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ON NONLINEAR SCALARIZATION IN VECTOR OPTIMIZATION

A.Cambini, A.Marchi, L.Martein

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Abstract

For a multiobjective concave optimization problem P linear scalarization holds in the sense that an efficient point for problem P turns out to be an optimal solution of a scalar problem whose objective function is a suitable weighted sum of objective functions of P.

Since this nice property does not hold when P is not concave, in this paper we will consider a scalar parametric problem of exponential kind P(λ,μ) with two parameters λ,μ and we will find conditions under which an efficient point for P is an optimal solution for P(λ,μ).

The suggested approach based on separation between two suitable sets allow us to obtain nonlinear scalarization for wide classes of problems containing some subclasses of generalized concave problem.

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** Department of Statistics and Applied Mathematics.
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1. Introduction

In multiobjective concave optimization, linear scalarization allows to characterize an efficient point as an optimal solution of a scalar problem whose objective function is a suitable weighted sum of the objective functions of the multiobjective problem P. Linear scalarization is strictly related to the existence of a hyperplane which separates the nonnegative orthant and the so called "conic extension"; such a hyperplane does not exist when the conic extension is not convex, so that linear scalarization does not hold.

The idea of this paper is to consider a non linear separation function, that is a function whose nonnegative levels contain the nonnegative orthant and whose nonpositive levels contain the "conic extension". Such a separation function will play the role of the objective function of a scalar problem P* and, furthermore, an efficient point for P turns out to be an optimal solution of P*. In this paper we propose a nonlinear separation function of exponential kind, which has the nice property that local separation implies global separation. Several theoretical results are obtained which allow us to characterize wide classes of multiobjective problems containing some subclasses of generalized concave ones for which nonlinear scalarization holds, that is problems for which an efficient point is also an optimal solution of the scalar problem whose objective function is of exponential kind.

2. Statement of the problem

Consider the following vector optimization problem:

$$\max\ (\varphi_1(x), \varphi_2(x), \ldots, \varphi_s(x)), \quad x \in X$$

where $\varphi_i : A \subset \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, s$ are continuous functions defined on the open set A containing the compact set X.

Let x^0 be a feasible point and set $f_i(x) = \varphi_i(x)-\varphi_i(x^0)$, $i=1, 2,...., s$. Taking into account these positions in the following we will refer to the multiobjective problem:

$$P : \max f(x)=(f_1(x), f_2(x),...., f_s(x)), \quad x \in X$$

It is well known that when P is a concave multiobjective problem, the conic extension $\mathbf{K}- \mathbb{R}^s_+$, with $\mathbf{K}=f(X)$, is a convex set; furthermore if x^0 is an efficient point for P, then the two convex sets $\mathbf{K}- \mathbb{R}^s_+$ and $\mathbb{R}^s_+$ can be linearly separated, that is there exists a linear function $l: \mathbb{R}^s \to \mathbb{R}$, $l(u) =\lambda^* u=\sum_{i=1}^{s} \lambda_i^* u_i$ such that $\lambda^* u \geq 0 \ \forall u \in \mathbb{R}^s_+$ and $\lambda^* u \leq 0 \ \forall u \in \mathbf{K}$ or, equivalently,

(2.1) $$\max_{u \in \mathbf{K}} \lambda^* u = \max_{x \in X} \lambda^* f(x) = 0 \leq \lambda^* u \quad \forall u \in \mathbb{R}^s_+.$$

In other words, linear separation (that is the existence of a hyperplane which separates $\mathbf{K}$ and $\mathbb{R}^s_+$) implies linear scalarization in the sense that an efficient point x^0 turns out to be an optimal solution for the following scalar problem

$$P(\lambda) : \max \sum_{i=1}^{s} \lambda_i f_i(x), x \in X.$$

Linear separation can be reinterpreted as the existence of an element l_{λ^*} in the set $L=\{l_\lambda : \mathbb{R}^s \to \mathbb{R}, l_\lambda(u) = \lambda u, \lambda \geq 0$[1]$\}$ which satisfies (2.1).

In order to be able to study vector optimization problem when the conic extension is not a convex set, we can generalize this kind of approach substituting L with a class of nonlinear functions whose nonnegative levels contain $\mathbb{R}^s_+$. More exactly, we consider the class of functions

$$W = \{w_{\lambda\mu} : \mathbb{R}^s \to \mathbb{R}, \lambda,\mu \in \mathbb{R}^s_+\}$$

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[1] Let $a= (a_1,a_2,..,a_k)$; $a \geq 0$ means $a \neq 0$ and $a_j \geq 0$, $j=1,..,k$, while $a \geqq 0$ means $a_j \geq 0, j=1,..,k$.

where

(2.2a) $\qquad w_{\lambda\mu}(0) = 0 \quad \forall\lambda \geq 0, \ \forall\mu \geqq 0$

(2.2b) $\qquad w_{\lambda\mu}(u) > 0 \quad \forall u \in \mathbb{R}^s_+ \setminus \{0\}, \forall\lambda \geq 0, \ \forall\mu \geqq 0.$

The existence of a function $w_{\lambda^*\mu^*} \in W$ satisfying (2.2.c) with

(2.2c) $\qquad w_{\lambda^*\mu^*}(u) \leq 0 \quad \forall u \in \mathbf{K}.$

implies $\max\limits_{u \in \mathbf{K}} w_{\lambda^*\mu^*}(u) = \max\limits_{x \in X} w_{\lambda^*\mu^*}(f(x)) = 0$,

so that the efficient point x^0 turns out to be an optimal solution of the scalar problem

$$P(\lambda^*, \mu^*) : \max \ w_{\lambda^*\mu^*}(\ f(x)\ ), \quad x \in X.$$

We will refer to $w_{\lambda^*\mu^*}$ as a separation function.

In choosing a class W of functions verifying (2.2a), (2.2b), we suggest the following one whose elements are of the exponential kind:

(2.3) $$w_{\lambda\mu}(u) = \sum_{i=1}^{s} \lambda_i u_i e^{-\mu_i u_i}, \ \lambda \geq 0, \mu \geqq 0$$

This choice is motivated by the properties stated in the following theorem.

Theorem 2.1:

i) If $0 \leq \mu^1 \leq \mu^2$ then $\Gamma^+_{\lambda\mu_1} \supset \Gamma^+_{\lambda\mu_2}$;

ii) For any $\lambda > 0$, $\bigcap\limits_{\mu \geq 0} \Gamma^+_{\lambda\mu} = \mathbb{R}^s_+$,

where $\Gamma^+_{\lambda\mu} = \{u : w_{\lambda\mu}(u) \geq 0\}$.

Proof: i) Since $\mu^1 \leq \mu^2$ implies $\mu^1_i \leq \mu^2_i$ and $\exists$ j such that $\mu^1_j < \mu^2_j$, we have: $\lambda_i \ u_i \ e^{-\mu^1_i u_i} \geq \lambda_i \ u_i \ e^{-\mu^2_i u_i}$, $\forall \ i \neq j$ and $\lambda_j \ u_j e^{-\mu^1_j u_j} > \lambda_j u_j \ e^{-\mu^2_j u_j}$, so that $\sum\limits_{i=1}^{s} \lambda_i \ u_i \ e^{-\mu^1_i u_i} > \sum\limits_{i=1}^{s} \lambda_i \ u_i \ e^{-\mu^2_i u_i}$, $\forall \ u \geq 0$;

ii) By the definition of the class obviously we have: $\bigcap\limits_{\mu \geq 0} \Gamma^+_{\lambda\mu} \supset \mathbb{R}^s_+$. On the other hand, when u has at least one negative component (for instance $u_k < 0$), choosing $\bar{\mu} \geq 0$ such that $\bar{\mu}_i=0$, $\forall$ $i \neq k$ and $\bar{\mu}_k>0$ we have $u \notin \Gamma^+_{\lambda\bar{\mu}}$. This completes the proof. ♦

The following theorem will allow us to consider the class of functions W with a scalar exponential parameter instead of a vector exponential parameter.

Theorem 2.2: If $w_{\lambda^*\mu^*}$ is a separation function, then $w_{\lambda^*\mu}$ is a separation function for any $\mu \geq \mu^*$.

Proof: Since $\mu_i \geq \mu_i^*$ $\forall i$, we have $e^{-\mu_i u_i} \leq e^{-\mu_i^* u_i}$ for every i such that $u_i \geq 0$ and $e^{-\mu_i u_i} \geq e^{-\mu_i^* u_i}$ for every i such that $u_i < 0$; as a consequence it results $\sum_{i=1}^{s} \lambda_i \; u_i \; e^{-\mu_i u_i} \leq \sum_{i=1}^{s} \lambda_i \; u_i \; e^{-\mu_i^* u_i} \leq 0$ so that (2.3) holds and this completes the proof. ♦

Remark 2.1: If $w_{\lambda^*\mu^*}$ is a separation function, setting $\mu' = \max_i \mu_i^*$ and $\mu=(\mu',...,\mu')^T$, Theorem 2.2 implies that $w_{\lambda^*\mu}$ is a separation function too. For this reason from now on, we will consider the class of functions (2.3) where the components of the exponential parameter are equal to each other, that is:

$$(2.4) \qquad w_{\lambda^*\mu^*}(u) = \sum_{i=1}^{s} \lambda_i \; u_i \; e^{-\mu \, u_i}, \;\; \lambda \geq 0, \; \mu \geq 0.$$

3. Exponential scalarization

As outlined in the previous section, our aim is to find conditions under which there exist $\lambda \in \mathbb{R}^s_+$, $\lambda \neq 0$, $\mu \in \mathbb{R}_+$, such that an efficient point x^0 for problem P is also an optimal solution for the scalar problem:

$$P(\lambda,\mu) : \max \; w_{\lambda\mu}(f(x)) = \sum_{i=1}^{s} \lambda_i \; f_i(x) \; e^{-\mu f_i(x)}, \;\; x \in X;$$

taking into account that $f_i(x^0) = 0$, $i = 1,...,s$, this is equivalent to find conditions under which there exists a function in the class W which ensures separation, that is such that the following inequality is verified:

(3.1) $$w_{\lambda\mu}(f(x)) \leq 0, \; x \in X.$$

From now on we assume that f_i, $i=1,2,...,s$ are differentiable functions.

Let us note that the efficiency of x^0 implies that x^0 is an optimal solution for any scalar problem P_i, $i=1,2,..,s$:

$$P_i : \begin{cases} \max f_i(x) \\ f_j(x) \geq 0 \quad j=1,...,s \quad j \neq i \\ x \in X \end{cases}$$

Applying Fritz John conditions to problem P_i we find $\bar{\lambda} \geq 0$ such that

(3.2) $$\sum_{i=1}^{s} \bar{\lambda}_i \nabla f_i(x^0) = 0.$$

We will refer to any vector $\bar{\lambda}$ satisfying (3.2) as a vector of Lagrange multipliers associated with x^0.

Since $\nabla w_{\lambda\mu}(f(x)) = \sum_{i=1}^{s} \lambda_i \, (1-\mu f_i(x)) \, e^{-\mu f_i(x)} \nabla f_i(x)$

and $f_i(x^0)=0$, $i=1,2..,s$, x^0 turns out to be a critical point for the function $w_{\bar{\lambda}\mu}$ $\forall \mu \in \mathbb{R}$ and, in particular, for the function:

(3.3) $$w_{\bar{\lambda}0}(f(x)) = \sum_{i=1}^{s} \bar{\lambda}_i f_i(x)$$

In other words, in the suggested approach, in order to find λ,μ such that x^0 is an optimal solution of the scalar problem $P(\lambda,\mu)$, it is sufficient to find the value of the scalar exponential parameter μ, since a vector $\bar{\lambda}$ is associated, in a natural way by means of Fritz John conditions, with the efficient point x^0.

3.1 Linear separation

The previous remarks, together with (3.3),(3.2), allow us to characterize some classes of problems for which linear scalarization holds. Taking into account that for any class of functions for which the critical point x^0 is an optimal solution for problem $P(\bar{\lambda},0)$ we have the validity of (3.1) with $\lambda=\bar{\lambda}$, $\mu=0$.

From known properties of generalized concave functions we have the following theorem:

Theorem 3.1: Let x^0 be an efficient point for the differentiable problem P, where X is a convex set and let $\bar{\lambda}$ be a vector of Lagrange multipliers associated with x^0. If one of the following conditions holds:

i) $w_{\bar{\lambda}0}$ is an incave function;

ii) $w_{\bar{\lambda}0}$ is a pseudoconcave function;

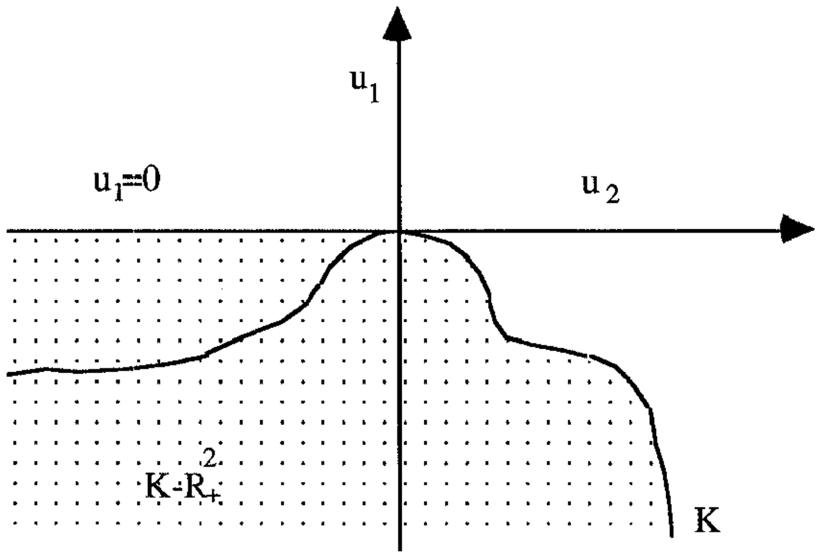
iii) $w_{\bar{\lambda}0}$ is a concave function;

iv) f_i is a concave function, $i=1,2,...,s$;

then x^0 is an optimal solution for the scalar problem $P(\bar{\lambda},0)$.

The following example shows that we have linear separation when the conic extension $\mathbf{K}\text{-}\mathbb{R}^s_+$ is not a convex set but a generalized concavity assumption holds.

Example 3.1: Consider problem P, where $A=\mathbb{R}$, $X=[-1,1]$ and $\varphi_1(x)=-1-\sqrt[3]{3x-1}\cdot e^x$, $\varphi_2(x)=x$. It is easy to verify that $x^0=0$ is an efficient point for P and that (3.2) is verified for $\bar{\lambda}=(1,0)$. The following picture shows that $\mathbf{K}\text{-}\mathbb{R}^2_+$ is not convex, but, since the function $w_{\bar{\lambda}0}(f(x))=\varphi_1(x)$ is an incave function, according to i) of Theorem 3.1, there exists a hyperplane, whose equation is $u_1=0$, which separates $\mathbf{K}$ and $\mathbb{R}^2_+$.



In the following we will see the role played by the scalar exponential parameter μ in finding a separation function of exponential kind instead of a linear separation function when assumptions of Theorem 3.1 are not satisfied.

The idea of the suggested approach is to study, first of all, conditions which ensure local separation, that is conditions under which there exists $\mu^*>0$ such that $w_{\lambda^*\mu^*}(f(x)) \leq 0$ for any x belonging to a suitable neighbourhood of x^0 and, successively, to increase the value of the exponential parameter in such a way global separation holds.

With this aim in the next we will study local exponential separation.

3.2 Local esponential separation

Now we will find conditions which ensure the existence of $\bar{\lambda} \geq 0$, $\bar{\mu} \in \mathbb{R}_+$ such that (3.1) holds in a suitable neighbourhood I_{x^0} of x^0, i.e.

(3.4) $$w_{\bar{\lambda}\bar{\mu}}(f(x)) \leq 0, \ \forall x \in I_{x^0}$$

where the functions f_i i=1,2,..s, are twice continuosly differentiable and $\bar{\lambda}$ is a vector of Lagrange multipliers associated with x^0. Let us note that (3.4) implies that x^0 is a local maximum point for problem $P(\overline{\lambda},\overline{\mu})$.

Since x^0 is a critical point for $w_{\overline{\lambda}\mu}$, $\forall \mu \in \mathbb{R}_+$, a sufficient condition for the validity of (3.4) is the existence of $\overline{\mu} \in \mathbb{R}_+$ such that the Hessian matrix $H_{\overline{\mu}}$ of function $w_{\overline{\lambda}\,\overline{\mu}}$ is definite negative at x^0. It is easy to verify that such a matrix H_μ has the following form:

$$H_\mu = \sum_{i=1}^{s} \overline{\lambda}_i H_i - 2\mu \sum_{i=1}^{s} \overline{\lambda}_i (\nabla f_i(x^0)^T \nabla f_i(x^0))$$

where H_i is the Hessian matrix, evaluated at x^0, of function f_i, $i=1,2,\dots,s$; the quadratic form associated with H_μ is

$$z^T H_\mu z = \sum_{i=1}^{s} \overline{\lambda}_i (z^T H_i z) - 2\mu \sum_{i=1}^{s} \overline{\lambda}_i (z^T \nabla f_i(x^0))^2 =$$

$$= \sum_{i\in I^*} \overline{\lambda}_i (z^T H_i z) - 2\mu \sum_{i\in I^*} \overline{\lambda}_i (z^T \nabla f_i(x^0))^2$$

where $I^*=\{i: \overline{\lambda}_i>0\}$ (Let us note that $I^*\neq\emptyset$ from (3.2)).

If $\sum_{i\in I^*} \overline{\lambda}_i (z^T H_i z)<0\ \forall z \neq 0$ then H_μ is definite negative $\forall \mu \in \mathbb{R}_+$ so that choosing $\mu=0$ we have linear separation once again; otherwise, set $Z(x)=\{z\in \mathbb{R}^n: z^T \nabla f_i(x) = 0,\ \forall\, i\in I^*\}$.

The following theorem gives conditions which ensure the definite negativity of the Hessian matrix:

Theorem 3.2:

i) If $Z(x^0)=\{0\}$, then there exists $\overline{\mu}>0$ such that $H_{\overline{\mu}}$ is definite negative;

ii) If $Z(x^0)\neq\{0\}$ and $\sum_{i\in I^*} \overline{\lambda}_i (z^T H_i z)<0\ \forall z\in Z(x^0)$, $z\neq 0$ then there exists $\overline{\mu}>0$ such that $H_{\overline{\mu}}$ is definite negative.

Proof: We must prove that there exists $\overline{\mu}>0$ such that:

(3.5) $$\sum_{i\in I^*} \overline{\lambda}_i (z^T H_i z) - 2\overline{\mu} \sum_{i\in I^*} \overline{\lambda}_i (z^T \nabla f_i(x^0))^2 <0\ \forall z\neq 0.$$

Set $\Psi(z) = \dfrac{\sum_{i\in I^*}\bar\lambda_i(z^T H_i z)}{2\sum_{i\in I^*}\bar\lambda_i(z^T\nabla f_i(x^0))^2}$ and $L(x^0) = \sup_{z\notin Z(x^0)} \Psi(z) = \sup_{\substack{z\notin Z(x^0)\\ \|z\|=1}} \Psi(z)$

i) Since $Z(x^0)=\{0\}$, $\Psi(z)$ is a continuous function on the unit ball $B = \{z\in \mathbb{R}^n: \|z\|=1\}$ so that $L(x^0)$ is reached as a maximum; it is easy to verify that the theorem holds for any $\mu \geq L(x^0)$ and hence for a suitable $\bar\mu > 0$;

ii) It is sufficient to prove that $L(x^0)$ is finite. When z belongs to $Z(x^0)\setminus\{0\}$, the assumption implies the validity of (3.5) for any $\mu\geq 0$, while when z does not belong to $Z(x^0)$, (3.5) is verified if $\mu>\Psi(z)$ that is if $\mu>L(x^0)$; so that if $L(x^0)$ is finite there exists a suitable $\mu>0$ verifying (3.5), $\forall z\neq 0$. Now it remains to prove that $L(x^0)$ is finite. Consider a sequence $\{z_n\}\subset B$ such that $\lim_{n\to+\infty} \Psi(z_n) = L(x^0)$. Since $\{z_n\}$ is a bounded sequence there exists a subsequence which converges to $z_0\in B$; we can suppose, without loss of generality (substituting $\{z_n\}$ with a suitable subsequence, if necessary) that $z_n\to z_0$. The thesis is achieved if $z_0 \notin Z(x^0)$ since, in such a case, $L(x^0)=\Psi(z_0)$. If $z_0 \in Z(x^0)$, then $\bar\lambda_i(z^T H_i z)<0$ so that there exists an index $\bar n$ such that $\bar\lambda_i(z_n^T H_i z_n) < 0$ $\forall n>\bar n$ and necessarily we have $L(x^0)=-\infty$ and this is absurd. This completes the proof. ♦

Remark 3.1: As regard assumption stated in i) of Theorem 3.2, let us note that $Z(x^0)=\{0\}$ if and only if rank $\{\nabla f_i(x^0), \forall i\in I^*\} = n$. As a consequence if the number of the objective functions is greater than the number n of the indipendent variables and if there exist n linearly indipendent gradients at x^0 then we have local separation.

Remark 3.2: The assumption stated in ii) of Theorem 3.2 points out that local separation is related to the behaviour of the restriction of the quadratic form associated with the Hessian matrix at x^0 of the function $\sum_{i\in I^*} \overline{\lambda}_i f_i(x)$ on the linear subspace $Z(x^0)$. As a consequence the assumption becomes weaker and weaker to the decreasing of the dimension of $Z(x^0)$.

The following Corollary characterizes a class of generalized concave multiobjective problems satisfying assumptions of Theorem 3.2.

Corollary 3.1: If $Z(x^0)\neq \{0\}$ and f_i, $i\in I^*$ are twice continuously differentiable quasi-concave functions with at least one strongly quasi-concave[2] then there exists $\overline{\mu} > 0$ such that $H_{\overline{\mu}}$ is definite negative.

Proof: The generalized concavity assumptions imply [2] $\sum_{i\in I^*} \overline{\lambda}_i (z^T H_i z)<0$ $\forall z \in Z(x^0)$, $z\neq 0$. The thesis follows from ii) of Theorem 3.2. ♦

The following example shows that the quasi-concavity of all functions f_i is not a sufficient condition to have an exponential separation function.

Example 3.2: Consider problem P, where $A=\mathbb{R}$, $X=[-1,1]$, $f_1(x)=x^3$ and $f_2(x)= =-x$. It is easy to verify that $x^0=0$ is an efficient point for problem P and (3.2) is verified for $\overline{\lambda}_2=0$ and $\overline{\lambda}_1>0$. As a consequence, we have $Z(0)=\mathbb{R}$ and $z^T H_\mu z$ is equal to zero for any $\mu\in \mathbb{R}$, so that there does not exist a $\mu \geq 0$ such that (3.4) is verified.

4. Global exponential separation

We have just outlined that in order to achieve global separation, we must require local separation. Since the image under the function f of a neighbourhood of x^0 is not equal, in general, to a neighbourhood of $f(x^0)$ in the image space, the following two conditions (4.1), (4.2) are not equivalent:

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[2] We recall that a twice continuously differentiable function h defined over an open set S is strongly quasi-concave if and only if $x^0\in S$, $\|v\|=1$, $v^T\nabla h(x^0)=0$ implies $v^T H_h v<0$.

(4.1) $\quad w_{\bar{\lambda}\,\bar{\mu}}(f(x)) \le 0, \ \forall\, x \in I_{x^0}$

(4.2) $\quad w_{\bar{\lambda}\,\bar{\mu}}(f(x)) \le 0, \ \forall\, u \in U_0 \cap \mathbf{K}$

where U_0 is a suitable neighbourhood of $f(x^0)$. As a consequence (4.1) does not imply that $w_{\bar{\lambda}\,\bar{\mu}}$ separates **K** and $\mathbb{R}^s_+$ locally.

The following example points out that some difficulties arise in achieving local separation in the image space:

Example 4.1: Consider problem P where $f_1(x)=x^2-2x$ and $f_2(x)=x(2-x)^3$, $x \in X=[-2,4]$. It is easy to verify that $x^0=0$ and $x^*=2$ are efficient points such that $f(x^0)=f(x^*)=(0,0)$; furthermore at $x^0=0$ we have local separation with $\lambda^*=(4,1)$, $\mu^*=0$, while at $x^*=2$ we do not have local separation since $\lambda^*=(0,1)$, $w_{\lambda^*\mu}(f(x))=-x(x-2)^3e^{\mu x(x-2)^3}$, so that $w_{\lambda^*\mu}(f(2))=0$ and in each neighbourhood of $x^*=2$ the function $w_{\lambda^*\mu}(f(x))$ assumes positive and negative values for any μ.

In order to achieve local separation in the image space, we must require, as pointed out in the previous example, to have local separation for any point of the subset $E(x^0)$ of efficient points:

$$E(x^0)=\{x \in X: f_i(x)=f_i(x^0),\ i=1,2,..s\}.$$

Such a requirement guarantees local separation in the image space as is stated in the following theorem:

Theorem 4.1: If for any $x \in E(x^0)$, we have $Z(x)=\{0\}$ or $\sum_{i \in I^*} \bar{\lambda}_i (z^T H_i z)<0$, $\forall z \in Z(x^0)$, $z \neq 0$, then there exists $\bar{\mu}>0$ verifying (4.2).

Proof: Set $L= \sup_{x \in E(x^0)} L(x) = \sup_{x \in E(x^0)} \ \sup_{\substack{z \notin Z(x) \\ \|z\|=1}} \dfrac{\sum_{i \in I^*} \bar{\lambda}_i (z^T H_i z)}{2 \sum_{i \in I^*} \bar{\lambda}_i (z^T \nabla f_i (x^0))^2}$.

It is sufficient to prove that L is finite, since in such a case it is easy to verify that any $\mu>L$ satisfies (3.2). Consider the unit ball B and the sequences

$\{x_n\}\subset E(x^0)$, $\{z_n\}\subset B$, $z_n\notin Z(x_n)$ such that $L=\lim\limits_{n\to+\infty}\dfrac{\sum\limits_{i\in I^*}\bar\lambda_i(z_n^T H_i z_n)}{2\sum\limits_{i\in I^*}\bar\lambda_i(z_n^T\nabla f_i(x_n))^2}$. Let us note that $E(x^0)$ is a compact set, since the functions f_i, $i=1,2,..,s$ are continuous and X is compact; as a consequence, taking into account that B is compact too, there exist a subsequence of $\{x_n\}$ and a subsequence of $\{z_n\}$ converging to $\bar{x}\in E(x^0)$ and $z_0\in B$ respectively. If $\sum\limits_{i\in I^*}\bar\lambda_i(z_0^T\nabla f_i(\bar{x}))^2=0$ then $z_0\in Z(\bar{x})$, which implies $\sum\limits_{i\in I^*}\bar\lambda_i(z_0^T H_i(\bar{x})z_0)<0$ so that $L=-\infty$ and this is absurd. As a consequence, necessarily we have $\sum\limits_{i\in I^*}\bar\lambda_i(z_0^T\nabla f_i(\bar{x}))^2\neq 0$ and $L=\Psi(z_0)$. ♦

The following theorem points out that the choice of the exponential class of functions $w_{\lambda\mu}$ allows us to state that if there exists a local separation function then there exists also a global separation function.

Theorem 4.2: If there exist $\bar\lambda,\bar\mu\geq 0$ such that $w_{\bar\lambda\bar\mu}(u)\leq 0$, $\forall u\in U_0\cap\mathbf{K}$, then there exists $\mu^*>0$ such that (4.3) holds:

(4.3) $\qquad w_{\bar\lambda\mu^*}(u)\leq 0,\ \forall u\in\mathbf{K}.$

Proof: Suppose that there does not exist μ^* verifying (4.3). Then there exist $\{\mu_n\}\subset \mathbb{R}_+$, $\{u^n=(u_1^n,\ldots,u_s^n)\}\subset\mathbf{K}$ such that $\mu_n\to+\infty$, $u^n\in\Gamma^+_{\bar\lambda\mu_n}$. Since X is a compact set and f is continuous, then $\mathbf{K}=f(X)$ is compact too, so that there exists a subsequence of $\{u^n\}$ which converges to $\bar{u}$. We prove that $\bar{u}\in Fr\mathbb{R}^s_+$. If $\bar{u}\notin Fr\mathbb{R}^s_+$, then there exists an index k such that $\bar{u}_k<0$, so that $\exists\ \bar{n}: \forall n>\bar{n},\ u_k^n<0$; as a consequence, $\lim\limits_{n\to+\infty}\sum\limits_{i\in I^*}\bar\lambda_k\ u_k^n\ e^{-\mu_n u_k^n}=-\infty$ and $\lim\limits_{n\to+\infty}\sum\limits_{i\in I^*}\bar\lambda_i\ u_i^n\ e^{-\mu_n u_i^n}=-\infty$ and this contradicts $u^n\in\Gamma^+_{\bar\lambda\mu_n}$. Since

$K \cap \mathbb{R}^s_+ = \{0\}$, necessarily we have $\bar{u}=0$ so that $\exists\ \bar{n}: \forall\ n>\bar{n}$, u^n belongs to a neighbourhood of the origin and this absurd since $w_{\bar{\lambda}\bar{\mu}}$ separates K and $\mathbb{R}^s_+$ locally. ♦

The obtained results allow us to characterize classes of multiobjective problems for which an efficient point is also an optimal solution of a scalar parametric problem whose objective function is of exponential kind.

Theorem 4.3: Let x^0 be an efficient point for the vector problem P where the objective functions are twice continuosly differentiable and let $\bar{\lambda}$ be a vector of Lagrange multipliers associated with x^0. If for any $x \in E(x^0)$, we have $Z(x)=\{0\}$ or $\sum_{i\in I^*} \bar{\lambda}_i (z^T H_i z)<0\ \forall z \in Z(x),\ z\neq 0$, then there exists $\bar{\mu}>0$ such that x^0 is an optimal solution for scalar problem

$$P(\bar{\lambda},\bar{\mu}) : \max w_{\bar{\lambda}\bar{\mu}}(f(x)) = \sum_{i=1}^{s} \bar{\lambda}_i\ f_i(x)\ e^{-\bar{\mu} f_i(x)},\ x \in X;$$

Corollary 4.1: Let x^0 be an efficient point for the vector problem P where X is a convex set, the objective functions are twice continuosly differentiable quasi-concave with at least one strongly quasi-concave and let $\bar{\lambda}$ be a vector of Lagrange multipliers associated with x^0. Then there exists $\bar{\mu}>0$ such that x^0 is an optimal solution for scalar problem $P(\bar{\lambda},\bar{\mu})$.

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