix} A + \frac{b\, d^T}{d_0} \\ -\frac{d^T}{d_0} \end{bmatrix} \text{ and } b' = \begin{bmatrix} \frac{b}{d_0} \\ -\frac{1}{d_0} \end{bmatrix}$$

Taking into account previous theorems, we can interpret the result of the procedure applied to problem (3.1), for solving problem (1.1), in this way:

1) if problem (3.1) has optimal solution $z_{ott}$, then two cases can occur:
   a) $d^T z_{ott} \neq 1$; the corresponding optimal solution of problem (1.1) is given by: $x_{ott} = \dfrac{z_{ott}\, d_0}{1 - d^T z_{ott}}$
   b) $d^T z_{ott} = 1$ (therefore $t = 1/(d^T x + d0) = 0$); problem (1.1) is lower bounded on S on ray $z_{ott}$.
2) if problem (3.1) is not lower bounded on T, it means that problem (1.1) is not lower bounded on S,
3) if problem (3.1) is lower bounded on T, but inf is not attained, it means that problem (1.1) is lower bounded on S.

Let us remember that problem (1.1) is non-differentiable in x=0. The following remark give us a condition to remove this problem.

**Remark 2:** If point $x = 0$ belongs to the feasible region S (i.e. $b \leq 0$), from the condition $d^T x + d_0 > 0\ \forall\ x \in S$, it follows that $d_0$ is positive. As a consequence, problem (1.1) can be transformed into problem (3.1) and procedure SO can be applied without any modification.

## 4. An example

Let us consider the following problem:

(4.1) $$\inf_{x\in S} f(x) = \frac{x_1 + x_2 + 3 + \sqrt{2x_1^2 + x_2^2}}{\frac{1}{2}x_1 + \frac{1}{2}x_2 - 1}$$

where $S = \{ (x_1, x_2) \in IR^2 : x_1 - x_2 \geq 0,\ x_1 + x_2 \geq 3,\ x_2 \geq 0\}$

The feasible region S is depicted in fig.1.

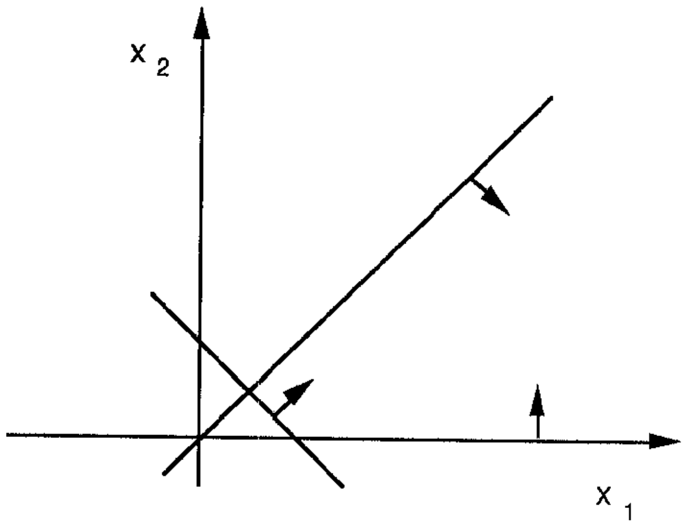


Fig.1

By applying the Charnes Cooper transformations, the problem becomes:

$$\inf_{(y_1, y_2, t) \in R} \varphi(y,t) = y_1 + y_2 + 3t + \sqrt{2y_1^2 + y_2^2}$$

where $R = \{ (y_1, y_2, t) \in IR^3 : y_1 - y_2 \geq 0,\ y_1 + y_2 - 3t \geq 3,$
$y_2 \geq 0,\ \frac{1}{2}y_1 + \frac{1}{2}y_2 - t = 1,\ t > 0\}$

Taking into account that $d_0 = -1 \neq 0$, by setting $t = \frac{1}{2}y_1 + \frac{1}{2}y_2 - 1$, the problem becomes:

(4.2) $$3 + \inf_{y\in S'} \varphi(y) = \frac{5}{2}y_1 + \frac{5}{2}y_2 + \sqrt{2y_1^2 + y_2^2}$$

where $S' = \{ (y_1, y_2) \in IR^2 : y_1 - y_2 \geq 0,\ -\frac{1}{2}y_1 - \frac{1}{2}y_2 \geq -3,\ y_2 \geq 0,\ \frac{1}{2}y_1 + \frac{1}{2}y_2 \geq 1\}$

The feasible region S is depicted in fig.2.

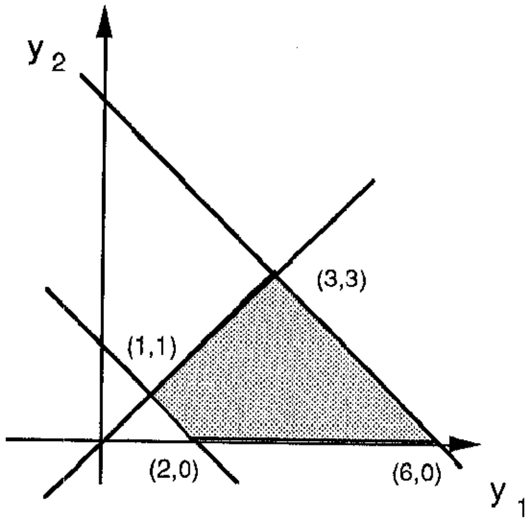


Fig.2

It is possible to apply the procedure SO to this problem, by setting:

$$z=y\,,\quad p=2\,,\ k_0=1,\quad q=\begin{bmatrix}\frac{5}{2}\\ \frac{5}{2}\end{bmatrix},\quad H=\begin{bmatrix}4 & 0\\ 0 & 2\end{bmatrix},\quad A'=\begin{bmatrix}1 & -1\\ -\frac{1}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\\ 0 & 1\end{bmatrix}\ \text{and}\ b'=\begin{bmatrix}0\\ -3\\ 1\\ 0\end{bmatrix}.$$

By applying the algorithm we obtain the following steps:

**Step 0**, we determine $H^{-1}=\begin{bmatrix}\frac{1}{4} & 0\\ 0 & \frac{1}{2}\end{bmatrix}$.

Since $(2q^TH^{-1}q)^{1/2} = 75/8 > k_0 = 1$ and $S'(\theta) = \{\theta\colon A'z(\theta) \geq b', \theta \leq 0\} = \emptyset$ where $z(\theta) = \frac{\theta}{q^T H^{-1} q} H^{-1}q = \theta\,[\,2/15,\ 4/15]^{T}$.

**Step 1**, set $k=1$, the optimal solution of the linear problem min $q^Tz$, $z \in S'$ is $z^{(1)} = (1,1) \neq 0$ and $S' \cap \{\, z : q^T z = q^T z^{(k)}\,\} = \{\,(1,1)\,\}$ then set $\theta^0 = q^T z^{(k)} = 5$.

**Step 2**, we solve the following system:

$$\begin{aligned}
4\,z_1 \quad - \mu_1 - \frac{5}{2}\mu_0 &= 0\\
2\,z_2 + \mu_1 - \frac{5}{2}\mu_0 &= 0\\
z_1 - z_2 &= 0\\
\frac{5}{2} z_1 + \frac{5}{2} z_2 &= 5 + \theta
\end{aligned}$$

from which the solution:

$$z_1(\theta) = 1 + \frac{1}{5}\theta$$

$$z_2(\theta) = 1 + \frac{1}{5}\theta$$

$$\mu_1^{(1)}(\theta) = 1 + \frac{1}{5}\theta$$

$$\mu_0^{(1)}(\theta) = \frac{6}{5} + \frac{1}{5}\theta$$

**Step 4**, note that $U(\theta) = \{0\}$ then $z_{ott} = z^{(1)}$ is the optimal solution for problem (5.2) and the procedure stops.

Since $d^T z_{ott} = 1$ then the inf f(x) is not attained and $z_{ott} = (1,1)$ is an optimal extreme ray for which $L = 2 + \sqrt{3}$.

## REFERENCES


[1] Cambini, A. and C.Sodini, Un algoritmo per un problema di programmazione frazionaria non lineare derivante da un problema di selezione del portafoglio, Atti del V Convegno AMASES, Perugia ott.81;

[2] Patkar, V. and I.M.Stancu-Minasian, Approaches for solving a class of non-differentiable non linear fractional Programming Problems, Nat. Acad. Sci. Letters, Vol.4, n.12,pp.477–479,1981;

[3] Stancu-Minasian, I.M., Studii Si Cercet. Mat., n.28, 1976;

[4] Stancu-Minasian, I.M., Programarea Stacastica cu mai multe functii Objectiv, Editura Acad. R.S.R., Bucuresti1980;

[5] Sodini, C., Minimizing the sum of a linear function and the square root of a convex quadratic form, X Symposium of Op. Res. Methods of Op. Res., n.53,pp.171–182, 1985.