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Generalized concavity and optimality conditions in Vector and Scalar optimization

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Abstract

The aim of this paper is to carry on the study of optimality in the vector and in the scalar case jointly, by studying the disjunction of suitable sets in the image space.
A cone is introduced which allows us to find necessary and/or sufficient optimality conditions in the image space and in the decision space both.
Key words: generalized concavity, optimality conditions.

1. Introduction

The aim of this paper is to establish necessary and/or sufficient optimality conditions for a vector optimization problem where the objective functions and the constraints may be directionally differentiable, differentiable and/ or generalized concave functions.
The general framework within which we establish our results is the image space where optimality can be carried on studying the disjunction between two suitable sets **K** and **H**. More exactly since a feasible point x_0 is a local efficient point for P if and only if $\mathbf{K} \cap \mathbf{H} = \emptyset$, any logical consequence of such a disjunction becomes a necessary optimality condition, while any condition which ensures $\mathbf{K} \cap \mathbf{H} = \emptyset$ becomes a sufficient optimality condition.

[+] The paper has been discussed jointly by the authors. Cambini has developed sections 1,2,8,9 ; Martein has developed sections 3,4,5,6,7 .

Since **K** does not have in general properties which are useful in the study of such a disjunction, some authors [2, 3, 4, 5, 7, 8,10] have introduced suitable sets instead of **K** with different aims .
In this order of ideas, we will define a suitable tangent cone $\mathbf{T_1}$, which allows us to find necessary and/or sufficient optimality conditions in the image space.
The obtained results can be used in deducing necessary and/or sufficent optimality conditions in the decision space, whenever a characterization of $\mathbf{T_1}$ is established.
Furthermore we point out that the image space seems to be appropriate in order to study generalized concavity since it is possible to obtain several optimality conditions in a general form.

2. Statement of the problem

Consider the following vector extremum problem

$$P:\ U\text{-max } \varphi(x),\ x \in S = \{x \in X: g(x) \in V\}$$

where $X \subset \mathbf{R}^n$ is an open set , $\varphi = (\varphi_1 \ldots \varphi_s): X \rightarrow \mathbf{R}^s$, $g = (g_1 \ldots g_m) : X \rightarrow \mathbf{R}^m$ are continuous functions, $s \geq 1$, $m \geq 1$, and $U \subset \mathbf{R}^s$,$V \subset \mathbf{R}^m$ are closed, convex cones with vertices at the origin such that $\text{int}U \neq \emptyset$, $\text{int}V \neq \emptyset$.
A point $x_0 \in S$ is said to be a <u>local efficient point</u> for problem P if there is no a feasible x belonging to a suitable neighbourhood of x_0 such that

$$\varphi(x) \in \varphi(x_0) + U^0 \tag{2.1}$$

where $U^0 = U \setminus \{0\}$.
We say that x_0 is an <u>efficient point</u> for P if (2.1) holds for every $x \in S$.
Let us note that when $s=1$, $U=\mathbf{R}_+$, $V = \mathbf{R}_+^m$, problem P reduces to a scalar optimization problem and (2.1) collapses to the ordinary definition of a local maximum point.

Let x_0 be a feasible point; from now on we assume that $g(x_0)=0$ (when $V=\mathbf{R}_+^m$, $g(x_0)=0$ means , obviously, that x_0 is binding at all the constraints so that such an assumption is not restrictive taking into account the continuity of φ and g).

Set

$f(x)=\varphi(x)-\varphi(x_0)$, $F(x)=(f(x), g(x))$, $\mathbf{K}=F(X)$, $\mathbf{H}=U^0\times V$

We will refer to $\mathbf{R}^n$ as the <u>decision space</u> and to $\mathbf{R}^{s+m}$ as the <u>image space</u>.
It is easy to prove that x_0 is either an efficient point or a maximum point ($s=1$, $U=\mathbf{R}_+$, $V=\mathbf{R}_+^m$) if and only if

$$\mathbf{K}\cap\mathbf{H}=\emptyset \tag{2.2}$$

Furthermore (2.2) is equivalent to state that x_0 is either a local efficient point or a local maximum point when X is a suitable neighbourhood of x_0.
Let us note that the study of the disjunction between $\mathbf{K}$ and $\mathbf{H}$ in the image space will allow us to carry on jointly the study of optimality in the vector case and in the scalar case.
More exactly any logical consequence of (2.2) becomes a necessary optimality condition, while any condition which ensures (2.2) becomes a sufficient optimality condition.

3. Some classes of generalized concave functions

Now we introduce some classes of generalized concave multiobjective functions which will allow us to establish, in the following sections , necessary and/or sufficient optimality conditions for the vector extremum problem P.

Let X be an open set of $\mathbf{R}^n$, $h: X\to\mathbf{R}^t$ be a function and let $W\subset\mathbf{R}^t$ be a cone with vertex at the origin $0\in W$. Set $W^0=W\setminus\{0\}$.

A feasible set $A\subset X$ is said to be <u>locally star shaped at x_0</u> if there exists a convex neighbourhood I of x_0 such that for all $x\in I\cap A$ we have

$$[x, x_0]=\{tx+(1-t)x_0:\ t\in[0,1]\}\subset A.$$

Definition 3.1

The function h is said to be W-concave at $x_0 \in A$ (with respect to the locally star shaped set A at x_0) if:

$$h(x_0+\lambda(x-x_0)) \in h(x_0)+\lambda(h(x)-h(x_0))+W \quad \forall\lambda \in (0, 1), \ \forall x \in A$$

Definition 3.2

The function h is said to be W-semistrictly quasiconcave (W-s.s.q.cv.) at $x_0 \in A$ (with respect to the locally star shaped set A at x_0) if:

$$x \in A \,,\, h(x) \in h(x_0) + W^0 \ \Rightarrow h(x_0+\lambda(x-x_0)) \in h(x_0) + W^0 \quad \forall\lambda \in (0, 1)$$

Definition 3.3

The function h is said to be W-quasiconcave (W-q.cv.) at $x_0 \in A$ (with respect to the locally star shaped set A at x_0) if:

$$x \in A \,,\, h(x) \in h(x_0) + W \ \Rightarrow \ h(x_0+\lambda(x-x_0)) \in h(x_0) + W \qquad \forall\lambda \in (0, 1)$$

Definition 3.4

Let h be directionally differentiable at $x_0 \in A$; h is said to be W-weakly pseudoconcave (W-w.p.cv) at $x_0 \in A$ (with respect to the locally star shaped set A at x_0) if:

$$x \in A \,,\ h(x) \in h(x_0) + W^0 \ \Rightarrow \frac{\partial h}{\partial d}(x_0) \in W^0, \ d=\frac{x-x_0}{\|x-x_0\|}$$

Definition 3.5

Let h be directionally differentiable at $x_0 \in A$ and assume that $\text{int}W \neq \emptyset$; h is said to be W-pseudoconcave (W-p.cv) at $x_0 \in A$ (with respect to the locally star shaped set A at x_0) if:

$$x \in A \,,\, h(x) \in h(x_0) + W^0 \quad \Rightarrow \frac{\partial h}{\partial d}(x_0) \in \text{int}W, \ d=\frac{x-x_0}{\|x-x_0\|}$$

Let us note that when $s=1$, $W=\mathbf{R}_+$, definitions 3.1, 3.2, 3.3 are the ordinary definitions of concave function, semistrictly quasiconcave function and quasiconcave function at a point x_0, while definitions 3.4, 3.5 collapse to the ordinary definition of pseudoconcave function at x_0 [9].

The following Theorem holds:

Theorem 3.1

Let A be locally star shaped at x_0 and let W be a convex cone.

i) if h is W-concave at x_0 then h is W-q.cv. at x_0

ii) if h is W-concave at x_0 and W is pointed, then h is W-s.s.q.cv. at x_0.

proof.

i) Assume that $h(x) \in h(x_0) + W$, that is $h(x) - h(x_0) \in W$. Since h is W-concave at x_0 we have $h(x_0+\lambda(x-x_0)) \in h(x_0)+\lambda(h(x)-h(x_0))+W \subset h(x_0)+W \quad \forall\lambda\in(0,1)$, so that h is W-q.cv at x_0.

ii) Assume that $h(x) \in h(x_0) + W^0$, that is $h(x) - h(x_0) \in W^0$. Since h is W-concave at x_0 we have $h(x_0+\lambda(x-x_0)) \in h(x_0)+\lambda(h(x)-h(x_0))+W$. The thesis follows by noting that for a pointed cone, $W^0 + W = W^0$. □

The following example shows that ii) of Theorem 3.1 is false if W is not pointed.

Example 3.1

Consider the function $h(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}$ and the non-pointed cone $W=\mathbf{R}$. It is easy to verify that h is W-concave at x_0 for every $x_0 \in \mathbf{R}$ but h is not W-s.s.q.cv. at $x_0=1$, since for $x^*=-1$ we have:

$h(-1) \in h(1)+\mathbf{R}\setminus\{0\}$ and $h(x_0 + \frac{1}{2}(x^*-x_0)) = h(0)=0 \notin h(x_0)+\mathbf{R}\setminus\{0\}$.

Theorem 3.2

Consider problem P where S is locally star shaped at x_0; if f, g are U-s.s.q.cv. and V-q.cv. at x_0, respectively, then $F=(f,g)$ is $\mathbf{H}$-s.s.q.cv. at x_0.

proof.

If $F(x) \in F(x_0)+\mathbf{H}$, we have $f(x) \in f(x_0)+U^0$, $g(x) \in g(x_0)+V$; it follows $f(x_0+\lambda(x-x_0)) \in f(x_0)+U^0 \quad \forall\lambda\in(0,1)$, $g(x_0+\lambda(x-x_0)) \in g(x_0)+V \quad \forall\lambda\in(0,1)$ so that $F(x_0+\lambda(x-x_0)) \in F(x_0)+\mathbf{H} \quad \forall\lambda\in(0,1)$. □

Corollary3.1

Consider problem P where S is locally star shaped at x_0 and U is a pointed cone.

i) If f is U-concave at x_0 and g is V-q.cv. at x_0 , then F=(f,g) is **H**-s.s.q.cv. at x_0.

ii) if f, g are U-concave and V-concave , respectively, at x_0 , then F=(f,g) is **H**-s.s.q.cv. at x_0.

<u>proof.</u>

i) from ii) of Theorem 3.1 f turns out to be U-s.s.q.cv, so that the thesis follows from Theorem 3.2

ii) It follows from Theorem 3.2, taking into account Theorem 3.1. □

In order to point out that the class of **H**-s.s.q.cv. functions is more general than the class of functions F=(f,g) where f is U-s.s.q.cv. and g is V-q.cv. , consider the case where $U=\mathbf{R}_+$, $V=\mathbf{R}_+^m$; if x_0 is a strict local maximum point for at least one of the functions f_i or for at least one of the functions g_i , then trivially F is **H**-s.s.q.cv. at x_0 whatever the functions f, g are (since the condition $F(x)\in F(x_0)+\mathbf{H}$ is not verified for every x).

A non trivial example is the following one:

<u>Example</u> 3.2

Let us consider the function F: $\mathbf{R}^2 \rightarrow \mathbf{R}^3$, F(x,y)=(f(x,y), g(x,y)) where $f(x,y)=x^2-y$, $g(x,y)=(x-y, -x^2-y)$ and the set $S=\{(x,y)\in\mathbf{R}^2: x \geq 0\}$. Set $U=\mathbf{R}_+$, $V=\mathbf{R}_+^2$, $x_0=(0,0)$. It can be verified that F is **H**-s.s.q.cv. at x_0 but f is not U-s.s.q.cv. at x_0.

Consider now the case where F is a directionally differentiable function; the following theorem holds:

Theorem 3.3

Consider problem P where S is locally star shaped at x_0. If one of the following conditions holds

i) f and g are U-w.p.cv. and V-w.p.cv. at x_0 , respectively;
ii) f is U-w.p.cv. at x_0, g is V-q.cv. and directionally differentiable at x_0;
iii) f is U-w.p.cv. at x_0, g is V-concave and directionally differentiable at x_0 ;
then F=(f,g) is **H**-w.p.cv. at x_0.

<u>proof</u>.
i) It follows directly from definition 3.4
ii) Assume that $F(x) \in F(x_0)+\mathbf{H}$; then $f(x) \in f(x_0)+U^0$ so that $\frac{\partial f}{\partial d}(x_0) \in U^0$, $d=\frac{x-x_0}{\|x-x_0\|}$ and $g(x) \in g(x_0)+V$. We have $\frac{g(\ x_0+\lambda(x-x_0)) - g(x_0)}{\lambda} \in V$ and this implies $\frac{\partial g}{\partial d}(x_0) \in V$, $d=\frac{x-x_0}{\|x-x_0\|}$ since V is a closed cone.
iii) It follows from ii) and from i) of theorem 3.1. This completes the proof. □

<u>Remark</u> 3.1
Example 3.2 shows that the class of **H**-p.cv. functions is more general than the class of functions F=(f,g) where f is U-p.cv. and g is V-p.cv. .

4. Some properties of a multiobjective generalized concave problem

The classes of the generalized concave functions introduced in the previous section, allow us to investigate relationships between local and global optima and between local efficiency at a point x_0 and local efficiency with respect to every feasible direction at x_0[1].
Let $\mathbf{K}_I$ be the image of $F(X \cap I)$, where I is a suitable neighbourhood of x_0.

[1] x_0 is said to be a local efficient point for problem P with respect to the direction u if there exists $\varepsilon>0$ such that $\varphi(x) \notin \varphi(x_0)+U^0 \quad \forall x$: $x=x_0+tu,\ 0 \le t < \varepsilon$.

The following Theorem holds:

Theorem 4.1

Consider problem P where S is locally star shaped at x_0 and $F=(f,g)$ is **H**-s.s.q.cv. at x_0. Then $\mathbf{K}_I \cap \mathbf{H}= \emptyset$ implies $\mathbf{K} \cap \mathbf{H}= \emptyset$.

proof.

Suppose $\mathbf{K} \cap \mathbf{H} \neq \emptyset$; then there exists $x^* \in S$ such that $F(x^*) \in F(x_0)+\mathbf{H} = \mathbf{H}$, taking into account that $F(x_0)=0$. Since F is **H**-s.s.q.cv. at x_0, we have $F(x_0+\lambda(x-x_0)) \in H \ \ \forall \lambda \in (0, 1)$ so that there exists $\lambda^*>0$, such that $x_0+\lambda^*(x-x_0) \in S \cap I$ and $F(x_0+\lambda^*(x-x_0)) \in H$ and this contradicts the assumption. □

Let us note that the previous theorem characterizes a class of problems for which a local efficient point is efficient too.

Corollary 4.1

Consider problem P where S is locally star shaped at x_0. If one of the following conditions hold

i) f is U-s.s.q.cv. at x_0 and g is V-q.cv. at x_0;

ii) U is a pointed cone, f is U-concave at x_0 and g is V-q.cv. at x_0;

iii) U is a pointed cone, f is U-concave at x_0 and g is V-concave at x_0;

then if x_0 is a local efficient point for P, it follows that it is also efficient.

As is known, the property for which a local efficient point with respect to every feasible direction of a star shaped set is also a local efficient point for problem P, does not hold for every function F; it needs of some assumptions of generalized concavity on F:

Theorem 4.2

Consider problem P where S is locally star shaped at x_0 and F is **H**-s.s.q.cv. at x_0. If x_0 is a local efficient point for all feasible directions at x_0, then x_0 is a local efficient point for P.

proof.

Suppose that there exists $x^* \in S \cap I$, where I is a suitable neighbourhood of x_0, such that $F(x^*) \in F(x_0)+\mathbf{H}-\mathbf{H}$. Then $F(x_0+\lambda(x^*-x_0)) \in \mathbf{H}$ $\forall \lambda \in (0, 1)$ so that there exists $\lambda^*>0$, such that $x_0+\lambda^*(x^*-x_0) \in S \cap I$ and $F(x_0+\lambda^*(x-x_0)) \in \mathbf{H}$ and this contradicts the assumption. □

Taking into account the previous theorem and corollary we have:

Corollary 4.2

Consider problem P where S is locally star shaped at x_0. If one of the following conditions hold

i) f is U-s.s.q.cv. at x_0 and g is V-q.cv. at x_0;

ii) U is a pointed cone, f is U-concave at x_0 and g is V-q.cv. at x_0;

iii) U is a pointed cone, f is U-concave at x_0 and g is V-concave at x_0;

then if x_0 is a local efficient point for all feasible directions at x_0, it follows that it is also efficient.

The following theorem extends to a multiobjective problem the result given in [6] for a scalar problem:

Theorem 4.3

Consider problem P where S is locally star shaped at x_0, f is U-p.cv. at x_0 and g is V-q.cv. at x_0. If x_0 is a local efficient point for all feasible directions at x_0, then x_0 is a local efficient point for P.

proof.

If x_0 is not a local efficient point for P, there exists $x^* \in S$ such that $f(x^*) \in f(x_0)+U^0$ and , for the pseudo-concavity of f and the V-quasiconcavity of g, we have

$\frac{\partial f}{\partial d}(x_0) \in \text{int}U$, $d=\frac{x^*-x_0}{\|x^*-x_0\|}$, $g(x_0+\lambda(x^*-x_0)) \in V$ $\forall \lambda \in (0,1)$, and this implies the existence of $t^*>0$, $x^*=x_0+t^*d$, such that $f(x^*) \in f(x_0)+\text{int}U$, $g(x^*) \in V$ and this contradicts the efficiency of x_0 with respect to the direction d. □

5. Optimality conditions in the image space

As just outlined, the study of optimality in the image space can be carried on by studying the disjunction between the sets **K** and **H**. Since **K** does not have in general properties which are useful in studying such a disjunction, some authors [2,3,4,5,7,8,10] have introduced suitable sets instead of **K** with different aims. On the other hand, in working in the image space we must pay attention in establishing conditions which permit also to deduce some results in the decision space; from this point of view it seems to be appropriate to consider the following cone $\mathbf{T_1}$, whose properties are studied in [4]:

$$\mathbf{T_1} = \{ t : \exists\ \alpha_n \to +\infty,\ x_n \to x_0 \text{ with } \alpha_n F(x_n) \to t \}$$

The following Theorem establishes a necessary optimality condition:

Theorem 5.1 Let x_0 be a local efficient point for problem P. Then

$$\mathbf{T_1} \cap \text{int}\mathbf{H} = \emptyset \tag{5.1}$$

proof.
Assume $t^* \in \mathbf{T_1} \cap \text{int}\mathbf{H}$, that is $t^* > 0$; then there exist a sequence $\{x_n\} \subset X$ with $F(x_n) \to F(x_0) = 0$ and a sequence $\alpha_n \to +\infty$, such that $\alpha_n F(x_n) \to t^*$. Hence $\exists\ m : \alpha_m F(x_m) > 0$ and this implies $F(x_m) > 0$, that is $\mathbf{K} \cap \mathbf{H} \neq \emptyset$ and this contradicts the efficiency of x_0. □

The following example shows that $\mathbf{T_1} \cap \text{int}\mathbf{H} = \emptyset$ is a necessary but not sufficient optimality condition.

Example 5.1
Consider problem P where $s=1$, $\varphi(x) = x^2$, $m=1$, $g(x) = x$, $x_0=0$, $U=V=\mathbf{R}_+$. It is easy to show that $\mathbf{T_1} = \{ \lambda(0,1), \lambda \in \mathbf{R} \}$ so that condition $\mathbf{T_1} \cap \text{int}\mathbf{H} = \emptyset$ holds but $x_0=0$ is not an optimal solution for P.

The following Theorem gives a sufficient optimality condition.

Theorem 5.2 Consider problem P. If

$$\mathbf{T_1} \cap \mathrm{cl}\mathbf{H} = \{0\} \tag{5.2}$$

then x_0 is a local efficient point for P.

<u>proof.</u>
If x_0 is not optimal for P, then there exists a sequence $(x_n) \subset S$, $x_n \rightarrow x_0$ such that $F(x_n) \in \mathbf{H}$.
Since the unit ball is a compact set, we can suppose[2] that the sequence $\frac{F(x_n)}{\|F(x_n)\|}$ converges at $t^* \neq 0$, $t^* \in \mathbf{T_1}$. On the other hand $\frac{F(x_n)}{\|F(x_n)\|} \in \mathbf{H}$ so that $t^* \in \mathrm{cl}\mathbf{H}$ and this is a contradiction. □

The following example shows that (5.2) is not a necessary optimality condition.

<u>Example 5.2</u>
Consider problem P where $s=1$, $\varphi(x) = -x^2$, $m=1$, $g(x) = x$, $x_0=0$.
It is easy to verify that $\mathbf{T_1}=\{\lambda(0,1),\ \lambda \in \mathbf{R}\}$ so that $\mathbf{T_1} \cap \mathrm{cl}\mathbf{H} \neq \{0\}$ but $x_0=0$ is the optimal solution of problem P.

The following Theorem states a necessary and sufficient optimality condition.

Theorem 5.3 Consider problem P. The feasible point x_0 is a local efficient point for P if and only if condition I holds:

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[2]Since in a finite dimensional space any bounded sequence (z_n) has a convergent subsequence, we will assume without loss of generality (substituting (z_n) with a suitable subsequence, if necessary), that $z_n \rightarrow z$.

Condition I: $\forall\ t \in \mathbf{T_1} \cap \mathrm{cl}\mathbf{H}$, $t \neq 0$, and for any sequence $x_n \to x_0$ such that there exists $\alpha_n \to +\infty$ with $\alpha_n F(x_n) \to t$, we have $F(x_n) \notin \mathbf{H}\ \ \forall n$.

proof.
if. The thesis follows immediately from (2.2).
only if. The proof is similar to the one given in Theorem 5.2. □

6. Characterizations of T_1

When in the problem P, φ and g are differentiable at x_0, it can be shown [4] that the tangent cone $\mathbf{T_1}$ can be characterized as

$$\mathbf{T_1} = \mathbf{K_L} \cup A \qquad (6.1)$$

where

$\mathbf{K_L} = \{ J_F(x-x_0),\ x \in \mathbf{R}^n \}$, J_F is the Jacobian matrix of F at x_0,

$$A = \{ t \in \mathbf{T_1} \setminus \{0\} : \exists\ x_n \to x_0,\ \alpha_n \to +\infty \text{ with } \alpha_n F(x_n) \to t,\ \frac{x_n - x_0}{\|x_n - x_0\|} \to d \text{ and } J_F(d) = 0 \}$$

Now we consider problem P where φ and g are directionally differentiable and locally lipschitzian at x_0.
Set

$$\mathbf{K_{\mathcal{D}}} = \{ k \frac{\partial F}{\partial d}(x_0),\ d \in \mathbf{R}^n, \|d\| = 1, k > 0 \}$$

$$A^* = \{ t \in \mathbf{T_1} \setminus \{0\} : \exists\ x_n \to x_0,\ \alpha_n \to +\infty \text{ with } \alpha_n F(x_n) \to t,\ \frac{x_n - x_0}{\|x_n - x_0\|} \to d \text{ and } \frac{\partial F}{\partial d}(x_0) = 0 \}$$

where $\frac{\partial F}{\partial d}(x_0)$ denotes the directional derivative of F at x_0.

In order to achieve a characterization of $\mathbf{T_1}$ we establish, first of all, the following Lemma:

Lemma 6.1

Let F be a function directionally differentiable and locally lipschitzian at x_0. Then for any sequence $\{x_n\}$, $x_n \to x_0$, there exists a subsequence $x_{n_k} \to x_0$, such that

$$\lim_{x_{n_k}\to x_0} \frac{x_{n_k}-x_0}{\|x_{n_k}-x_0\|} = d \tag{6.2a}$$

$$\lim_{x_{n_k}\to x_0} \frac{F(x_{n_k})-F(x_0)}{\|x_{n_k}-x_0\|} = \frac{\partial F}{\partial d}(x_0) \tag{6.2b}$$

proof.

Set $d_n = \frac{x_n - x_0}{\|x_n-x_0\|}$, $t_n = \|x_n - x_0\|$; we have $\frac{F(x_n)-F(x_0)}{\|x_n-x_0\|} = \frac{F(x_0+t_n d_n)-F(x_0)}{t_n}$.

Since $\{x_n\}$ is a bounded sequence, there exists a subsequence verifying (6.2.a), so that

$$\frac{F(x_0+t_{n_k}d_{n_k})-F(x_0)}{t_{n_k}} = \frac{F(x_0+t_{n_k}d)-F(x_0)}{t_{n_k}} + \frac{F(x_0+t_{n_k}d_{n_k})-F(x_0+t_{n_k}d)}{t_{n_k}}$$

Taking into account that F is locally lipschitzian function at x_0, we have

$$\left\| \frac{F(x_0+t_{n_k}d_{n_k})-F(x_0+t_{n_k}d)}{t_{n_k}} \right\| \le K \, \| d_{n_k} - d \| \quad \text{, and thus}$$

$$\lim_{x_{n_k}\to x_0} \frac{F(x_{n_k})-F(x_0)}{\|x_{n_k}-x_0\|} = \lim_{x_{n_k}\to x_0} \frac{F(x_0+t_{n_k}d)-F(x_0)}{t_{n_k}} = \frac{\partial F}{\partial d}(x_0) \ .$$

This completes the proof. □

The following Theorem holds:

Theorem 6.1

Consider problem P where φ and g are directionally differentiable and locally lipschitzian at x_0 . Then

$$\mathbf{T_1} = \mathbf{K}_{\varphi} \cup A^* \cup \{0\} \tag{6.3}$$

proof.

First of all we prove that $\mathbf{T_1} \supset \mathbf{K}_{\mathcal{D}}$, that is $\frac{\partial F}{\partial d}(x_0) \in \mathbf{T_1}$, $\forall\, d \in \mathbf{R}^n$, $\|d\|=1$.

With this regard, taking into account that $\frac{\partial F}{\partial d}(x_0) = \lim_{\lambda \to 0^+} \frac{F(x_0+\lambda d)-F(x_0)}{\lambda}$,

it is sufficient to choose $x_n = x_0 + \frac{1}{n} d$ and $\alpha_n = n$.

Since $A^* \subset \mathbf{T_1}$ and $0 \in \mathbf{T_1}$, it results $\mathbf{T_1} \supset \mathbf{K}_{\mathcal{D}} \cup A^* \cup \{0\}$.

Now we will prove that $\mathbf{T_1} \setminus \{0\} \subset \mathbf{K}_{\mathcal{D}} \cup A^*$ since $0 \in \mathbf{T_1}$.

Let $0 \neq t \in \mathbf{T_1}$; then there exist a sequence $x_n \to x_0$ and a sequence $\alpha_n \to +\infty$ such that $t = \lim_{n\to+\infty} \alpha_n F(x_n) = \lim_{n\to+\infty} \alpha_n (F(x_n) - F(x_0)) =$

$$\lim_{n\to+\infty} \alpha_n \|x_n - x_0\| \frac{F(x_n) - F(x_0)}{\|x_n - x_0\|} \tag{6.4}$$

Taking into account Lemma (6.1), there exists a subsequence of $\{x_n\}$, which we can suppose to be the same sequence$^{(1)}$, such that

$$\lim_{n\to+\infty} \frac{F(x_n) - F(x_0)}{\|x_n - x_0\|} = \frac{\partial F}{\partial d}(x_0),$$

$d = \lim_{n\to+\infty} \frac{x_n - x_0}{\|x_n - x_0\|}$. If $\frac{\partial F}{\partial d}(x_0) = 0$, then $t \in A^*$, otherwise (6.4) implies that

$\alpha_n \|x_n - x_0\|$ converges to $k \neq 0$ and $t = k \frac{\partial F}{\partial d}(x_0)$. □

In the following sections we will show how the previous characterizations of the tangent cone $\mathbf{T_1}$ can be useful in stating necessary and/or sufficient optimality conditions.

7. Some optimality conditions for a generalized concave vector problem

As we have just outlined in section 5, $\mathbf{T_1} \cap \mathrm{int}\mathbf{H} = \emptyset$ is a necessary but not sufficient optimality condition.

The following Theorem states that such a condition becomes sufficient too, under a suitable generalized concavity assumption.

Theorem 7.1

Consider problem P where S is locally star shaped at x_0 and F is **H**-p.cv at $x_0 \in S$.

If $\mathbf{T_1} \cap \text{int}\mathbf{H} = \emptyset$, then x_0 is a local efficient point for P.

<u>proof</u>.

Suppose that there exists $x^* \in S$ such that $F(x^*) \in F(x_0)+\mathbf{H}$. Since F is **H**-p.cv. at $x_0 \in S$, we have $\frac{\partial F}{\partial d}(x_0) \in \text{int}\mathbf{H}$, $d = \frac{x^*-x_0}{\|x^*-x_0\|}$, and this implies $\mathbf{T_1} \cap \text{int}\mathbf{H} \neq \emptyset$ because of (6.3). □

Taking into account the characterizations of $\mathbf{T_1}$ given in section 6, we have the following results in the image space:

Theorem 7.2

Consider the vector extremum problem P where F is directionally differentiable and locally lipschitzian at x_0.

i) If x_0 is a local efficient point for P then $\mathbf{K}_{\mathscr{D}} \cap \text{int}\mathbf{H} = \emptyset$

ii) assume that S is locally star shaped at x_0 and F is **H**-p.cv. at x_0. If $\mathbf{K}_{\mathscr{D}} \cap \text{int}\mathbf{H} = \emptyset$, then x_0 is a local efficient point for P.

<u>proof</u>.

i) it follows from (5.1) taking into account (6.3)

ii) the proof is similar to the one given in Theorem 7.1 □

Theorem 7.3

Consider the vector extremum problem P where F is differentiable at x_0.

i) If x_0 is a local efficient point for P then $\mathbf{K_L} \cap \text{int}\mathbf{H} = \emptyset$

ii) assume that S is locally star shaped at x_0 and F is **H**-p.cv. at x_0. If

$$\mathbf{K_L} \cap \text{int}\mathbf{H} = \emptyset \tag{7.1}$$

then x_0 is a local efficient point for P.

<u>proof</u>.

i) it follows from (5.1) taking into account (6.1)

ii) the proof is similar to the one given in Theorem 7.1 □

As a consequence of Theorems 7.2 and 7.3 we obtain the following optimality conditions stated in the decision space:

Theorem 7.4

Consider the vector optimization problem P where F is directionally differentiable and locally lipschitzian at x_0.

i) if x_0 is a local efficient point for P, then $\frac{\partial F}{\partial d}(x_0) \notin \mathrm{int}\mathbf{H}$ $\forall d \in \mathbf{R}^n$, $\|d\|=1$

ii) assume that S is locally star shaped at x_0 and F is $\mathbf{H}$-p.cv at x_0. If $\frac{\partial F}{\partial d}(x_0) \notin \mathrm{int}\mathbf{H}$ $\forall d \in \mathbf{R}^n$, $\|d\|=1$, then x_0 is a local efficient point for P.

Theorem 7.5

Consider the vector optimization problem P where F is differentiable at x_0.

i) if x_0 is a local efficient point for P, then $J_F(x-x_0) \notin \mathrm{int}\mathbf{H}$ $\forall x \in \mathbf{R}^n$

ii) assume that S is locally star shaped at x_0 and F is $\mathbf{H}$-p.cv at x_0. If $J_F(x-x_0) \notin \mathrm{int}\mathbf{H}$ $\forall x \in \mathbf{R}^n$, then x_0 is a local efficient point for P.

As we have point out, the study of optimality is based on the disjunction between $\mathbf{K}$ and $\mathbf{H}$; substituting $\mathbf{K}$ with $\mathbf{T_1}$ we have obtained some necessary and/or sufficient optimality conditions; now we see how the behaviour of $\mathbf{T_1}$ with respect to $\mathbf{H}$ together with the given characterizations of $\mathbf{T_1}$, allows us to deduce some others results.

Theorem 7.6

Consider problem P where S is locally star shaped at x_0 and F is $\mathbf{H}$-w.p.cv at x_0. If $\mathbf{T_1} \cap \mathbf{H} = \emptyset$, then x_0 is a local efficient point for P.

proof.

The proof is similar to the one given in Theorem 7.1 □

Theorem 7.7
Consider problem P where S is locally star shaped at x_0 and F is $\mathbf{H}$-w.p.cv at x_0. If one of the following conditions holds
i) if $\mathbf{K}_{\mathcal{D}} \cap \mathbf{H} = \emptyset$;
ii) F is differentiable at x_0 and

$$\mathbf{K}_{\mathbf{L}} \cap \mathbf{H} = \emptyset \tag{7.2}$$

then x_0 is a local efficient point for P.

Corollary 7.1
Consider problem P where S is locally star shaped at x_0 and assume that one of the following conditions holds
a) f, g are U-w.p.cv. and V-q.cv. at x_0, respectively;
b) f is U-w.p.cv. at x_0, g is V-q.cv. and directionally differentiable at x_0;
then
i) $\mathbf{T}_1 \cap \mathbf{H} = \emptyset$ implies that x_0 is a local efficient point for P
ii) $\mathbf{K}_{\mathcal{D}} \cap \mathbf{H} = \emptyset$ implies that x_0 is a local efficient point for P
iii) Assume that f, g are differentiable at x_0; then $\mathbf{K}_{\mathbf{L}} \cap \mathbf{H} = \emptyset$ implies that x_0 is a local efficient point for P.

proof.
It follows directly from Theorems 7.6, 7.7, taking into account Theorem 3.3. □

From Theorem 7.7 and from ii), iii) of Corollary 7.1 we can deduce, immediately, the following optimality conditions stated in the decision space:

Corollary 7.2
Consider problem P where S is locally star shaped at x_0 and F is $\mathbf{H}$-w.p.cv at x_0. If one of the following conditions holds
i) $\frac{\partial F}{\partial d}(x_0) \notin \mathbf{H} \quad \forall d \in \mathbf{R}^n, \|d\|=1$;
ii) F is differentiable at x_0 and $J_F(x-x_0) \notin \mathbf{H} \quad \forall x \in \mathbf{R}^n$;
then x_0 is a local efficient point for P.

Corollary 7.3

Consider problem P where S is locally star shaped at x_0 and assume that one of the following conditions holds:

a) f, g are U-w.p.cv. and V-q.cv. respectively, at x_0

b) f is U-w.p.cv. at x_0 , g is V-q.cv. and directionally differentiable at x_0

c) f is U-w.p.cv. at x_0 , g is V-concave and directionally differentiable at x_0

Then

i) if $\frac{\partial F}{\partial d}(x_0)\notin \mathbf{H}\ \ \forall d\in \mathbf{R}^n, \|d\|=1$, then x_0 is a local efficient point for P.

ii) if F is differentiable at $x_0\in S$ and $J_F(x-x_0)\notin \mathbf{H}\ \ \forall x\in \mathbf{R}^n$, then x_0 is a local efficient point for P.

By introducing some other classes of generalized concave functions with respect to suitable cones, we can deduce in a similar way some other optimality conditions; for instance the following sufficient optimality condition, stated directly in the decision space, could be deduced introducing a suitable definition of generalized concavity with respect to the cone intU×V.

Theorem 7.8

Consider the vector extremum problem P where S is locally star shaped at x_0 and f , g are U-p.cv. and V-q.cv. at x_0, respectively. If one of the following conditions hold

i) $$\mathbf{K}_{\mathcal{D}} \cap \text{intU}\times\text{V} = \emptyset \quad (7.3)$$

ii) $$\mathbf{K}_{\mathbf{L}} \cap \text{intU}\times\text{V} = \emptyset \quad (7.4)$$

then x_0 is a local efficient point for P.

<u>proof.</u>

i) Suppose that there exists $x^*\in S$ such that $f(x^*)\in f(x_0)+U^0$. The assumptions of generalized concavity for f and g imply $\frac{\partial f}{\partial d}(x_0)\in \text{intU}$, $\frac{\partial g}{\partial d}(x_0)\in V$, $d=\frac{x^*-x_0}{\|x^*-x_0\|}$, and this contradicts (7.3). In a similar way ii) can be proven. □

In order to point out the role played by separation theorems in stating some classic optimality conditions, consider the case where U and V are polyhedral cones of $\mathbf{R}^s$ and $\mathbf{R}^m$, respectively, and P is a differentiable problem.
Since $\mathbf{K}_L$ is a linear subspace and $\mathbf{H}$ is a convex set, using suitable separation theorems [11] we have that:

- $\mathbf{K}_L \cap \mathrm{int}\mathbf{H} = \emptyset$ if and only if (7.5) holds

$$\exists\ 0 \neq \alpha = (\alpha_f, \alpha_g) \in \mathbf{H}^*,\ \alpha_f \in U^*,\ \alpha_g \in V^* :\ \alpha_f^t J_f(x_0) + \alpha_g^t J_g(x_0) = 0 \qquad (7.5)$$

- $\mathbf{K}_L \cap \mathrm{int} U \times V = \emptyset$ if and only if (7.6) holds

$$\exists\ 0 \neq \alpha = (\alpha_f, \alpha_g) \in \mathbf{H}^*,\ \alpha_f \in U^* \setminus \{0\},\ \alpha_g \in V^* :\ \alpha_f^t J_f(x_0) + \alpha_g^t J_g(x_0) = 0 \qquad (7.6)$$

- $\mathbf{K}_L \cap \mathbf{H} = \emptyset$ if and only if (7.7) holds

$$\exists\ 0 \neq \alpha = (\alpha_f, \alpha_g) \in \mathbf{H}^*,\ \alpha_f \in \mathrm{int} U^*,\ \alpha_g \in V^* :\ \alpha_f^t J_f(x_0) + \alpha_g^t J_g(x_0) = 0 \qquad (7.7)$$

Taking into account that $\mathbf{K}_L \cap \mathrm{int}\mathbf{H} = \emptyset$ is a necessary optimality condition, we have the following theorem which states the Fritz-John optimality conditions for a vector optimization problem:

Theorem 7.9

If x_0 is a local efficient point for problem P, then:

$$\exists\ 0 \neq \alpha = (\alpha_f, \alpha_g) \in \mathbf{H}^*,\ \alpha_f \in U^*,\ \alpha_g \in V^* :\ \alpha_f^t J_f(x_0) + \alpha_g^t J_g(x_0) = 0 \qquad (7.5)$$

Under suitable assumptions of generalized concavity conditions (7.6), (7.7), become sufficient optimality conditions, so that we have the following :

Theorem 7.10

i) Consider the vector extremum problem P where S is locally star shaped at x_0 and f , g are U-p.cv. and V-q.cv. at x_0, respectively.

If the following conditions hold:

$$\exists\ 0\neq\alpha=(\alpha_f,\alpha_g)\in H^*,\ \alpha_f\in U^*\setminus\{0\},\ \alpha_g\in V^*:\ \alpha_f^t J_f(x_0)+\alpha_g^t J_g(x_0)=0 \qquad (7.6)$$

then x_0 is a local efficient point for P.

ii) Consider the vector extremum problem P where S is locally star shaped at x_0, f is U-w.p.cv. at x_0, and g is V-q.cv at x_0.

If the following conditions hold:

$$\exists\ 0\neq\alpha=(\alpha_f,\alpha_g)\in H^*,\ \alpha_f\in \text{int}U^*,\ \alpha_g\in V^*:\ \alpha_f^t J_f(x_0)+\alpha_g^t J_g(x_0)=0 \qquad (7.7)$$

then x_0 is a local efficient point for P.

Remark 7.1

Relation (7.5) can be interpreted as a general formulation of the F. John conditions for a vector optimization problem while (7.6) and (7.7) can be interpreted as two possible formulations of the Kuhn-Tucker conditions for a multiobjective problem since in the scalar case (s=1) they collapse to them; as a consequence $\mathbf{K_L} \cap (\text{int}U\times V)=\emptyset$ and $\mathbf{K_L} \cap \mathbf{H}=\emptyset$, can be viewed play the role of regularity conditions.

For instance, when $U=\mathbf{R}^s_+$, and $V=\mathbf{R}^m_+$, the condition $\mathbf{K_L} \cap \mathbf{H}=\emptyset$ is equivalent to state that x_0 is a properly efficient point in the sense of Kuhn-Tucker [13].

In section 5 we have seen that $\mathbf{T_1} \cap \text{cl}\mathbf{H}=\{0\}$ is a sufficient optimality condition; taking into account relation $\mathbf{T_1}=\mathbf{K_L} \cup A$, we obtain the following:

Theorem 7.11

If $A=\emptyset$ and $\mathbf{K_L} \cap \text{cl}\mathbf{H}=\{0\}$, then x_0 is a local efficient point for problem P.

Corollary 7.4

Assume that condition (7.7) holds. If $\text{rank}J=n$ then x_0 is a local efficient point for problem P.

proof.
The assumption $\text{rank}J = n$ implies $\mathbf{T_1}=\mathbf{K_L}$ [4], so that $A=\emptyset$;
on the other hand the validity of (7.7) implies $\mathbf{K_L} \cap \text{cl}\mathbf{H}=\{0\}$. The thesis follows from Theorem 7.11. □

8. Some particular cases

First of all we see how some of the results given in the previous sections can be deepened when the feasible region of problem P is defined by linear constraints or when P is a linear multiobjective problem.
The following theorem holds:

Theorem 8.1
Consider problem P when φ is differentiable at x_0 and g is linear. If x_0 is a local efficient point for P then $\mathbf{K_L} \cap (\text{int}U\times V)=\emptyset$.

proof.
Suppose that there exists x^* such that $(J_f(x^*-x_0), J_g(x^*-x_0)) \in \text{int}U\times V$; then $(J_f d, J_g d) \in \text{int}U\times V$, $d=\dfrac{x^*-x_0}{\|x^*-x_0\|}$. Consider the sequence
$x_n=x_0+\frac{1}{n}d$. We have $F(x_n)-F(x_0)=J_F(x_n-x_0)+\sigma(x_n,x_0)$ with
$\lim\limits_{n\to+\infty}\dfrac{\sigma(x_n,x_0)}{\|x_n-x_0\|}=0$, so that $\dfrac{F(x_n)-F(x_0)}{\|x_n-x_0\|}$ converges to $J_F d \in \text{int}U\times V$. As a consequence there exists n^* such that
$\forall\, n>n^*$, $f(x_n)\in \text{int}U$ with $g(x_n)\in V$ and this contradicts the efficiency of x_0. □

Taking into account (7.6) and the previous theorem we have the following

Corollary 8.1
Consider problem P when f is differentiable at x_0 and g is linear.
If x_0 is a local efficient point for P then

$$\exists\ \alpha_f \in U^*\setminus\{0\}, \alpha_g \in V^* :\ \alpha_f^t J_f(x_0) + \alpha_g^t J_g(x_0) = 0 \qquad (8.1)$$

<u>Remark</u> 8.1

When $U = \mathbb{R}^s_+$, $V = \mathbb{R}^m_+$, Corollary 8.1 points out that when the constraints are linear functions , at least one of the components of α_f is positive (not necessary all); in the scalar case this means that Kuhn-Tucker conditions hold without any constraint qualification.

Consider now the case where P is a linear multiobjective optimization problem i.e. f and g are linear functions.
Obviously we have $\mathbf{K} = \mathbf{K_L}$, thus (2.2) is equivalent to state that $\mathbf{K_L} \cap \mathbf{H} = \emptyset$
Taking into account (7.7) and that F=(f,g) is **H**-w.p.cv. at x_0 for every $x_0 \in \mathbf{R}^n$, we have the following classic result:

Theorem 8.2

Consider the linear multiobjective optimization problem P. Then x_0 is an efficient point for P if and only if (8.2) holds:

$$\exists\ \alpha_f \in \mathrm{int}U^*, \alpha_g \in V^* :\ \alpha_f^t J_f + \alpha_g^t J_g = 0 \qquad (8.2)$$

Let us note that Theorem 8.2 implies that for a linear multiobjective optimization problem an efficient point for P is also strictly efficient.

9. Further suggestions

As we have outlined in the previous sections, the approach in the image space is based on the study of the disjunction between **K** and **H**. Since **K**=F(X), the obtained results involve any point belonging to a suitable neighbourhood of x_0, so that , if we are interested to deepen the behaviour of the objective functions on the feasible region S or on a suitable set strictly related to S, we must consider the image F(S) instead

of F(X); now we see how in this way is possible to establish some other kinds of optimality conditions.
With this aim, consider the following subset of $\mathbf{T_1}$:
$\mathbf{T_1}(S) = \{ t : \exists\ \alpha_n \to +\infty\ , x_n \to x_0 , (x_n) \subset S, \text{ with } \alpha_n F(x_n) \to t \}$
The following theorem holds:

Theorem 9.1
Let x_0 be a local efficient point for problem P. Then

$$\mathbf{T_1}(S) \cap (\text{int}U \times V) = \emptyset . \qquad (9.1)$$

proof.
If (9.1) does not hold, there exist $(x_n) \subset S$, $x_n \to x_0$, $\alpha_n \to +\infty$, such that $\lim_{n \to +\infty} \alpha_n f(x_n) \in \text{int } U$ and $\lim_{n \to +\infty} \alpha_n g(x_n) \in V$. Since $(x_n) \subset S$, $g(x_n) \in V$ and furthermore there exists n^* such that $f(x_n) \in \text{int } U \quad \forall n > n^*$; consequently $(f(x_n), g(x_n)) \in (\text{int}U \times V) \quad \forall n > n^*$ and this contradicts the efficiency of x_0. □

The following example shows that the necessary optimality condition (9.1) cannot be extended to the tangent cone $\mathbf{T_1}$.

Example 9.1

Consider problem P where $s=1$, $\varphi(x) = -\sqrt[3]{x}$, $m=1$, $g(x) = x$, $x_0=0$, $U=V=\mathbf{R}_+$. It is easy to show that $x_0=0$ is an optimal solution for P, $\mathbf{T_1}=\{ \lambda(1,0) , \lambda \in \mathbf{R} \}$, $\mathbf{T_1}(S)=\{ \lambda(-1,0) , \lambda \geq 0 \}$, so that $\mathbf{T_1} \cap (\text{int}U \times V) \neq \emptyset$, while $\mathbf{T_1}(S) \cap (\text{int}U \times V) = \emptyset$.
This example points out that if we limit ourselves on considering the image of the feasible region, it is possible to obtain conditions in a more general form.

Theorem 9.2
Consider problem P. If

$$\mathbf{T_1}(S) \cap \text{cl}\mathbf{H} = \{0\} \qquad (9.2)$$

then x_0 is a local efficient point for P.

proof. Similar to the one given in Theorem 5.2. □

When in problem P, f and g are directionally differentiable and locally lipschitzian at x_0 or differentiable at x_0, we can characterize the tangent cone $\mathbf{T_1}(S)$ in the following way (the proofs are similar to the ones given in section 6):

$$\mathbf{T_1}(S) = \mathbf{K}_{\mathcal{D}}(S) \cup A^*(S) \cup \{0\} \qquad (9.3\ a)$$

$$\mathbf{T_1}(S) = \mathbf{K_L}(S) \cup A(S) \cup \{0\} \qquad (9.3\ b)$$

where

$$\mathbf{K}_{\mathcal{D}}(S) = \{ k \frac{\partial F}{\partial d}(x_0)\ ,\ d \in T(S,x_0),\ \|d\|=1,\ k>0\} \subset \mathbf{K}_{\mathcal{D}}$$

$$A^*(S) = \{ t \in \mathbf{T_1} \setminus \{0\} :\ \exists\ x_n \to x_0\ ,\ (x_n) \subset S,\ \alpha_n \to +\infty \text{ with } \alpha_n F(x_n) \to t\ ,$$

$$\frac{x_n - x_0}{\|x_n - x_0\|} \to d \in T(S,x_0) \text{ and } \frac{\partial F}{\partial d}(x_0) = 0 \} \subset A^*$$

$$\mathbf{K_L}(S) = \{ J_F(x-x_0)\ ,\ x \in S \} \subset \mathbf{K_L},$$

$$A(S) = \{ t \in \mathbf{T_1}/\{0\} :\ \exists\ x_n \to x_0\ ,\ (x_n) \subset S,\ \alpha_n \to +\infty \text{ with } \alpha_n F(x_n) \to t\ ,$$

$$\frac{x_n - x_0}{\|x_n - x_0\|} \to d \in T(S,x_0) \text{ and } J_F(d) = 0 \} \subset A.$$

and where $T(S,x_0)$ denotes the tangent cone to S at x_0.

The given characterization of $\mathbf{T_1}(S)$ allow us to obtain the following necessary optimality conditions:

Theorem 9.3

Let x_0 be a local efficient point for P.

i) if F is directionally differentiable and locally lipschitzian at x_0, then

$$\mathbf{K}_{\mathcal{D}}(S) \cap \text{int} U \times V = \emptyset \qquad (9.4\ a)$$

ii) if F is differentiable at x_0, than

$$\mathbf{K_L}(S) \cap \text{int} U \times V = \emptyset \qquad (9.4\ b)$$

The following corollary states necessary optimality conditions with respect to the directions of the tangent cone $T(S,x_0)$:

Corollary 9.1

Let x_0 be a local efficient point for P.

i) if f, g are directionally differentiable and locally lipschitzian at x_0, then

$$\frac{\partial f}{\partial d}(x_0) \notin \text{int}U \quad \forall d \in T(S,x_0),\ d \neq 0 \tag{9.5 a}$$

ii) if f, g are differentiable at x_0, then

$$J_f(d) \notin \text{int}U \quad \forall d \in T(S,x_0) \tag{9.5 b}$$

proof.

(9.5 a) and(9.5 b) follow immediately from (9.4 a) and(9.4 b) taking into account that there exists a sequence $(x_n) \subset S$, $x_n \to x_0$, $\frac{x_n - x_0}{\|x_n - x_0\|} \to d$ such that $\frac{\partial g}{\partial d}(x_0) - \lim_{n \to +\infty} \frac{g(x_n) - g(x_0)}{\|x_n - x_0\|} \in V$. □

The following theorem states sufficient optimality conditions:

Theorem 9.4

Consider the vector optimization problem P. If one of the following conditions hold

i) F is directionally differentiable , locally lipschitzian at x_0 and

$$\mathbf{K}_{\mathcal{D}}(S) \cap \text{cl}\mathbf{H} = \emptyset \tag{9.6 a}$$

ii) F is differentiable at x_0 and

$$\mathbf{K}_{\mathbf{L}}(S) \cap \text{cl}\mathbf{H} = \emptyset \tag{9.6 b}$$

then x_0 is a local efficient point for P.

proof.

It easy to verify that (9.6 a) implies $A^*(S)=\emptyset$; thus for (9.3 a) we have $\mathbf{T_1}(S)=\{0\}$ and consequently $\mathbf{T_1}(S) \cap \text{cl}\mathbf{H} = \{0\}$, so that the thesis follows from Theorem 9.2.

ii) the proof is similar to the one given in i). □

The following corollary states sufficient optimality conditions in the decision space:

Corollary 9.2

Consider the vector extremum problem P. If one of the following conditions holds

i) f, g are directionally differentiable , locally lipschitzian at x_0 and

$$\frac{\partial f}{\partial d}(x_0) \notin U \quad \forall d \in T(S,x_0),\ d \neq 0 \tag{9.7 a}$$

ii) f, g are differentiable at x_0 and

$$J_f(d) \notin U \qquad \forall d \in T(S,x_0),\ d \neq 0 \tag{9.7 b}$$

then x_0 is a local efficient point for P.

proof.

Similar to the one given in Corollary 9.1. □

Remark 9.1

When in problem P s=1, V= $\mathbb{R}^m_+$, the optimality conditions (9.5), (9.7) collapse to the ones given in [6].

Remark 9.2

When the feasible region S is a closed convex cone with vertex at x_0, if we set $g(x)=x-x_0$ and $V=S-\{x_0\}$, taking into account that $T(S,x_0)=S-\{x_0\}$, the optimality conditions (9.5 a) and (9.7 a) reduces to the following ones:

$$\frac{\partial f}{\partial d}(x_0) \notin \text{int}U \quad \forall d \in V \tag{9.8 a}$$

$$\frac{\partial f}{\partial d}(x_0) \notin U \quad \forall d \in V \tag{9.8 b}$$

When S is a polyhedral set and x_0 is a vertex of S , (9.8 b) states a sufficient condition for a vertex x_0 to be an efficient point for P; this result generalizes the ones given in [5,6]

At last , let us note that the results obtained in this last section point out once more how the image space can be viewed as a general framework within which different kinds of optimality conditions can be obtained .

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