Report n.72

An Approach to Optimality Conditions in Vector and Scalar Optimization

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Pisa, 1993
This research is partially supported by the Italian Ministry of Education

An Approach to Optimality Conditions in Vector and Scalar Optimization

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The implication of concavity in economics have suggested in the scalar case several kinds of generalization starting from the pioneering work of Arrow-Enthoven (1961)
The aim of this paper is to point out the role played by generalized concavity and by the tangent cone to the feasible region at a point, in stating several necessary and/or sufficient optimality conditions for a vector and scalar optimization problem.
Furthermore, in deriving F.John optimality conditions, the role of separations theorems is analyzed in order to suggest suitable formulations of Kuhn-Tucker conditions and a way for studying regularity conditions.

1 Introduction

The implication of concavity in economics have suggested in the scalar case several kinds of generalization starting from the pioneering work of Arrow-Enthoven (1961) where, for the first time, the earlier concavity assumption on utility and production functions was relaxed to quasiconcavity.

Nevertheless the study of generalized concavity of a vector valued function is not yet sufficiently explored and some classes with related properties have been suggested mainly for the paretian case.

In this paper we introduce some classes of generalized concave functions with respect to any cone in order to obtain sufficient optimality conditions in a general form, which can be specified both in the paretian case and in the scalar case.

The followed approach points out the role played by the tangent cone to the feasible region at a point in deriving, for a differentiable and non differentiable multiobjective problem, necessary and/or sufficient optimality conditions which can be expressed by means of the directions belonging to such a tangent cone.

Furthermore, in deriving F.John optimality conditions, the role of separations theorems is analyzed in order to suggest suitable formulations of Kuhn-Tucker conditions and a way for studying regularity conditions.

2 Statement of the problem

Consider the following vector optimization problem:

$$P : U\text{-max } F(x)\ ,\ x \in S \subseteq X$$

where X be an open set of $\mathbf{R}^n$, $F : X \to \mathbf{R}^s$, and $U \subset \mathbf{R}^s$ is a non trivial cone with vertex at the origin $0 \in U$.

Set $U^0 = U \setminus \{0\}$.

A point $x_0 \in S$ is said to be a <u>(global) efficient point</u> for problem P with respect to the cone U if

$$F(x) \notin F(x_0) + U^0 \qquad , \ \forall\, x \in S \tag{2.1a}$$

If (2.1a) is verified in a suitable neighbourhood I of x_0, i.e.

[1] The paper has been discussed jointly by the authors. Cambini has developed sections 1,2,7; Martein has developed sections 3,4,5,6.

$$F(x) \notin F(x_0) + U^0, \quad \forall x \in I \cap S \tag{2.1b}$$

x_0 is said to be a local efficient point.
If there is no points x in (2.1b) such that $F(x)=F(x_0)$, i.e.

$$F(x) \notin F(x_0) + U, \ \forall x \in I \cap S, \ x \neq x_0 \tag{2.1c}$$

x_0 is said to be a strict local efficient point
Let us note that in the scalar case ($s=1$, $U=\mathbf{R}_+$) (2.1b), (2.1c) collapse to the ordinary definitions of a local maximum and a strict local maximum point, respectively, while when $U=\mathbf{R}^s_+$, problem P reduces to a vector Pareto problem.
In the next sections we will establish several optimality conditions involving generalized concavity and/or semidifferentiability and/or differentiability of the objective function.

3. Generalized concave functions

In finding conditions under which a local maximum point is also global, an important role is played by the concept of generalized concavity at a point introduced by Mangasarian (1969) for a scalar optimization problem.

Let us note that thare are different way in generalizing to the vector case the definitions of generalized concave functions given in the scalar case. For the aim of this paper we limit ourselves to define some classes of generalized concave vector valued functions which will allow us to state several optimality conditions.

With this aim, let us consider an open set X of the n-dimensional space $\mathbf{R}^n$, a function $F:X\to\mathbf{R}^s$ and a non trivial cone $U \subset \mathbf{R}^s$ with vertex at the origin $0 \in U$. Set $U^0 = U \setminus \{0\}$.
A set $S \subset X$ is said to be locally star shaped at $x_0 \in S$ if there exists a neighbourhood I of x_0 such that for every $x \in I \cap S$ we have:

$$[x, x_0] = \{tx+(1-t)x_0 : t \in [0,1]\} \subset S.$$

Definition 3.1
The function F is said to be U-concave at x_0 (with respect to the locally star shaped set S at x_0) if:

$$F(x_0+\lambda(x-x_0)) \in F(x_0)+\lambda(F(x)-F(x_0))+U \quad \forall \lambda \in (0, 1), \ \forall x \in S$$

Definition 3.2
The function F is said to be U-quasiconcave (U-q.cv.) at x_0 (with respect to the locally star shaped set S at x_0) if:

$$x \in S, \ F(x) \in F(x_0) + U \Rightarrow F(x_0+\lambda(x-x_0)) \in F(x_0) + U \quad \forall \lambda \in (0, 1)$$

Definition 3.3
The function F is said to be U-semistrictly quasiconcave (U-s.s.q.cv.) at x_0 (with respect to the locally star shaped set S at x_0) if:

$$x \in S, \ F(x) \in F(x_0) + U^0 \Rightarrow F(x_0+\lambda(x-x_0)) \in F(x_0) + U^0 \quad \forall \lambda \in (0, 1)$$

Definition 3.4
Let F be directionally differentiable at x_0; F is said to be U-weakly pseudoconcave (U-w.p.cv) at x_0 (with respect to the locally star shaped set S at x_0) if:

$$x \in S, \; F(x) \in F(x_0) + U^0 \Rightarrow \frac{\partial F}{\partial d}(x_0) \in U^0, \; d=\frac{x-x_0}{\|x-x_0\|}$$

Definition 3.5
Let F be directionally differentiable at x_0 and assume that $\text{int}U \neq \emptyset$; F is said to be U-pseudoconcave (U-p.cv) at x_0 (with respect to the locally star shaped set S at x_0) if:

$$x \in S, \; F(x) \in F(x_0) + U^0 \Rightarrow \frac{\partial F}{\partial d}(x_0) \in \text{int}U, \; d=\frac{x-x_0}{\|x-x_0\|}$$

Let us note that when s=1 ed $U=\mathbf{R}_+$, definitions 3.1, 3.2, 3.3 are the ordinary definitions of concave function, quasiconcave function and semistictly quasiconcave function at a point x_0, respectively, while definitions 3.4, 3.5 collapse to the ordinary definition of a pseudoconcave function at x_0 (see Mangasarian (1969)).

In the scalar case an upper semicontinuous and semistrictly quasiconcave function is also quasiconcave; this property is lost for a vector valued function as is shown in the following example:

Example 3.1
Consider the star shaped set $S=\mathbf{R}_+$ at $x_0=0$ and the function $F: \mathbf{R} \rightarrow \mathbf{R}^2$ defined as follows:
$F(x)=(x \text{ sen}\frac{1}{x}, - x \text{ sen}\frac{1}{x})$ if $x\neq 0$; $F(x)=0$ if $x=0$.

It easy to verify that F is continuous and $\mathbf{R}^2_+$ - s.s.q.cv. at x_0 but it is not $\mathbf{R}^2_+$ - q.cv. at x_0.

Remark 3.1
It follows immediately from the given definitions that a linear function F is U-concave and U-weakly pseudoconcave with respect to every cone U with vertex at the origin $0 \in U$ but it is not U-pseudoconcave.

Remark 3.2
With respect to the paretian cone $U= \mathbf{R}^s_+$ it is easy to verify that F is U-concave at x_0 if and only if all its components is concave at x_0; such a property does not hold for the other given classes of generalized concave functions.

With this regards it is sufficient to note that if at least one component of F has a strict local maximum point at x_0, then F verifies definitions 3.2, 3.3, 3.4, 3.5 without any other requirement on the other components of F. A non trivial example is the following one:

Example 3.2
Let us consider the function $F(x_1,x_2)=(x_1- x_2, x_1^2- x_2, -x_1^2- x_2)$, $x_0=(0,0)$, $S=\{(x_1,x_2): x_1\geq 0\}$ and the cone $U= \mathbf{R}^3_+$.

We can verify that $f_2(x_1,x_2)= x_1^2- x_2$ is not s.s.q.cv. , q.cv. , p.cv. at x_0 while F turns out to be $\mathbf{R}^3_+$- s.s.q.cv. , $\mathbf{R}^3_+$- q.cv. ,$\mathbf{R}^3_+$- p.cv. and $\mathbf{R}^3_+$-w.p.cv.at x_0.

The following theorem establishes ralationships among some classes of generalized concave functions:

Theorem 3.1
Let S be a locally star shaped set at x_0 and let U be a convex cone.
i) if F is U-concave at x_0 then F is U-q.cv. at x_0
ii) if F is U-concave at x_0 and U is a pointed cone then F is U-s.s.q.cv. at x_0.
proof
i) Assume that $F(x) \in F(x_0) + U$, that is $F(x)- F(x_0) \in U$. Since F is U-concave at x_0 we have
$F(x_0+\lambda(x-x_0)) \in F(x_0)+\lambda(F(x)-F(x_0))+U \subset F(x_0)+U \quad \forall \lambda \in (0,1)$, so that F is U-q.cv at x_0.
ii) Assume that $F(x) \in F(x_0) + U^0$, that is $F(x)- F(x_0) \in U^0$. Since F is U-concave at x_0 we have $F(x_0+\lambda(x-x_0)) \in F(x_0)+\lambda(F(x)-F(x_0))+U$. The thesis follows taking into account that for a pointed cone the property $U^0 + U = U^0$ holds. ◆

The following example shows that ii) of Theorem 3.1 is false if U is not pointed.

Example 3.3
Consider the function $F(x)=\begin{cases} 0 & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}$ and the non pointed cone $U=\mathbf{R}$. It is easy to verify that F is U-concave at x_0 for every $x_0 \in \mathbf{R}$ but F is not U-s.s.qv. at $x_0=1$, since for $x^*=-1$ we have:
$F(-1) \in F(1)+\mathbf{R}\setminus\{0\}$, $F(x_0+\frac{1}{2}(x^*-x_0)) = F(0)=0 \notin F(x_0)+\mathbf{R}\setminus\{0\}$.

4. Some properties of a vector generalized concave problem

The classes of the generalized concave functions introduced in the previous section, allow us to investigate relationships between local and global optima and between local efficiency at a point x_0 and local efficiency with respect to every feasible direction at x_0.
The following theorem shows , as it happens in the scalar case, that the semistrictly quasiconcavity or the pseudoconcavity of the objective function implies that a local efficient point is also global; such a property holds for a quasiconcave function only on respect to a strict local efficient point:

Theorem 4.1
Let us consider problem P where S is a locally star shaped set at x_0.
i) if x_0 is a local efficient point and F is U-s.s.q.cv. at x_0, then x_0 is an efficient point for P
ii) if x_0 is a strict local efficient point and F is U-q.cv. at x_0, then x_0 is an efficient point for P
iii) if x_0 is a local efficient point , $\text{int}U \neq \emptyset$ and F is U-p.cv. at x_0, then x_0 is an efficient point for P.
proof.
i) Ab absurdo suppose that there exists $x^* \in S$ such that $F(x^*) \in F(x_0)+U^0$. Since F is U-s.s.q.cv. at x_0, we have $F(x_0+\lambda(x^*-x_0)) \in F(x_0)+U^0 \quad \forall \lambda \in (0,1)$ and such a relation implies , choosing λ small enough,the non local efficiency of x_0.
ii) Ab absurdo suppose that there exists $x^* \in S$ such that $F(x^*) \in F(x_0)+U^0$. Since F is U-q.cv. at x_0, we have $F(x_0+\lambda(x^*-x_0)) \in F(x_0)+U \quad \forall \lambda \in (0,1)$ and such a relation implies , choosing λ small enough,the non strict local efficiency of x_0.
iii) Ab absurdo suppose that there exists $x^* \in S$ such that $F(x^*) \in F(x_0)+U^0$.Since F is

U-p.cv. at x_0, we have $\frac{\partial F}{\partial d}(x_0) \in \text{int}U$, $d=\frac{x^*-x_0}{\|x^*-x_0\|}$, that is $\lim_{t\to 0^+} \frac{F(x_0+td)-F(x_0)}{t} \in \text{int}U$ and this implies the existence of a suitable $\varepsilon>0$, such that $F(x_0+td)-F(x_0) \in \text{int}U \ \forall\, t\in(0,\varepsilon)$.

Set $t=\lambda\|x^*-x_0\|$; we have $F(x_0+\lambda(x^*-x_0)) \in F(x_0)+\text{int}U \quad \forall\, \lambda\in(0, \frac{\varepsilon}{\|x^*-x_0\|})$ and this contradicts the local efficiency of x_0. ♦

Remark 4.1

Property iii) of Theorem 4.1 cannot be extended to the class of U-w.p.cv. functions even if x_0 is a strict local efficient point; with this regard consider problem P where $U=\mathbf{R}^2_+$, $F(x)=(x^2, -x^2+2x)$ and $S=\{x\in\mathbf{R}: x\geq 0\}$. It is easy to verify that $x_0=0$ is a strict local efficient point for P but it is not efficient with respect to S; furthermore F is $\mathbf{R}^2_+$-w.p.cv. but not $\mathbf{R}^2_+$-p.cv. at x_0.

Corollary 4.1

Let us consider problem P where S is locally star shaped at x_0, U is a pointed cone and F is U-concave at x_0. Then a local efficient point x_0 is an efficient point too.

proof.

It follows immediately from Theorems 3.1, 4.1. ♦

Corollary 4.2

Let us consider problem P where S is locally star shaped at x_0, and F is linear.
Then a local efficient point x_0 is an efficient point too.

proof.

It follows from Corollary 4.1 taking into account Remark 3.1. ♦

As is known, the property for which a local efficient point with respect to every feasible direction of a star shaped set is also a local efficient point for P, does not hold for every function F (for instance consider the function $F(x,y)=(y-x^4)(x^2-y)$ and the star shaped set $S=\mathbf{R}^2_+$ at $x_0=(0,0)$).

Now we investigate the relationships between the local efficiency of x_0 and the local efficiency of x_0 with respect to all directions starting from x_0. With this aim we give the following definition:

A point x_0 is said to be a local efficient point (strict local efficient point) with respect to the direction $d=\frac{x-x_0}{\|x-x_0\|}$, $x\in S$ and with respect to the cone U if there exists $t^*>0$ such that

$$F(x) \notin F(x_0)+U,\ \forall\, x=x_0+t\,d,\ t\in(0,t^*)$$

$$(F(x) \notin F(x_0)+U,\ \forall\, x=x_0+t\,d,\ t\in(0,t^*)).$$

The following theorem holds:

Theorem 4.2
Let us consider problem P where S is locally star shaped at x_0.

i) if x_0 is a local efficient point for every direction $d=\frac{x-x_0}{\|x-x_0\|}$, $x \in S$ and F is U-s.s.q.cv. at x_0, then x_0 is a local efficient point for P.

ii) if x_0 is a strict local efficient point for every direction $d=\frac{x-x_0}{\|x-x_0\|}$, $x \in S$ and F is U-q.cv. at x_0, then x_0 is a local efficient point for P.

iii) if x_0 is a local efficient point for every direction $d=\frac{x-x_0}{\|x-x_0\|}$, $x \in S$, $\text{int}U \neq \emptyset$ and F is U-p.cv. at x_0, then x_0 is a local efficient point for P.

proof.
Similar to the one given in Theorem 4.1. ♦

5. Optimality conditions (non differentiable case)

In this section and in the following one, we state some necessary and/or sufficient optimality conditions stated by means of a general approach involving the directions belonging to the tangent cone to the feasible region at x_0.

We recall that the tangent cone to the set S at $x_0 \in S$, is the set:

$$T(S,x_0)=\{ v: \exists \ \{\alpha_n\} \subset \mathbf{R}, \{x_n\} \subset S, \ \alpha_n \to +\infty, \ x_n \to x_0 \text{ con } \alpha_n(x_n-x_0) \to v \}.$$

Let us note that $T(S,x_0)=\{0\}$ if and only if x_0 is an isolated point and in such a case x_0 is obviously an efficient point for problem P. For this reason throughout this paper it is assumed that $T(S,x_0) \neq \{0\}$.
The following Lemma points out when the directional derivative of the function F with respect to the direction v can be obtained by means of a limit involving a suitable sequence converging to x_0.

Lemma 5.1
Let F be directionally differentiable at x_0 and locally lipschitzian at x_0. Then for any sequence $\{x_n\}$, $x_n \to x_0$, there exists a subsequence $x_{n_k} \to x_0$, such that

$$\lim_{x_{n_k} \to x_0} \frac{x_{n_k}-x_0}{\|x_{n_k}-x_0\|} = v \qquad (5.1a)$$

$$\lim_{x_{n_k} \to x_0} \frac{F(x_{n_k})-F(x_0)}{\|x_{n_k}-x_0\|} = \frac{\partial F}{\partial v}(x_0) \qquad (5.1b)$$

proof
Set $v_n=\frac{x_n-x_0}{\|x_n-x_0\|}$, $t_n=\|x_n-x_0\|$; we have
$$\frac{F(x_n)-F(x_0)}{\|x_n-x_0\|}=\frac{F(x_0+t_n v_n)-F(x_0)}{t_n}.$$
Since $\{\frac{x_{n_k}-x_0}{\|x_{n_k}-x_0\|}\}$ is a bounded sequence, there exists a subsequence verifyng (2.2a). Since

$$\frac{F(x_0+t_{n_k}v_{n_k})-F(x_0)}{t_{n_k}} = \frac{F(x_0+t_{n_k}v)-F(x_0)}{t_{n_k}} + \frac{F(x_0+t_{n_k}v_{n_k})-F(x_0+t_{n_k}v)}{t_{n_k}}$$

and taking into account that the lipschitzianity of function F implies

$$\left\| \frac{F(x_0+t_{n_k}v_{n_k})-F(x_0+t_{n_k}v)}{t_{n_k}} \right\| \leq K \| v_{n_k}-v\| \quad , \text{ we have}$$

$$\lim_{x_{n_k}\to x_0} \frac{F(x_{n_k})-F(x_0)}{\|x_{n_k}-x_0\|} = \lim_{x_{n_k}\to x_0} \frac{F(x_0+t_{n_k}v)-F(x_0)}{t_{n_k}} = \frac{\partial F}{\partial v}(x_0) \ . \qquad \blacklozenge$$

The following theorem states a necessary optimality condition:

Theorem 5.1
Let us consider problem P where intU $\neq \emptyset$ and F is directionally differentiable and locally lipschitzian at x_0.
If x_0 is a local efficient point for P than

$$\frac{\partial F}{\partial v}(x_0) \notin \text{intU} \ , \quad \forall\ v \in T(S,x_0) \ , \quad v \neq 0. \tag{5.2}$$

proof
It is sufficient to prove (5.2) for every direction $v\in T(S,x_0)$ such that $\|v\|=1$; Let $\{x_n\}\subset S$, $x_n \to x_0$, be a sequence such that $\lim_{x_n\to x_0} \frac{x_n-x_0}{\|x_n-x_0\|} = v$. From Lemma 5.1 we have $\lim_{x_n\to x_0} \frac{F(x_n)-F(x_0)}{\|x_n-x_0\|} = \frac{\partial F}{\partial v}(x_0)$; on the other hand the local efficiency of x_0 implies $\frac{F(x_n)-F(x_0)}{\|x_n-x_0\|} \notin U^0 \ \forall\ n$ so that $\lim_{x_n\to x_0} \frac{F(x_n)-F(x_0)}{\|x_n-x_0\|} = \frac{\partial F}{\partial v}(x_0) \notin \text{intU}$. ♦

The following theorem states a sufficient optimality condition:

Theorem 5.2
Let us consider problem P where U is a closed cone and F is directionally differentiable and locally lipschitzian at x_0. A sufficient condition for x_0 to be a local efficient point for P is

$$\frac{\partial F}{\partial v}(x_0) \notin U \ , \quad \forall\ v \in T(S,x_0) \ , \quad v \neq 0. \tag{5.3}$$

proof
Ab absurdo suppose that x_0 is not a local efficient point. Then there exists a sequence $\{x_n\}\subset S$, $x_n \to x_0$ such that $F(x_n) \in F(x_0)+U^0$. From Lemma 5.1 there exists a subsequence $\{x_{n_k}\}$ such that

$$\lim_{x_{n_k}\to x_0} \frac{F(x_{n_k})-F(x_0)}{\|x_{n_k}-x_0\|} = \frac{\partial F}{\partial v}(x_0) \quad \text{with} \quad \lim_{x_{n_k}\to x_0} \frac{x_{n_k}-x_0}{\|x_{n_k}-x_0\|} = v .$$

Since $\frac{F(x_{n_k})-F(x_0)}{\|x_{n_k}-x_0\|} \in U^0$, we have $\lim_{x_{n_k}\to x_0} \frac{F(x_{n_k})-F(x_0)}{\|x_{n_k}-x_0\|} = \frac{\partial F}{\partial v}(x_0) \in \text{cl}U=U$ and this contradicts (5.3). ♦

The optimality conditions (5.2), (5.3) can be specialized with respect to the feasible region S.

If S is a closed convex cone with vertex x_0 the tangent cone $T(S,x_0)$ reduces to the set of all feasible directions $D=S-\{x_0\}$ and consequently (5.2), (5.3) become:

$$\frac{\partial F}{\partial v}(x_0) \notin \text{int}U , \quad \forall\, v\in D . \qquad (5.4\text{ a})$$

$$\frac{\partial F}{\partial v}(x_0) \notin U , \quad \forall\, v\in D . \qquad (5.4\text{ b})$$

When S is a polyhedral set and x_0 is a vertex of S , (5.4 b) states a sufficient condition for a vertex to be an efficient point for P; this result generalizes the ones given in Cambini-Martein (1991) and in Cambini R. (1992).

If $S=\mathbf{R}^n$ (5.4 b) states the following sufficient optimality condition for an interior point, which generalizes the one given in the scalar case by Ben-Tal and Zowe (1985):

$$\frac{\partial F}{\partial v}(x_0) \notin U , \quad \forall\, v\in \mathbf{R}^n,\ v\neq 0 \qquad (5.5)$$

Furthermore , as a direct consequence of the given definitions 3.4, 3.5, we have the following sufficient optimality conditions :

Theorem 5.3
Let us consider problem P where F is directionally differentiable and locally lipschitzian at x_0.
i) if S is locally star shaped at x_0, $\text{int}U \neq \emptyset$ and F is U-pseudoconcave at x_0, then (5.6) is a sufficient condition for x_0 to be a local efficient point for P

$$\frac{\partial F}{\partial v}(x_0) \notin \text{int}U , \quad \forall\, v\in D \qquad (5.6)$$

ii) if S is locally star shaped at x_0, U is a closed cone and F is U-weakly pseudoconcave at x_0, then (5.7) is a sufficient condition for x_0 to be a local efficient point for P

$$\frac{\partial F}{\partial v}(x_0) \notin U^0 , \quad \forall\, v\in D \qquad (5.7)$$

6. Optimality conditions (differentiable case)

In this section we consider problem P where the objective function F is differentiable at x_0. In such a case F is also directionally differentiable with respect to any direction v and it results $J_{F_{x_0}}(v)= \frac{\partial F}{\partial v}(x_0)$ where $J_{F_{x_0}}$ denotes the Jacobian matrix of F at x_0.

In order to apply the results given in section 5 to the differentiable case , we need to extend

Lemma 5.1 since F is not necessarily lipschitzian at x_0:

Lemma 6.1
Let F be differentiable at x_0. Then for any sequence $\{x_n\}$, $x_n \to x_0$, there exists a subsequence $x_{n_k} \to x_0$, such that

$$\lim_{x_{n_k} \to x_0} \frac{x_{n_k}-x_0}{\|x_{n_k}-x_0\|} = v \quad , \quad \lim_{x_{n_k} \to x_0} \frac{F(x_{n_k})-F(x_0)}{\|x_{n_k}-x_0\|} = J_{F_{x_0}}(v) \tag{6.1}$$

<u>proof</u>

Since $F(x_n)-F(x_0)= J_{x_0}(x_n-x_0)+\sigma(x_n,x_0)$, $\lim_{x_n \to x_0} \frac{\sigma(x_n,x_0)}{\|x_n-x_0\|} =0$, we have

$$\frac{F(x_n)-F(x_0)}{\|x_n-x_0\|} = J_{x_0}\left(\frac{x_n-x_0}{\|x_n-x_0\|}\right) + \frac{\sigma(x_n,x_0)}{\|x_n-x_0\|} .$$

Taking into account that $\{\frac{x_n-x_0}{\|x_n-x_0\|}\}$ is a bounded sequence, there exists a convergent subsequence $\{ \frac{x_{n_k}-x_0}{\|x_{n_k}-x_0\|} \}$ verifying (6.1). ♦

The previous Lemma allows us to restate Theorems 5.1, 5.2, , in the following way:

Teorema 6.1
Let us consider problem P where $intU \neq \emptyset$ and F is differentiable at x_0 . If x_0 is a local efficient point for P then

$$J_{F_{x_0}}(v) \notin intU , \quad \forall\, v \in T(S,x_0) , \quad v \neq 0. \tag{6.2}$$

Theorem 6.2
Let us consider problem P where U is a closed cone and F is differentiable at x_0 . A sufficient condition for x_0 to be a local efficient point for P is

$$J_{F_{x_0}}(v) \notin U , \quad \forall\, v \in T(S,x_0) , \quad v \neq 0. \tag{6.3}$$

Some particular cases

- <u>S is a closed cone with vertex at x_0</u> .

Since $T(S,x_0) = S-\{x_0\} = D$, (6.2), (6.3) hold $\forall\, v \in D$ and Theorem 5.3 can be restate as follows:

Theorem 6.3
Let us consider problem P where F is differentiable at x_0 .
i) if S is locally star shaped at x_0 , $intU \neq \emptyset$ and F is U-pseudoconcave at x_0, then (6.4) is a sufficient condition for x_0 to be a local efficient point for P

$$J_{F_{x_0}}(v) \notin intU , \quad \forall\, v \in D \tag{6.4}$$

ii) if S is locally star shaped at x_0, U is a closed cone and F is U-weakly pseudoconcave at x_0, then (6.5) is a sufficient condition for x_0 to be a local efficient point for P

$$J_{F_{x_0}}(v) \notin U^0, \quad \forall\, v \in D \qquad (6.5)$$

- <u>S is an open set</u> (unconstrained problem)

Let $U^*=\{\alpha : \alpha^t u \geq 0,\ \forall\, u \in U\}$ be the (positive) polar cone of U.

The following theorem holds:

Theorem 6.4

Let us consider the unconstrained problem P (S is open) where U is a convex cone with $\text{int}U \neq \emptyset$ and F is differentiable at x_0.

If x_0 is a local efficient point for P, then :

$$\exists\ \alpha \in U^* \setminus \{0\} \text{ such that } \alpha^t J_{F_{x_0}} = 0 \qquad (6.6)$$

<u>proof</u>.

Since $T(S,x_0) = \mathbf{R}^n$, condition (6.2) is equivalent to (6.7)

$$J_{F_{x_0}}(x-x_0) \notin \text{int}U, \quad \forall\, x \in \mathbf{R}^n, \ x \neq x_0. \qquad (6.7)$$

Let W be the linear manifold $W=\{z=J_{F_{x_0}}(x-x_0),\ x \in \mathbf{R}^n\}$. Condition (6.7) is equivalent to $W \cap \text{int}U = \emptyset$ so that there exists an hyperplane which separates properly the convex sets W and intU, such that $\alpha^t J_{F_{x_0}}(x-x_0) = 0$, $\forall\, x \in \mathbf{R}^n$, $\alpha \in U^* \setminus \{0\}$, that is $\alpha^t J_{F_{x_0}} = 0$. ♦

The following example shows that (6.6) is not, in general, a sufficient optimality condition:

<u>Esempio 6.1</u>

Consider the problem U-max $(x_1^3 + x_2, -x_2)$, $(x_1,x_2) \in \mathbf{R}^2$, $U = \mathbf{R}^2_+$ and the feasible point $x_0 =(0,0)$. We have

$J_{F_{x_0}} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$, $\alpha^t J_{F_{x_0}} = 0$, $\alpha^t=(1,1)$ so that (6.6) is verified but x_0 is not a local efficient point since $F(x_1,0)=(x_1^3,0) \in \mathbf{R}^2_+\ \forall\, x_1>0$.

The following theorem points out the different roles played by weakly pseudoconcavity and pseudoconcavity:

Theorem 6.5

Let us consider the unconstrained problem P where S is a star shaped set and F is differentiable at x_0.

i) if condition (6.6) holds, $\text{int}U \neq \emptyset$ and F is U-p.cv. at x_0, then x_0 is a local efficient point

ii) if condition (6.6) holds with $\alpha \in \text{int}U^*$ and F is U-w.p.cv. at x_0, then x_0 is a local efficient point

<u>proof</u>.

i) Ab absurdo suppose that there exists $x^* \in S$ such that $F(x^*) \in F(x_0)+U^0$. Since F is U-p.cv. at x_0, we have $J_{F_{x_0}}(d) \in \text{int}U$, $d=\frac{x^*-x_0}{\|x^*-x_0\|}$, so that $\alpha^t(J_{F_{x_0}}(d)) > 0$ and this contradicts (6.6).

ii) Ab absurdo suppose that there exists $x^* \in S$ such that $F(x^*) \in F(x_0)+U^0$. Since F is U-w.p.cv. at x_0, we have $J_{F_{x_0}}(d) \in U^0$, $d=\frac{x^*-x_0}{\|x^*-x_0\|}$, so that $\alpha^t(J_{F_{x_0}}(d)) > 0$ and this contradicts (6.6). ♦

When P is a linear multiobjective problem, ii) of Theorem 6.5 can be specified by means of the following theorem:

Theorem 6.6
Let us consider the unconstrained linear multiobjective problem P where U is a closed, convex, pointed cone.
Then x_0 is an efficient point for P if and only if

$$\exists \ \alpha \in \text{int}U^* \text{ such that } \alpha^t J_{F_{x_0}} = 0 \quad (6.8)$$

proof.
If (6.8) holds, the thesis follows from ii) of Theorem 6.5 and from Corollary 3.2 taking into account that F is U-w.p.cv. at x_0.
If x_0 is an efficient point then $F(x) \notin F(x_0) + U^0$, $\forall x \in \mathbf{R}^n$, and this condition, for the linearity of F, is equivalent to $J_{F_{x_0}}(x-x_0) \notin U^0$, $\forall x \in \mathbf{R}^n$, so that setting $W=\{z=J_{F_{x_0}}(x-x_0),\ x \in \mathbf{R}^n\}$, we have $W \cap U^0 = \emptyset$.
Since W-U is a closed convex cone (Rockafellar 1970) such that $(W-U) \cap U^0 = \emptyset$, applying a separation theorem given by Martein (1989) to the convex sets W and U^0, we obtain (6.8). ♦

Remark 6.1
Let us note that in Theorem 6.6 condition (6.8) is indipendent from x_0 since $J_{F_{x_0}}=F$; as a consequence a linear function does not have interior efficient points (with respect to a closed, pointed, convex cone) or every point of $\mathbf{R}^n$ is efficient.

Consider now the sufficient optimality condition (6.3) which, in the unconstrained case, becomes:

$$J_{F_{x_0}}(x-x_0) \notin U, \quad \forall x \in \mathbf{R}^n, \quad x \neq x_0. \quad (6.9)$$

In the scalar case ($s=1$, $U=\mathbf{R}_+$), (6.9) is inconsistent since relation $\nabla F(x_0)(x-x_0) < 0 \ \forall x \neq x_0$, cannot be verified, but this does not happen when $s>1$.
A class of problems for which (6.9) holds is characterized in the following theorem:

Teorema 6.7
Let us consider the unconstrained problem P (S is open) where U is a closed cone and F is differentiable at x_0. If

i) $s>n$, rank $J_{F_{x_0}}=n$

ii) $\exists \ \alpha \in \text{int}U^*$ such that $\alpha^t J_{F_{x_0}} = 0$

then x_0 is a local efficient point for P.
proof.
ii) implies that $J_{F_{x_0}}(x-x_0) \notin U^0 \ \forall x \in \mathbf{R}^n$, and i) implies that the linear system $J_{F_{x_0}}(x-x_0)=0$ has the unique solution $x= x_0$, so that (6.9) is verified $\forall x \neq x_0$. ♦

The following example shows that the class of problems verifying i), ii) of Theorem 6.7 is non empty:

Example 6.2

Consider problem P where $S=\mathbf{R}^2$, s=3, $U=R^3_+$, $F(x_1,x_2)=(x_1+x_2,x_1+2x_2, -2x_1-3x_2)$.

We have $J_F=\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix}$, $\mathrm{rank}J_F=2=n < s=3$. Condition ii) is verified for $\alpha^t=(1, 1, 1)$.

7. F.John and Kuhn-Tucker generalized conditions

In this section we consider the following vector optimization problem P:

$$P:\ U\text{-max } F(x),\ x \in S=\{x \in X: G(x) \in V\}$$

where $X \subset \mathbf{R}^n$ is an open set, $F: X \to \mathbf{R}^s$, $G: X \to \mathbf{R}^m$ are continuous functions, $s \geq 1$, $m \geq 1$, and $U\subset \mathbf{R}^s$, $V\subset \mathbf{R}^m$ are closed, pointed, convex cones with verteces at the origin such that $\mathrm{int}U\neq\emptyset$, $\mathrm{int}V\neq\emptyset$.

Let x_0 be a feasible point and assume that $G(x_0)=0$ (when $V=\mathbf{R}^m_+$, $G(x_0)=0$ means that x_0 is binding at all the constraints so that such an assumption is not restrictive taking into account the continuity of F and G).

In order to point out the role played by separation theorems in stating optimality and regularity conditions, consider the linear subspace $W=\{z=\begin{bmatrix} J_{F_{x_0}} \\ J_{G_{x_0}} \end{bmatrix}(x-x_0),\ x\in \mathbf{R}^n\}$ and the cones $\mathrm{int}U\times \mathrm{int}V$, $\mathrm{int}U\times V$, $U^0\times V$.

The following lemma holds:

Lemma 7.1

i) $W\cap(\mathrm{int}U\times \mathrm{int}V)=\emptyset$ if and only if

$$\exists\ 0\neq(\alpha_F,\alpha_G),\ \alpha_F\in U^*,\ \alpha_G\in V^*:\ \alpha_F^t J_{F_{x_0}}+\alpha_G^t J_{G_{x_0}}=0 \tag{7.1}$$

ii) $W\cap(\mathrm{int}U\times V)=\emptyset$ if and only if

$$\exists\ 0\neq(\alpha_F,\alpha_G),\ \alpha_F\in U^*\backslash\{0\},\ \alpha_G\in V^*:\ \alpha_F^t J_{F_{x_0}}+\alpha_G^t J_{G_{x_0}}=0 \tag{7.2}$$

iii) $W\cap(U^0\times V)=\emptyset$ if and only if

$$\exists\ 0\neq(\alpha_F,\alpha_G),\ \alpha_F\in \mathrm{int}U^*,\ \alpha_G\in V^*:\ \alpha_F^t J_{F_{x_0}}+\alpha_G^t J_{G_{x_0}}=0 \tag{7.3}$$

proof.

Consider the set $W-(U\times V)$ which turns out to be a closed convex cone (Rockafellar 1970). It easy to prove that

$W\cap(\mathrm{int}U\times \mathrm{int}V)=\emptyset$, $W\cap(\mathrm{int}U\times V)=\emptyset$, $W\cap(U^0\times V)=\emptyset$, implies

$(W-(U\times V))\cap(\mathrm{int}U\times \mathrm{int}V)=\emptyset$, $(W-(U\times V))\cap(\mathrm{int}U\times V)=\emptyset$, $(W-(U\times V))\cap(U^0\times V)=\emptyset$, respectively. Since $\mathrm{int}U\times \mathrm{int}V$, $\mathrm{int}U\times V$, $U^0\times V$, are convex, pointed cones, the thesis follows applyng a separation theorem given by Martein (1989). ♦

Lemma 7.2
Let x_0 be a local efficient point for P. Then:

i) $W \cap (\text{int}U \times \text{int}V) = \emptyset$

ii) if G is a linear function $W \cap (\text{int}U \times V) = \emptyset$

iii) if F and G are linear functions $W \cap (U^0 \times V) = \emptyset$.

proof

i) Ab absurdo suppose that there exists $x^* \in X$ such that $J_{F_{x_0}}(x^*-x_0) \in \text{int}U$, $J_{G_{x_0}}(x^*-x_0)) \in \text{int}V$. Since

$$\lim_{t\to 0^+} \frac{F(x_0+t(x^*-x_0))-F(x_0)}{t} = J_{F_{x_0}}(x^*-x_0) , \lim_{t\to 0^+} \frac{G(x_0+t(x^*-x_0))-G(x_0)}{t} = J_{G_{x_0}}(x^*-x_0) ,$$

there exists $\varepsilon>0$ such that

$\left(\frac{F(x_0+t(x^*-x_0))-F(x_0)}{t}, \frac{G(x_0+t(x^*-x_0))-G(x_0)}{t}\right) \in \text{int}U \times \text{int}V \quad \forall\, t \in (0,\varepsilon)$, and this implies $x=x_0+t(x^*-x_0) \in S \quad \forall\, t \in (0,\varepsilon)$, $F(x) \in F(x_0)+\text{int}U$ and this contradicts the local efficiency of x_0.

ii) and iii) follow in a similar way taking into account the linearity of F and G. ♦

Theorem 7.1 (F.John optimality conditions)
Let us consider the vector optimization problem P where F, G are differentiable at x_0. If x_0 is a local efficient point for P , then (7.1) holds.

proof.

It follows from i) of Lemma 7.2 and from i) of Lemma 7.1. ♦

Remark 7.1

- (7.1) can be interpreted as a general formulation of the F.John conditions for a vector optimization problem while (7.2), (7.3) can be interpreted as two possible formulations of the Kuhn-Tucker conditions since in the scalar case (s=1) they collapse to them; as a consequence $W \cap (\text{int}U \times V) = \emptyset$ and $W \cap (U^0 \times V) = \emptyset$ can be viewed play the role of regularity conditions, since any condition which ensure such disjunctions allows us to obtain Kuhn-Tucker conditions.

The following theorem characterizes some classe of functions for which (7.2), (7.3) become necessary optimality conditions :

Theorem 7.2
Let us consider the vector optimization problem P where F, G are differentiable at x_0.

i) if x_0 is a local efficient point for P and G is a linear function , then (7.2) holds

ii) if x_0 is a local efficient point for P and F, G are linear functions, then (7.3) holds.

proof.

It follows from ii) and iii) of lemmas 7.2, 7.1. ♦

Remark 7.2

- in the paretian case $U= \mathbf{R}^s_+$, $V= \mathbf{R}^m_+$, i) of Theorem 7.2 implies that at least one of the component of α_F is strictly positive in (7.2); in the scalar case (s=1) this means that when the feasible region is defined by linear constraints , the Kuhn-Tucker conditions hold without any constraint qualification

- ii) of Theorem 7.2 implies that for a linear multiobjective problem an efficient point is also strictly efficient.

The following theorem points out the role of generalized concavity in stating sufficient optimality conditions:

Theorem 7.2
Let us consider the vector optimization problem P where S is a star shaped set at x_0 and F, G are differentiable at x_0.

i) if F is U-w.p.cv. at x_0, G is V-q.cv. at x_0, and (7.1) holds with $\alpha_F \in \text{int}U^*$, then x_0 is a local efficient poit for P.

ii) if F is U-p.cv. at x_0, G is V-q.cv. at x_0, and (7.1) holds with $\alpha_F \in U^* \setminus \{0\}$, then x_0 is a local efficient point for P.

proof

i) Suppose that there exists $x^* \in S$ such that $F(x^*) \in F(x_0)+U^0$. Since F is U-w.p.cv. at x_0 and G is V-q.cv. at x_0 we hâve, respectively, $J_{F_{x_0}}(x^*-x_0) \in U^0$, $J_{G_{x_0}}(x^*-x_0) \in V$ and thus

$\alpha_F^t J_{F_{x_0}}(x^*-x_0)>0$, $\beta_F^t J_{G_{x_0}}(x^*-x_0)\geq 0$ since $\alpha_F \in \text{int}U^*$ and $\alpha_G \in V^*$. Consequently

$\alpha_F^t J_{F_{x_0}}(x^*-x_0)+ \alpha_G^t J_{G_{x_0}}(x^*-x_0)>0$ and this contradicts (7.1).

ii) similar to the one given in i). ♦

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This research has been partially supported by the Ministry of Public Education (40%,60%)