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A Finite Algorithm for Generalized Linear Multiplicative Programming

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Abstract. The nonconvex problem of minimizing the sum of a linear function and the product of two linear functions over a convex polyhedron is considered. A finite algorithm is proposed which either finds a global optimum or shows that the objective function is unbounded from below on the feasible region. This is done by means of a sequence of primal and/or dual simplex iterations.

Keywords. Parametric simplex algorithm, linear multiplicative programming.

1. Introduction

In this paper we consider the problem

$$\begin{aligned} &\min \ f(x) = c^T x + (d^T x + d_0)(q^T x + q_0) \\ &\text{subject to} \quad x \in X \end{aligned} \tag{1.1}$$

where $X = \{x \in R^n : Ax = b \ , \ x \geq 0\}$, A is a mxn matrix, c, d, $q \in R^n$, $b \in R^m$, c_0, d_0, $q_0 \in R$. Problem (1.1) has been studied by Konno and T. Kuno [4] who have proposed an algorithm that solves the problem when X is bounded. For earlier methods see the references in [4]. The algorithm of Konno and Kuno is based on the idea of embedding the original n-dimensional problem into an (n+1)-dimensional master problem which is solved parametrically. More recently Konno, Yajima and Matsui [5] have shown that for a compact feasible region X problem (1.1), can be solved by means of a parametric linear programming problem. For a survey of these and related results see [6].

In the following we establish some theoretical properties of problem (1.1) which allow us to propose a finite algorithm that solves the problem also when X is unbounded. The algorithm is similar to the one proposed by Konno, Yajima and Matsui, but it uses different optimality conditions. As we will show in Section 5, even when X is compact, the two algorithms are not equivalent, i.e. starting from the same feasible vertex, the optimal solution is obtained through a different sequence of points and a different number of iterations.

2. Theoretical properties

Problem (1.1) is not a quasiconvex program in general [1]. This implies that a local minimum is not necessarily a global minimum.

The following theorem holds:

Theorem 2.1

If problem (1.1) has an optimal solution, then at least one optimal solution belongs to an edge of X.

<u>Proof</u>. If x' is an optimal solution of problem (1.1), then it is also an optimal solution of the linear program

$$\min \ f(x) = c^Tx + (d^Tx + d_0)\ (q^Tx + q_0) \qquad (2.1)$$
$$\text{subject to} \quad x \in X'$$

where $X'=X\cap\{x\in R^n : d^Tx = d^Tx'\}$. Clearly, at least one vertex of X' is an optimal solution of (2.1). Since a vertex of X' lies on an edge of X, the theorem is proved.

3. Optimality conditions

If we add the constraint $d^Tx + d_0 = \xi, \xi\in R$, to problem (1.1), the following linear program is obtained:

$$P(\xi) \qquad z(\xi) = \min \ c^Tx + \xi\ (q^Tx + q_0)$$
$$\text{subject to} \quad x\in X(\xi)$$

where $X(\xi)=X\cap\{x\in R^n : d^Tx + d_0 = \xi\}$. The parameter ξ is said to be a <u>feasible level</u> if the set $X(\xi)$ is nonempty. An optimal solution of problem $P(\xi)$ is called an <u>optimal level solution</u>, [1], [2], [3].

Clearly, problem (1.1) is equivalent to problem $P(\xi)$, when ξ is the level corresponding to an optimal solution of problem (1.1).

In this section we give some optimality conditions which allow us to detect if an optimal level solution is a local minimum of problem (1.1).

Let x' be an optimal basic solution of problem $P(\xi')$ and let A_B be the corresponding basis. Since x' is an optimal basic solution, we have

$(c'_N{}^T + \xi'\ q'_N{}^T) \geq 0$, where

$c'_N{}^T = c_N{}^T - c_B{}^TA_B{}^{-1}A_N$, $q'_N{}^T = q_N{}^T - q_B{}^TA_B{}^{-1}A_N$.

Let us consider the parametric program:

$$P(\xi'+\theta) \qquad z(\xi'+\theta) = \min \ c^Tx + (\xi'+\theta)\ (q^Tx + q_0)$$
$$\text{subject to} \quad x\in X(\xi'+\theta)$$

where $X(\xi'+\theta)=X\cap\{x\in R^n : d^Tx + d_0 = \xi'+\theta\}$.

Set

- $x'(\theta)^T = (x'_B(\theta), 0)^T$, $x'_B(\theta) = x'_B + \theta\,\alpha$, $\alpha = A_B^{-1}\, e^{m+1}$, $e^{m+1^T}=(0,0, ...,0,1)$;
- $f_N^T(\theta) = (c'_N{}^T + \xi'\, q'_N{}^T) + \theta\, q'_N{}^T$;
- $F = \{\ \theta\in R : x'_B + \theta\,\alpha \geq 0\ \}$, $O = \{\ \theta\in R : f_N^T(\theta) \geq 0\ \}$;
- $FO = F\cap O$.

Clearly, $x'(\theta)$ is an optimal level solution for $\theta\in FO$. Set $z'=z(\xi')$, $z(\theta)=z(\xi'+\theta)$. The following lemma gives an explicit form for the function $z(\theta)$, $\theta\in FO$.

Lemma 3.1

If $FO\neq\{0\}$, then $z(\theta)=\beta\theta^2+\gamma\theta+z'$ where $\beta=q_B{}^T\alpha$, $\gamma=c_B{}^T\alpha+\xi'q_B{}^T\alpha+q_B{}^Tx'_B+q_0$.

Proof. We have $z(\theta)=c_B{}^T(x'_B+\theta\alpha)+(\xi'+\theta)(q_B{}^T(x'_B+\theta\alpha)+q_0)$

$=c_B{}^Tx'_B+\theta c_B{}^T\alpha+\xi'q_B{}^Tx'_B+\theta\xi'q_B{}^T\alpha+\xi'q_0+\theta q_B{}^Tx'_B+\theta^2q_B{}^T\alpha+\theta q_0$

$=q_B{}^T\alpha\theta^2+(c_B{}^T\alpha+\xi'q_B{}^T\alpha+q_B{}^Tx'_B+q_0)\theta+z'$.

Now, the following lemma can be derived.

Lemma 3.2

If $\gamma>0$ ($\gamma<0$), then $z(\theta)$ is increasing (decreasing) at $\theta=0$.

Proof. We have $\frac{dz}{d\theta}(\theta)=2\beta\theta+\gamma$. Hence, $\frac{dz}{d\theta}(0)=\gamma$.

Set

$U = FO\cap[0,+\infty)$, if $\gamma<0$;

$U = FO\cap(-\infty,0]$, if $\gamma>0$;

$\theta'=-(\gamma/2\beta)$, if $\beta>0$.

The following theorem holds:

Theorem 3.1

a) If $\gamma=0$ and $\beta\geq0$, then x' is a local minimum for problem (1.1).

b) If $\theta'\in U$, then $x'(\theta')$ is a local minimum for problem (1.1).

Proof. a) $\gamma=0$ and $\beta\geq0$ imply $\frac{dz}{d\theta}(0)=0$ and $\frac{d^2z}{d\theta^2}(0)=2\beta\geq0$; hence $x'(0)=x'$ is a local minimum. b) We have $\frac{dz}{d\theta}(\theta')=0$ and $\frac{d^2z}{d\theta^2}(\theta')=2\beta>0$; this implies that $x'(\theta')$ is a local minimum for problem (1.1).

Though x' is a vertex of $X(\xi')$, it is not a vertex of X, in general. If x' is a vertex of X, then x' is a degenerate basic solution for problem $P(\xi'+\theta)$. It follows that different bases with nonnegative reduced cost correspond to the solution x'. A basis A_B is said to be feasible if the corresponding reduced cost are nonnegative. To point out the dependence of $z(\theta)$, F, etc. on the basis A_B, we write $z_B(\theta)$, F_B, etc..

If x' is a vertex of X, then the following theorem holds:

Theorem 3.2

a) If there are two different bases A_{B_1} and A_{B_2} such that either $\gamma_{B_1} > 0$, $\sup FO_{B_1} > 0$, $\gamma_{B_2} < 0$, $\inf FO_{B_2} < 0$ or $\gamma_{B_1} < 0$, $\inf FO_{B_1} < 0$, $\gamma_{B_2} > 0$, $\sup FO_{B_2} > 0$, then x' is a local minimum for problem (1.1).

b) If we have $U_B=\{0\}$ for any feasible basis A_B, then x' is a local minimum for problem (1.1).

Proof. a) In view of Lemma 3.2, condition $\gamma_{B_1}>0$, $\gamma_{B_2}<0$ ($\gamma_{B_1}<0$, $\gamma_{B_2}>0$) implies $z(\theta)\geq z'$ in a neighborhood of 0. Hence x' is a local minimum for problem (1.1). b) This follows directly from the definition of U_B.

4. A finite algorithm for problem (1.1)

Since problem (1.1) is nonconvex, in general, it is necessary to solve problem $P(\xi)$ for all feasible levels in order to find a global minimum, assuming one exists. In this section we will show that this can be done by means of a finite number of primal and/or dual simplex iterations, using the results of the previous section.

Let A_B be a basis corresponding to the optimal solution of problem $P(\xi')$ and suppose that x* is the incumbent global minimum for $\xi\leq\xi'$, i.e. x* is the best optimal level solution for $\xi\leq\xi'$. Clearly, UB=f(x*) is an upper bound for the value of $z(\xi)$ for $\xi>\xi'$.

Let

- $x'_B=A_B^{-1}(b^T, \xi'-d_0)^T$,
- $c'^T_N= c_N^T-c_B^T A_B^{-1}A_N$,
- $q'^T_N=q_N^T-q_B^T A_B^{-1}A_N$,
- $\alpha=A_B^{-1}e^{m+1}$,
- $\beta=q_B^T\alpha$,
- $\gamma=c_B^T\alpha+\xi' q_B^T\alpha+q_B^T x'_B+ q_0$,
- $z'=c_B^T x'_B+ \xi'(q_B^T x'_B+q_0)$,
- $\theta'=-(\gamma/2\beta)$ if $\beta>0$,
- $\xi_{max} = \sup \{d^T x + d_0 , x\in X\}$ (Of course ξ_{max} may be equal to $+\infty$).

Let us consider the parametric problem $P(\xi'+\theta)$ for $\theta\geq 0$ and determine the sets F, O, FO as well as sup F, sup O. If sup F and sup O are finite, let

$\sup F=-x'_{B_r}/\alpha_r = \min \{-x'_{B_i}/\alpha_i, \alpha_i< 0\}$ and

$\sup O = -(c'_{N_s}+q'_{N_s})/q'_{N_s} = \min \{-(c'_{N_j}+q'_{N_j})/q'_{N_j}, q'_{N_j}<0\}$.

For each $\theta \in O$, $z(\theta)$ is a lower bound for $P(\xi'+\theta)$, since $x'(\theta)$ is a dual basic solution (primal if $\theta \in F$). The following four cases can occur:

C1) $\beta>0, \gamma<0$; three subcases need to be considered:

C1a) $\theta' \in F$, $\sup O = +\infty$, problem (1.1) is solved; in fact, if $UB<z(\theta')$, then x^* is a global minimum; otherwise $x'(\theta')$ is a global minimum;

C1b) $\theta' \in F$, $\sup O = \theta'' > \theta'$; if $UB>z(\theta')$, then $x^*=x'(\theta')$ and $UB=z(\theta')$; in any case, set $\xi'=\xi'+\theta''$, $x'= x'(\theta'')$, and if $\xi' \geq \xi_{max}$, then problem (1.1) is solved and x^* is a global minimum; otherwise:
- if x' is feasible, do a primal simplex iteration with x_{N_s} as entering variable;
- if x' is infeasible, apply the dual simplex algorithm with x_{B_r} as leaving variable.

C1c) $\sup FO=\theta''<\theta'$; if $UB>z(\theta'')$, then $x^*=x'(\theta'')$ and $UB=z(\theta'')$; in any case, set $\xi'=\xi'+\theta''$, $x'= x'(\theta'')$, and if $\xi' \geq \xi_{max}$, then problem (1.1) is solved and x^* is a global minimum; otherwise:
- if $\theta''= \sup O$, do a primal simplex iteration with x_{N_s} as entering variable;
- if $\theta''= \sup F$, apply the dual simplex algorithm with x_{B_r} as leaving variable.

C2) $\beta \geq 0, \gamma \geq 0$; two subcases need to be considered:

C2a) $\sup O = +\infty$, problem (1.1) is solved; x^* is a global minimum;

C2b) $\sup O = \theta'' < +\infty$; set $\xi'=\xi'+\theta''$, $x'= x'(\theta'')$, and if $\xi' \geq \xi_{max}$, then problem (1.1) is solved and x^* is a global minimum; otherwise:
- if x' is feasible, do a primal simplex iteration with x_{N_s} as entering variable;
- if x' is infeasible, apply the dual simplex algorithm with x_{B_r} as leaving variable.

C3) $\beta \leq 0, \gamma<0$; two subcases need to be considered:

C3a) $\sup FO = +\infty$; problem (1.1) is unbounded, i.e. $\inf f(x)=-\infty$;

C3b) $\sup FO = \theta'' < +\infty$, set $\xi'=\xi'+\theta''$, $x'= x'(\theta'')$, and if $\xi' \geq \xi_{max}$, then problem (1.1) is solved and x^* is a global minimum; otherwise:
- if $\theta''= \sup O$, do a primal simplex iteration with x_{N_s} as entering variable;
- if $\theta''= \sup F$, apply the dual simplex algorithm with x_{B_r} as leaving variable.

C4) $\beta<0, \gamma \geq 0$; let $\sup F = \theta^1$, $\sup O = \theta^2$ and θ^* be the positive root of the equation $z(\theta)=UB$; four subcases need to be considered:

C4a) $\theta^1 < \theta^* \leq \theta^2$; set $\xi'=\xi'+\theta^*$, $x'= x'(\theta^*)$, and if $\xi' \geq \xi_{max}$, then problem (1.1) is solved and x^* is a global minimum; otherwise apply the dual simplex algorithm with x_{B_r} as leaving variable.

C4b) $\theta^*<\theta^1<\theta^2$; set $\xi'=\xi'+\theta^1$, $x'= x'(\theta^1)$, $x^*= x'(\theta^1)$, $UB= z(\theta^1)$, and if $\xi'\geq\xi_{max}$, then problem (1.1) is solved and x^* is a global minimum; otherwise do a dual simplex iteration with x_{B_r} as leaving variable;

C4c) $\theta^2<\theta^1$; set $\xi'=\xi'+\theta^2$, $x'= x'(\theta^2)$, do a primal simplex iteration with x_{N_s} as entering variable, and if $UB>z(\theta^2)$, then set $x^*=x'(\theta^2)$, $UB= z(\theta^2)$;

C4d) sup FO $=+\infty$; problem (1.1) is unbounded.

Starting from the solution x' and the level ξ', we arrive at one of the following situations:

i) x^* is an optimal solution;
ii) the problem is unbounded;
iii) a new optimal level solution corresponding to a level greater than ξ' has been found together with the best incumbent solution. The new solution corresponds to a new vertex or to a new edge of the feasible region.

In order to propose a finite algorithm to solve problem (1.1), it remains to consider an appropriate initialization.

Let us solve one of the following linear programs:

(P_1) min $d^Tx + d_0$, $x\in X$;
(P_2) min $q^Tx + q_0$, $x\in X$;
(P_3) max $d^Tx + d_0$, $x\in X$;
(P_4) max $q^Tx + q_0$, $x\in X$.

If x' is the unique optimal solution of (P_1) ((P_2)) and $\xi'=d^Tx+d_0$ ($\xi'=q^Tx+q_0$) is the corresponding level then $X(\xi')=\{x'\}$ and clearly x' is an optimal level solution; in this case $x^*=x'$ and only increasing values of ξ need to be considered. Analogously, if x' is the unique optimal solution of (P_3) ((P_4)) and $\xi'=d^Tx+d_0$ ($\xi'=q^Tx+q_0$) is the corresponding level, then $x^*=x'$ and only decreasing values of ξ need to be considered. In any of these cases, if x' is not a unique optimal solution, then $X(\xi')\neq\{x'\}$ and x' is not an optimal level solution in general; in this case we can start from the optimal solution x' of $P(\xi')$ setting $x^*=x'$. Otherwise we can start from the optimal level solution x' corresponding to a feasible level ξ'; also in this case $x^*=x'$; but it is necessary to consider either increasing or decreasing values of the parameter.

We give now a formal description of the algorithm, called SS, under the assumption that problem (P_1) has a finite optimal solution. The modifications necessary for an unbounded problem (P_1) are straightforward.

The algorithm SS

Step 0 (Initialization) Find an optimal level solution x' and the corresponding level

ξ' by solving problem (P_1). Set $x^*=x'$, $UB=f(x^*)$; go to Step 1.

Step 1 According to the four cases C1-C4 discussed above, if the problem is solved or unboundedness is found, then **STOP**; otherwise go to **Step 2.**

Step 2 Update x', x^*, ξ' and go to **Step 1.**

Clearly, the proposed algorithm is finite, since at each step a new vertex or a new edge is reached.

5. A comparison with the algorithm of Konno, Yajima and Matsui

The algorithm of Konno, Yajima and Matsui [5], called KYM, solves problem (1.1) in the case of a bounded feasible region. In this section we give a formal description of algorithm KYM in order to show how it differs from the algorithm SS. Let $\xi_{min} = \inf \{d^Tx + d_0 , x\in X\}$, $\xi_{max} = \sup \{d^Tx + d_0 , x\in X\}$; since X is bounded, it follows that ξ_{min} and ξ_{max} are finite.

The algorithm KYM

Step 0 (Initialization) Find an optimal level solution x' and the corresponding level ξ' by solving problem (P_1).Set $x^*=x'$, $UB=f(x^*)$; go to Step 1.

Step 1 If $\xi'=\xi_{max}$, then **STOP**; x^* is an optimal solution. Otherwise, let $\theta'=\sup FO$ and $z(\theta^*) = \min \{z(\theta), 0\leq\theta\leq\theta'\}$. Set $\xi'=\xi'+\theta'$, $x'=x'(\theta')$ and if $z(\theta^*)<UB$, $x^* =x'(\theta^*)$; go to **Step 2.**

Step 2 If $\theta'=\sup F$, then do a dual simplex iteration with x_{B_r} as leaving variable and go to **Step 1**; otherwise, do a primal simplex iteration with x_{N_s} as entering variable and go to **Step 1.**

Clearly, algorithm KYM finds the optimal solution by solving problem $P(\xi)$ for all the feasible levels from the minimum level ξ_{min} to the maximum level ξ_{max} and stops only when condition $\xi=\xi_{max}$ is verified.

The differences between algorithm SS and algorithm KYM are the following:

1) algorithm KYM works only for problems with a bounded feasible region while algorithm SS can also be used for the unbounded case;
2) algorithm KYM terminates when the level ξ_{max} is reached; algorithm SS can terminate in three different ways:
   i) case C1a;
   ii) case C2a;
   iii) condition $\xi'\geq\xi_{max}$;
3) algorithm KYM generates a sequence of feasible points while algorithm SS can generate also infeasible points.

The following examples demonstrate the differences between the two algorithms.

Example 1:

$$\begin{array}{ll} \min f(x) = & x_1 + (x_1 - x_2 + 10)(x_1 + x_2 - 6) \\ \text{subject to} & -x_1 + 2x_2 \le 18, \\ & 3x_1 + 4x_2 \ge 12, \\ & x_1 + x_2 \le 13, \\ & x_1 - 4x_2 \le 8, \\ & x_1 \ge 0,\ x_2 \ge 0. \end{array}$$

First, we apply algorithm SS. Starting from the optimal solution $x'=(0,9)$ of the linear program
$\min\ \{x_1 - x_2 + 10:\ -x_1 + 2x_2 \le 18,\ 3x_1 + 4x_2 \ge 12,\ x_1 \ge 0,\ x_2 \ge 0\}$,
we obtain the following steps:

- $x'=(0,9)$, $x'(\theta)=(0,9-\theta)$, $\sup O=+\infty$, $\sup F=6$, $z(\theta)=-\theta^2+2\theta+3$, $UB=3$, $\xi'=1$, $x^*=(0,9)$; hence $\theta^*=2$, $\sup F=6>\theta^*=2$, $\xi'=7$, $UB=z(6)=-21$, $x^*=x'(6)=(0,3)$ and a dual iteration is done;
- $x'=(0,3)$, $x'(\theta)=(4/7\theta,3-3/7\theta)$, $\sup O=+\infty$, $\sup F=7$, $z(\theta)=1/7\theta^2-10/7\theta-21$, $\theta'=5\in F$, $z(5)=-172/7 < UB$; hence $UB=-172/7$, $x^*=x'(5)=(20/7,6/7)$;
- $x^*=(20/7,6/7)$ is a global minimum while $(0,9)$ is a local minimum.

Now we apply algorithm KYM. Starting from the optimal solution $x'=(0,9)$ of the linear program
$\min\ \{x_1 - x_2 + 10:\ -x_1 + 2x_2 \le 18,\ 3x_1 + 4x_2 \ge 12,\ x_1 \ge 0,\ x_2 \ge 0\}$,
we obtain the following steps:

- $x'=(0,9)$, $x'(\theta)=(0,9-\theta)$, $UB=3$, $\xi'=1$, $x^*=(0,9)$, $\sup O=+\infty$, $\sup F=6$, $\theta'=\sup FO=6$, $z(6)=\min\{z(\theta)=-\theta^2+2\theta+3,\ 0\le\theta\le6\}=-21< UB$; hence $\xi'=7$, $x'=x'(6)=(0,3)$, $x^*=x'(6)=(0,3)$, $UB=z(6)=-21$ and a dual iteration is done;
- $x'=(0,3)$, $x'(\theta)=(4/7\theta,3-3/7\theta)$, $\xi'=7$, $\sup O=+\infty$, $\sup F=7$, $\theta'=\sup FO=7$, $z(5) = \min\{z(\theta) = 1/7\theta^2-10/7\theta-21,\ 0\le\theta\le7\} = -172/7 < UB$; hence $\xi'=14$, $x'=x'(7)=(4,0)$, $x^*=x'(5) =(20/7,6/7)$, $UB=-172/7$ and a dual iteration is done;
- $x'=(4,0)$, $x'(\theta)=(4+\theta,0)$, $\xi'=14$, $\sup O=+\infty$, $\sup F=4$, $\theta'=\sup FO=4$, $z(0)=\min\{z(\theta) = \theta^2+13\theta-24,\ 0\le\theta\le4\}=-24 > UB$; hence $\xi'=18$, $x'=x'(4)=(8,0)$ and a dual iteration is done;
- $x'=(8,0)$, $x'(\theta)=(8+4/3\theta,1/3\theta)$, $\xi'=18$, $\sup O=+\infty$, $\sup F=3$, $\theta'=\sup FO=3$, $z(0)=\min\{z(\theta)=5/3\theta^2+100/3\theta+44,\ 0\le\theta\le3\}=44>UB$; hence $\xi'=21$, $x'=x'(3)=(12,1)$ and a dual iteration is done;
- $x^*=(20/7,6/7)$ is a global minimum.

The paths followed by the two algorithms are depicted in fig. 1.

Algorithm SS finds the optimal solution generating the sequence of points (0,9), (0,3) with one iteration; algorithm KYM finds the optimal solution generating the sequence (0,9), (0,3), (4,0), (8,0), (12,1) with four iterations.

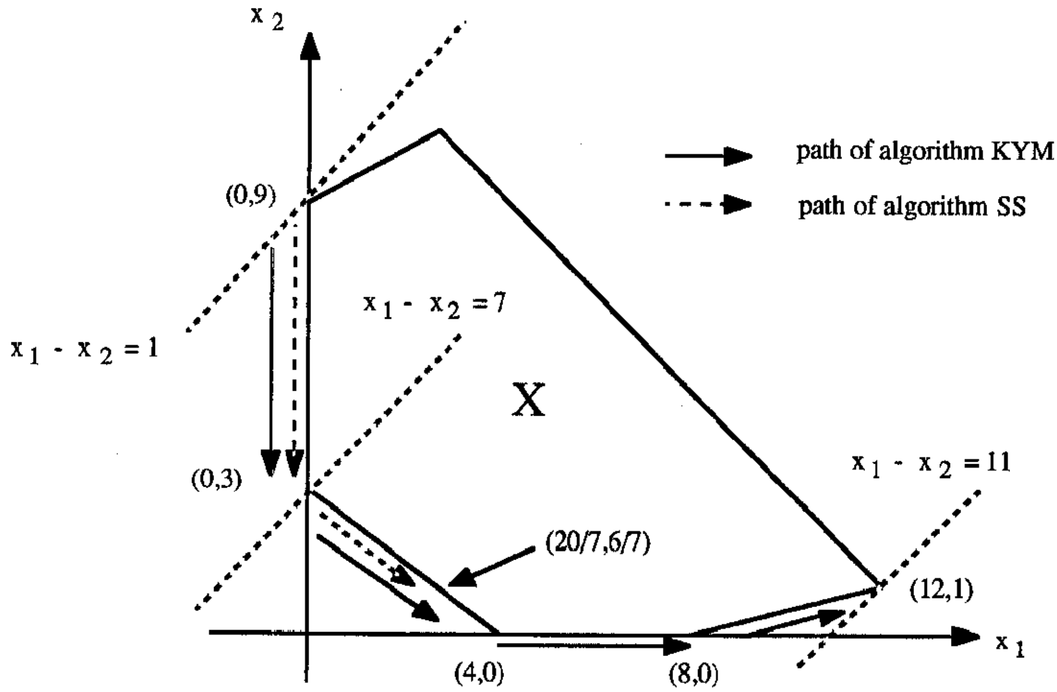

fig. 1

Example 2:

$$\min f(x) = x_1 + (2x_1 - 3x_2 + 13)(x_1 + x_2 - 1)$$

subject to

$$-x_1 + 2x_2 \leq 8,$$
$$x_2 \geq 3,$$
$$x_1 + 2x_2 \leq 12,$$
$$-x_1 + 2x_2 \geq 5,$$
$$x_1 \geq 0,\ x_2 \geq 0.$$

First, we apply algorithm SS. Starting from the optimal solution $x'=(0,4)$ of the linear program
$\min \{2x_1 - 3x_2+13: -x_1 + 2x_2 \leq 8,\ x_2 \geq 3, x_1 + 2x_2 \leq 12, -x_1 + 2x_2 \geq 5, x_1 \geq 0, x_2 \geq 0\}$,
we obtain the following steps:

- $x'=(0,4)$, $x'(\theta)=(0,4-1/3\theta)$, $\sup O=+\infty$, $\sup F=6$, $z(\theta)=-1/3\theta^2+8/3\theta+3$, UB=3, $\xi'=1$, $x^*=(0,4)$; hence $\theta^*=8$, $\sup F < \theta^*=8$, $\xi'=9$, $x'=x'(8)=(0,4/3)$;
- $x'=(0,4/3)$ is an infeasible point, by two dual iterations the new infeasible point (7,6) is obtained and condition $\xi' \geq \xi_{max}$ is verified;
- $x^*=(0,4)$ is a global minimum.

Now we apply algorithm KYM. Starting from the optimal solution x'=(0,4) of the linear program

$\min \{2x_1 - 3x_2 + 13: -x_1 + 2x_2 \leq 8,\ x_2 \geq 3, x_1 + 2x_2 \leq 12, -x_1 + 2x_2 \geq 5, x_1 \geq 0, x_2 \geq 0\}$,

we obtain the following steps:

- x'=(0,4) , x'(θ)=(0,4-1/3θ), UB=3, ξ'=1, x*=(0,4), sup O=+∞, sup F=3, θ'=sup FO=3, $z(0)=\min\{z(\theta)=-1/3\theta^2+8/3\theta+3,\ 0\leq\theta\leq3\}=3$; hence ξ'=4, x'=x'(3)=(0,3) and a dual iteration is done;
- x'=(0,3), x'(θ)=(1/2θ,3), ξ'=4, sup O=+∞, sup F=2, θ'=sup FO=2, $z(0) = \min\{z(\theta) = 1/2\theta^2+9/2\theta+8,\ 0\leq\theta\leq2\} = 8 > UB$; hence ξ'=6, x'=x'(2)=(1,3) and a dual iteration is done;
- x'=(1,3), x'(θ)=(1+2θ,3+θ), ξ'=6, sup O=+∞, sup F=5/4, θ'=sup FO=5/4, $z(0) = \min\{z(\theta) = 3\theta^2+23\theta+19,\ 0\leq\theta\leq4\} = 19 > UB$; hence ξ'=29/4, x'=x'(5/4)=(7/2,17/4) and a dual iteration is done;
- x*=(0,4) is a global minimum.

The paths followed by the two algorithms are depicted in fig. 2.

Algorithm SS finds the optimal solution generating the sequence of points (0,4), (0,4/3), (7,6) whith two iterations; algorithm KYM finds the optimal solution generating the sequence (0,4), (0,3), (1,3), (7/2,17/4) with three iterations.

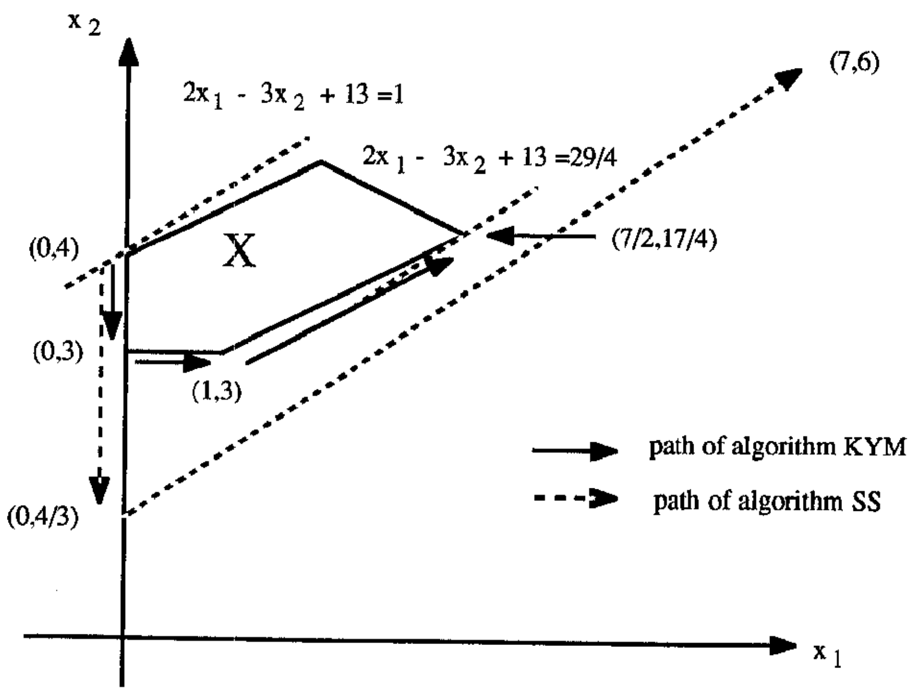

fig. 2

Example 3:

$$\begin{aligned} \min f(x) = \ & 20x_1 + 2x_2 + (x_1 + x_2 + 2)(-2x_1 + x_2 + 3) \\ \text{subject to} \ & -x_1 + x_2 \le 0, \\ & -x_1 + 3x_2 \ge 2, \\ & x_1 \ge 0,\ x_2 \ge 0. \end{aligned}$$

Applying algorithm SS starting from the optimal solution $x'=(1,1)$ of the linear program $\min \{x_1 + x_2 + 2:\ -x_1 + x_2 \le 0,\ -x_1 + 3x_2 \ge 2,\ x_1 \ge 0, x_2 \ge 0\}$, we obtain the following steps:

- $x'=(1,1)$, $x^*=(1,1)$, $x'(\theta)=(1+1/2\theta, 1+1/2\theta)$, sup $O=2$, sup $F=+\infty$, $UB=30$, $\xi'=4$, $z(\theta)=-1/2\theta^2+11\theta+30$, $\theta^*=22$; hence $\xi'=6$, $x'=x'(2)=(2,2)$ and a primal iteration is done;
- the point $x'=(5/2,3/2)$ is obtained with $x'(\theta)=(5/2+3/4\theta, 3/2+1/4\theta)$, sup $O=+\infty$, sup $F=+\infty$, sup $FO=+\infty$, $z(\theta)=-5/4\theta^2+15/2\theta+50$; hence the problem is unbounded and $x^*=(1,1)$ is a local minimum.

The path followed by the algorithm is depicted in fig. 3.

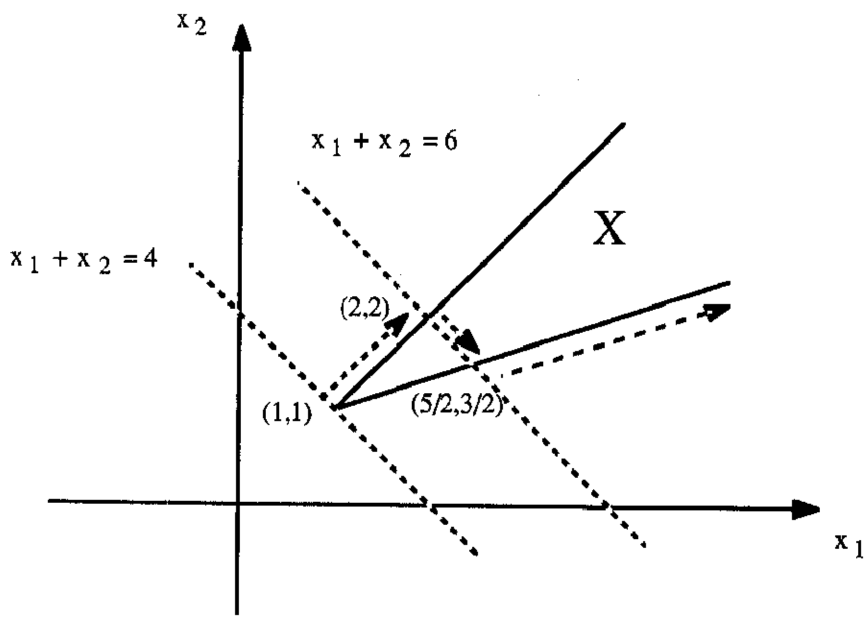

fig. 3

Algorithm KYM cannot be used in this case since the feasible region is unbounded.

6. Conclusion

For the problem of minimizing the sum of a linear function and the product of two linear functions over a convex polyhedron a finite algorithm is proposed that either finds a global minimum or detects the unboundedness of the objective function. A sequence of primal and/or dual simplex iterations is employed. The method differs from the one in

[5] in several ways.

In [5], also the problem of minimizing the sum of a linear function and the ratio of two linear functions is considered and an algorithm similar to the one in case of the product is suggested. In the same way, the algorithm in the present paper can be modified to solve the problem of minimizing the sum of a linear function and the ratio rather than product of two linear functions. In contrast to [5], we do not need to assume compactness of the feasible region. For related algorithms and applications of generalized linear fractional programming see [1], [3], [4], [6].

We mention that the problem of minimizing the sum of two ratios of linear functions can be reduced to a generalized linear fractional program with help of the variable transformation by Charnes-Cooper, as shown in [1], [5]. Hence the sum-of-two-ratios-problem can be solved by a method similar to the one proposed in the present paper as well.

References

[1] Cambini, A., Martein L. and S. Schaible (1990), On maximizing a sum of ratios, *Journal of Information and Optimization Sciences*, 10,65-79.

[2] Cambini, A., Martein L. and C. Sodini (1983), An algorithm for two particular non linear fractional programs, *Methods of Operations Research*, 45, 61-70.

[3] Martein, L. (1985), Massimo della somma tra una funzione lineare ed una funzione lineare fratta, *Rivista di matematica per le scienze economiche e sociali*, anno VIII, fasc. 1, 13-20.

[4] Konno, H. and T. Kuno (1990), Generalized linear multiplicative and fractional programming, *Annals of Operations Research*, 25, 147-162.

[5] Konno, H., Yajima Y. and T. Matsui (1991), Parametric simplex algorithms for solving a special class of nonconvex minimization problems, *Journal of Global Optimization*, 1, 65-81.

[6] Schaible, S. (1992), Some recent results in fractional programming, in Generalized concavity for economic applications, P. Mazzoleni (ed.), Proceeding of a Workshop held in Pisa, Italy on April 2, 1992, 7-14.