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Some optimality conditions in multiobjective programming

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Abstract

One of the aim of this paper is to introduce new classes of vector generalized concave functions and to point out their role in investigating local and global efficiency and in establishing sufficient optimality conditions for a vector optimization problem. Another aim is to stress the role of the Bouligand tangent cone at a point of the feasible region in deriving optimality conditions.

1 Introduction

In these last years several of articles dealing with scalar generalized concavity have appeared in scientific journals and numerous textbooks have specific chapters in this subject. On the contrary the role of vector generalized concavity in multiobjective optimization is not yet sufficiently explored; only occasionally, with the aim to extend to the vector case some properties of scalar generalized concavity, some authors have considered, in the Paretian case, componentwise generalized concavity or, in studying specific topics like as the connectedness of the set of all efficient points, have defined some special classes of vector generalized concave functions with respect to a cone [11,12,13,15,19,20].

For such a reason some classes of generalized concave multiobjective functions with their properties have been recently introduced and studied [4,5,7,8,9,16,17].

First of all in this paper, taking into account the results obtained in [8,9], we will introduce some classes of vector generalized concave functions pointing out their role in vector optimization, and successively we will investigate local and global efficiency and we will state several necessary and/or sufficient optimality conditions established by means of the Bouligand tangent cone to the feasible region at a point.

The obtained results generalize and extend the ones given in [4].

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2 Some classes of generalized concave multiobjective functions

In finding conditions under which a local maximum point is also global or a necessary optimality condition becomes sufficient too, an important role is played by the concept of generalized concavity at a point introduced in [14] for a scalar optimization problem.

Let us note that there are different way in generalizing to the vector case the definitions of generalized concave functions given in the scalar case. In this section, following [8,9], we introduce wide classes of vector generalized concave function which are more general than the ones suggested in [4,5,16].

With this aim, let us consider an open set X of the n-dimensional space $\mathbf{R}^n$, a function $F:X\to \mathbf{R}^s$ and a non trivial cone $U\subset \mathbf{R}^s$ with vertex at the origin $0\in U$ and with non empty interior, i.e. $intU\neq\emptyset$. Set $U^0= U\setminus\{0\}$.

We recall that a set $S\subset X$ is said to be <u>star shaped</u> at $x_0\in S$ if $x\in S$ implies

$$[x , x_0] = \{tx+(1-t)x_0 :\ t\in [0,1]\} \subset S.$$

Definition 2.1

Consider the cones $U^*,U^\#\in \{U, U^0, intU\}$.

The function F is said to be $(U^*,U^\#)$-<u>quasiconcave</u> (shortly $(U^*,U^\#)$-qcv) at x_0 (with respect to the star shaped set S at x_0) if:

$$x\in S\ ,\ x\neq x_0,\ F(x)\in F(x_0)+U^* \Rightarrow F(x_0+\lambda(x-x_0))\in F(x_0)+U^\# \quad \forall\lambda\in(0,1)$$

Let us note that when $U^*=U^\#=U$ we obtain the definition of a vector quasiconcave function given by Yahn [11], while when $U^*=U^\#=U^0$ we obtain the class of U-semistrictly quasiconcave vector functions introduced by Martein [16,17].

In the scalar case (s=1) Definition 2.1 reduces to the classical definition of a quasiconcave, strictly quasiconcave, semistrictly quasiconcave function when $U^*=U^\#=U=\mathbf{R}_+$, $U^*=\mathbf{R}_+$ $U^\#=\mathbf{R}_{++}$, $U^*=U^\#=\mathbf{R}_{++}$ respectively, where $\mathbf{R}_+$ is the set of the non negative real numbers and $\mathbf{R}_{++}$ is the set of the positive real numbers.

Definition 2.2

Consider the cones $U^*\in\{U, U^0, intU\}$, $U^\#\in\{U^0, intU\}$and let F be directionally differentiable at x_0.

F is said to be $(U^*,U^\#)$-<u>pseudoconcave</u> (shortly $(U^*,U^\#)$-pcv) at x_0 (with respect to the star shaped set S at x_0) if:

$$x\in S\ ,\ x\neq x_0,\ F(x)\in F(x_0)+U^* \Rightarrow \frac{\partial F}{\partial d}(x_0)\in U^\#,\ \ d=\frac{x-x_0}{\|x-x_0\|}$$

When $U^*=U^\#=U^0$ we obtain the class of U-weakly pseudoconcave vector functions introduced by Martein [16,17], while when $U^*=U^0$, $U^\#=\text{int}U$, we obtain the class of U-pseudoconcave vector functions introduced by Cambini-Martein [5]. Let us note that in the scalar case these last two definitions collapse to the ordinary definition of a pseudoconcave function while a (U, intU)-pcv collapse to the ordinary definition of a strictly pseudoconcave function.

Relationships among the defined classes of generalized concave functions are studied in [9].

3 Local and global efficiency

Consider the following vector optimization problem:

$$P : U\text{-max } F(x) , x \in S \subseteq X$$

where X is an open set of $\mathbf{R}^n$, $F : X \to \mathbf{R}^s$, and $U \subset \mathbf{R}^s$ is a non trivial cone with vertex at the origin $0 \in U$ and with nonempty interior.

Set $U^0= U \backslash \{0\}$.

We recall that a feasible point x_0 is said to be:

- weakly efficient if

$$F(x) \notin F(x_0) + \text{int}U, \qquad \forall x \in S \tag{3.1a}$$

- efficient if

$$F(x) \notin F(x_0) + U^0, \qquad \forall x \in S \tag{3.1b}$$

- strictly efficient if

$$F(x) \notin F(x_0) + U, \qquad \forall x \in S \tag{3.1c}$$

If (2.1a), (2.1b), (2.1c), are verified in $I \cap S$, where I is a suitable neighbourhood of x_0, the feasible point x_0 is said to be a local weak efficient point, a local efficient point and a local strict efficient point, respectively.

Obviously a (local) strict efficient point is also a (local) efficient point, and a (local) efficient point is also a (local) weak efficient point.

Let us note that in the scalar case ($s=1$, $U=\mathbf{R}_+$) (2.1a) and (2.1b) collapse to the ordinary definitions of a maximum point and (2.1c) collapse to the ordinary definition of a strict maximum point, while when $U=\mathbf{R}^s_+$, problem P reduces to a vector Pareto problem.

Let us note that definitions (3.1) can be rewritten in a unified way as follows:
let $U^* \in \{U, U^0, \text{int}U\}$; x_0 <u>is said to be a (local) U^*-efficient point</u> if

$$F(x) \notin F(x_0)+U^*, \qquad \forall x \in S \quad (\forall x \in I \cap S) \tag{3.2}$$

When $U^*=\text{int}U$, $U^*=U^0$, $U^*=U$, (3.2) collapses to definition (3.1a), (3.1b), (3.1.c), respectively.
The classes of generalized concave functions introduced in section 2, allow us to investigate relationships between local and global U^*-efficiency.
The following theorem holds:

Theorem 3.1
Consider problem P where S is a star shaped set at x_0 and F is $(U^*,U^\#)$-qcv) at x_0.
If x_0 is a local $U^\#$-efficient point, then x_0 is also U^*-efficient for P.
<u>Proof.</u>
i) Ab absurdo suppose that there exists $x^* \in S$ such that $F(x^*) \in F(x_0)+U^*$. Since F is $(U^*,U^\#)$-qcv) at x_0, we have $F(x_0+\lambda(x^*-x_0)) \in F(x_0)+U^\# \quad \forall \lambda \in (0,1)$ and such a relation implies, choosing λ small enough, the non local $U^\#$-efficiency of x_0. ♦

Specifying the cones U^* and $U^\#$ we obtain Theorem 4.1 in [4] and some other different kinds of results. For instance:
- $U^*=U^\#=U^0$: if x_0 is a local efficient point and F is (U^0,U^0)-qcv at x_0, then x_0 is an efficient point for P;
- $U^*=U^\#=\text{int}U$: if x_0 is a local weak efficient point and F is $(\text{int}U,\text{int}U)$-qcv at x_0, then x_0 is a weak efficient point for P;
- $U^*=\text{int}U$, $U^\#=U^0$: if x_0 is a local weak efficient point and F is $(\text{int}U,U^0)$-qcv at x_0, then x_0 is an efficient point for P;
- $U^*=\text{int}U$, $U^\#=U$: if x_0 is a local weak efficient point and F is $(\text{int}U,U)$-qcv at x_0, then x_0 is a strict efficient point for P.

Recalling that a function F is said to be <u>U-concave</u> at x_0 (with respect to the star shaped set S at x_0) if

$$F(x_0+\lambda(x-x_0)) \in F(x_0)+\lambda(F(x)-F(x_0))+U \quad \forall\lambda\in (0, 1), \ \forall x\in S$$

we have the following:

Corollary 3.1

Let us consider problem P where S is star shaped at x_0, U is a convex pointed cone and F is U-concave at x_0. Then a local efficient point x_0 is an efficient point too.

<u>Proof.</u>

It follows from Theorems 3.1 taking into account that a U-concave function at x_0 is also (U^0,U^0)-qcv at x_0, when U is pointed and convex. ♦

Corollary 3.2

Let us consider problem P where S is star shaped at x_0, and F is linear.
Then a local efficient point x_0 is an efficient point too.

<u>Proof.</u>

It is sufficient to note that a linear function is also (U^0,U^0)-qcv at every point. ♦

Theorem 3.2

Consider problem P where S is a star shaped set at x_0 and F is $(U^*, \text{int}U)$-pcv at x_0.
If x_0 is a local U^*-efficient point, then x_0 is also U^*-efficient for P.

<u>Proof.</u>

Ab absurdo suppose that there exists $x^*\in S$, $x^*\neq x_0$, such that $F(x^*) \in F(x_0)+U^*$.
Since F is $(U^*, \text{int}U)$-pcv at x_0, we have
$\frac{\partial F}{\partial d}(x_0) \in \text{int}U$, $d=\frac{x^*-x_0}{\|x^*-x_0\|}$, that is $\lim_{t\to 0^+} \frac{F(x_0+td)-F(x_0)}{t} \in \text{int}U$ and this implies the existence of a suitable $\varepsilon>0$, such that $F(x_0+td)-F(x_0) \in \text{int}U \ \ \forall t\in (0, \varepsilon)$.
Set $t=\lambda\|x^*-x_0\|$; we have $F(x_0+\lambda(x^*-x_0)) \in F(x_0)+\text{int}U \ \ \forall \lambda\in (0, \frac{\varepsilon}{\|x^*-x_0\|})$ and this contradicts the local U^*-efficiency of x_0. ♦

Let us note that specifying the cone U^* in Theorem 2 we obtain different kinds of results.

4 Efficiency along a direction and efficiency

As is known, the property for which a local efficient point with respect to every feasible direction of a star shaped set is also a local efficient point for P does not hold for every function F . For such a reason in this section we investigate by means of vector generalized concavity the relationships between the local U*-efficiency of x_0 and the local U*-efficiency of x_0 with respect to all directions starting from x_0.

With this aim we give the following definition:

A point x_0 is said to be a local U*-efficient point along the direction $d=\frac{x-x_0}{\|x-x_0\|}$, $x \in S$ and with respect to the cone U if there exists t*>0 such that:

$$F(x) \notin F(x_0) + U^*, \ \forall\, x = x_0 + t\,d \ , t \in (0, t^*)$$

Let $D = \{d=\frac{x-x_0}{\|x-x_0\|}, x \in S\}$ be the set of feasible directions at $x_0 \in S$.

The following theorem holds:

Theorem 4.1

Let us consider problem P where S is star shaped at x_0 and F is (U*,U*)-qcv at x_0. Then x_0 is a local U*-efficient point if and only if x_0 is local U*-efficient for every direction $d \in D$.

Proof.

if. If x_0 is a local U*-efficient point obviously it is also local U*-efficient for every $d \in D$.

only if. Ab absurdo suppose that there exists $x^* \in S$ such that $F(x^*) \in F(x_0)+U^*$. Since F is (U*,U*)-qcv at x_0, we have $F(x_0+\lambda(x^*-x_0)) \in F(x_0)+U^* \quad \forall \lambda \in (0,1)$ and such a relation implies, choosing λ small enough, the non local U*-efficiency of x_0 with respect to the direction $d=\frac{x^*-x_0}{\|x^*-x_0\|} \in D$. ♦

Theorem 4.2

Let us consider problem P where F is directionally differentiable and (U*,U*)-qcv at x_0 and let S be star shaped at x_0.

Then x_0 is a local U*-efficient point if and only if the following conditions hold:

i) $\frac{\partial F}{\partial d}(x_0) \notin \text{int}U \ \forall d \in D$;

ii) x_0 is a local U*-efficient point for every direction $d \in D$ such that $\frac{\partial F}{\partial d}(x_0) \in clU \setminus intU$.

Proof.

if. Since for any $x \in S$, $F(x_0+\lambda(x-x_0))-F(x_0) \in U^*$ $\forall \lambda \in (0, 1)$, obviously we have $\frac{\partial F}{\partial d}(x_0) \notin intU$, $d=\frac{x^*-x_0}{\|x^*-x_0\|} \in D$. The thesis follows taking into account Theorem 4.1.

only if.

Ab absurdo suppose that there exists $x^* \in S$, $x^* \neq x_0$, such that $F(x^*) \in F(x_0)+U^*$. Since F is (U*,U*)-qcv at x_0, we have $F(x_0+\lambda(x-x_0))-F(x_0) \in U^*$ $\forall \lambda \in (0, 1)$, so that setting $d=\frac{x^*-x_0}{\|x^*-x_0\|}$ we have $\frac{\partial F}{\partial d}(x_0) \in clU$ and this is absurd since contradicts i) or ii).♦

Let us note, once again that specifying the cone U* in Theorems 4.1, 4.2, we obtain different kinds of results.

Some other characterizations can be found in [8].

5 Optimality conditions

In this section and in the following one, we state some necessary and/or sufficient first-order optimality conditions for problem P stated by means of a general approach involving the directions belonging to the Bouligand tangent cone to the feasible region at a point x_0.

With this aim we need of the following definition:

Definition 5.1

Let $G: A \rightarrow \mathbf{R}^k$ be a function defined in the open set $A \subseteq \mathbf{R}^n$.

G is said to be regular directionally differentiable at $x_0 \in A$ if $\lim_{h_n \to 0} \frac{h_n}{\|h_n\|} = d$ implies

$$\lim_{h_n \to 0} \frac{G(x_0+h_n)-G(x_0)}{\|h_n\|} = \lim_{t \to 0^+} \frac{G(x_0+td)-G(x_0)}{t} \triangleq \frac{\partial G}{\partial d}(x_0) .$$

where $\{h_n\}$ is a sequence of directions converging to O.

Classes of functions satisfying the above definition are stated in the following Property:

Property 5.1

i) if G is directionally differentiable and locally lipschitzian at x_0, then G is regular directionally differentiable at x_0;

ii) if G is differentiable at x_0, then G is regular directionally differentiable at x_0.

Proof.

i) Let $\{h_n\}$ be a sequence of directions converging to O with $\lim_{h_n\to 0}\frac{h_n}{\|h_n\|}=d$. Set $\frac{h_n}{\|h_n\|}=d_n$.

We have $\frac{G(x_0+h_n)-G(x_0)}{\|h_n\|}=\frac{G(x_0+\|h_n\|d_n)-G(x_0+\|h_n\|d)}{\|h_n\|}+\frac{G(x_0+\|h_n\|d)-G(x_0)}{\|h_n\|}$

Since G is locally lipschitzian at x_0 there exists L>0 such that

$\left\| \frac{G(x_0+\|h_n\|d_n)-G(x_0+\|h_n\|d)}{\|h_n\|} \right\| < L\,\|d_n-d\|$,

so that $\lim_{h_n\to 0}\frac{G(x_0+\|h_n\|d_n)-G(x_0+\|h_n\|d)}{\|h_n\|}=0$,

and consequently $\lim_{h_n\to 0}\frac{G(x_0+h_n)-G(x_0)}{\|h_n\|}=\lim_{h_n\to 0}\frac{G(x_0+\|h_n\|d)-G(x_0)}{\|h_n\|}=\frac{\partial G}{\partial d}(x_0)$.

ii) It is a direct consequence of the assumption of the differentiability of the function. ♦

Consider now the Bouligand tangent cone to the set S at $x_0\in S$, that is the set:

$$T(S,x_0)=\{v:\ \exists\ \{\alpha_n\}\subset \mathbf{R},\{x_n\}\subset S,\ \alpha_n\to+\infty,\ x_n\to x_0 \text{ with } \alpha_n(x_n-x_0)\to v\}.$$

Let us note that $T(S,x_0)=\{0\}$ if and only if x_0 is an isolated point and in such a case x_0 is obviously an efficient point for problem P. For this reason, throughout this paper, it is assumed that $T(S,x_0)\neq\{0\}$.

Now we give some extensions and generalizations of the results stated in [4].

The following theorem states a necessary optimality condition for any kind of local efficient point:

Theorem 5.1

Let x_0 be a local U*-efficient point for P.

i) if F is regular directionally differentiable at x_0, then:

$$\frac{\partial F}{\partial v}(x_0)\notin \text{int}U,\quad \forall\ v\in T(S,x_0),\quad v\neq 0. \tag{5.1}$$

ii) if F is directionally differentiable and locally lipschitzian at x_0, then (5.1) holds;

iii) if F is differentiable at x_0, then:

$$J_{F_{x_0}}(v) \notin \text{int}U\ , \quad \forall\ v \in T(S,x_0)\ , \quad v \neq 0. \tag{5.2}$$

Proof.

i) It is sufficient to prove (5.1) for every direction $v \in T(S,x_0)$ such that $\|v\|=1$; Let $\{x_n\} \subset S$, $x_n \to x_0$, be a sequence such that $\lim\limits_{x_n \to x_0} \frac{x_n - x_0}{\|x_n - x_0\|} = v$. Since F is regular directionally differentiable at x_0, we have:

$$\lim_{x_n \to x_0} \frac{F(x_n)-F(x_0)}{\|x_n - x_0\|} = \frac{\partial F}{\partial v}(x_0)\ ;$$

on the other hand the local U*-efficiency of x_0 implies:

$$\frac{F(x_n)-F(x_0)}{\|x_n - x_0\|} \notin \text{int}U \quad \forall\ n \ \text{ so that } \quad \lim_{x_n \to x_0} \frac{F(x_n)-F(x_0)}{\|x_n - x_0\|} = \frac{\partial F}{\partial v}(x_0) \notin \text{int}U\ .$$

ii), iii) follow from i) taking into account Property 5.1. ♦

Condition (5.1) is a necessary but not sufficient optimality condition; it is easy to verify that it becomes sufficient too for the classes of vector generalized pseudoconcave functions stated in the following Theorem:

Theorem 5.2

Let us consider problem P where S is locally star shaped at x_0 and F is (U*, intU)-pcv at x_0.

i) if F is regular directionally differentiable at x_0, then x_0 is a local U*-efficient point for P if and only if (5.1) holds;

ii) if F is directionally differentiable and locally lipschitzian at x_0, then x_0 is a local U*-efficient point for P if and only if (5.1) holds;

iii) if F is differentiable at x_0, then x_0 is a local U*-efficient point for P if and only if (5.2) holds.

Remark 5.1

Let $U^+=\{\alpha : \alpha^t u \geq 0\ , \forall\ u \in U\}$ be the positive polar cone of U.

Let us note that if in problem P the cone U is closed convex and pointed, S is a convex closed set and F is differentiable at x_0, then the Bouligand tangent cone $T(S,x_0)$ becomes a closed convex cone, so that by applying a separation Theorem, relation (5.2) implies the following condition:

$$\exists\ \alpha \in U^+ \setminus \{0\} \text{ such that } \alpha^t J_{F_{x_0}}(v) \leq 0 \quad \forall\ v \in T(S,x_0)\ , \quad v \neq 0$$

This last relation reduces to condition (5.3) when x_0 is an interior point:

$$\exists \ \alpha \in U^{+}\backslash\{0\} \text{ such that } \alpha^t J_{F_{x_0}} = 0 \qquad (5.3)$$

Condition (5.3) is a necessary but not sufficient condition for an interior point x_0 to be U*-efficient; it becomes sufficient too under suitable assumption of generalized concavity. To this regards the following theorem which generalizes the result given in [4] points out the different roles played by some classes of generalized concave functions:

Theorem 5.3

Let us consider the unconstrained problem P where S is a star shaped set and F is differentiable at x_0.

i) if F is (U*, intU)-pcv at x_0, then (5.3) becomes a sufficient condition for x_0 to be a local U*-efficient point;

ii) if F is (U*, U^0)-pcv at x_0, then (5.3) becomes a sufficient condition for x_0 to be a local U*-efficient point if $\alpha \in \text{int}U^+$.

Proof.

i) Ab absurdo suppose that there exists $x^* \in S$ such that $F(x^*) \in F(x_0)+U^*$. Since F is (U*, intU)-pcv at x_0, we have $J_{F_{x_0}}(d) \in \text{int}U$, $d=\frac{x^*-x_0}{\|x^*-x_0\|}$, so that $\alpha^t(J_{F_{x_0}}(d)) > 0$ and this contradicts (5.3).

ii) Ab absurdo suppose that there exists $x^* \in S$ such that $F(x^*) \in F(x_0)+U^*$. Since F is (U*, U^0)-pcv at x_0, we have $J_{F_{x_0}}(d) \in U^0$, $d=\frac{x^*-x_0}{\|x^*-x_0\|}$, so that, taking into account the assumption $\alpha \in \text{int}U^+$, it results $\alpha^t(J_{F_{x_0}}(d)) > 0$ and this contradicts (5.3). ♦

In order to state some sufficient optimality conditions, we introduce the cone $K(d,\varepsilon)=\{x\in \mathbf{R}^n: x=\lambda y, \lambda\geq 0, \|y-d\|<\varepsilon\}$, where $d \in \mathbf{R}^n$, $\|d\|=1$, and $\varepsilon>0$ is a real number.

The following Theorem states a necessary and sufficient optimality condition with respect to any kind of efficient point.

Theorem 5.4

Let us consider problem P where F is a regular directionally differentiable function at x_0. Then x_0 is a local U*-efficient point for P if and only if the following conditions hold:

i) $\frac{\partial F}{\partial d}(x_0) \notin \text{int}U \;\; \forall d \in T(S,x_0)$

ii) for every $d \in T(S,x_0)$ such that $\frac{\partial F}{\partial d}(x_0) \in \text{cl}U \setminus \text{int}U$ there exists $\varepsilon>0$ such that x_0 is a local U*-efficient point with respect to the region $S \cap (x_0+K(d,\varepsilon))$

Proof.

if. i) follows by Theorem 5.1 while ii) follows by noting that $S \cap (x_0+K(d,\varepsilon)) \subset S$.

only if. Assume that x_0 is not a local U*-efficient point ; then there exist a sequence $\{x_k\} \subset S$, $x_k \to x_0$ such that $F(x_k) \in F(x_0)+U^* \;\; \forall k$ and $\lim_{k\to+\infty} \frac{x_k-x_0}{\|x_k-x_0\|} = \tilde{d} \in T(S,x_0)$.

Since $\frac{F(x_k)-F(x_0)}{\|x_k-x_0\|} \in U^*$ and F is regular directionally differentiable, taking into account of the assumption $\frac{\partial F}{\partial d}(x_0) \notin \text{int}U \;\; \forall d \in T(S,x_0)$, we have:

$$\lim_{k\to+\infty} \frac{F(x_k)-F(x_0)}{\|x_k-x_0\|} = \frac{\partial F}{\partial \tilde{d}}(x_0) \in \text{cl}U \setminus \text{int}U.$$

For ii) we have $\tilde{d} \in \text{int}(x_0+K(d,\varepsilon)$ and this implies the existence of n^* such that $x_k \in S \cap (x_0+K(\tilde{d},\varepsilon)) \;\; \forall k>n^*$, and this contradicts the local U*-efficiency of x_0 with respect to $S \cap (x_0+K(d,\varepsilon))$. ♦

As a direct consequence of the previous Theorem we obtain the following sufficient optimality condition which holds for any kind of efficient point:

Corollary 5.1

Let us consider problem P where U is a closed cone.

i) if F is regular directionally differentiable at x_0 then (5.4) is a sufficient condition for x_0 to be a local U*-efficient point for P

$$\frac{\partial F}{\partial v}(x_0) \notin U , \quad \forall \; v \in T(S,x_0) , \quad v \neq 0. \tag{5.4}$$

ii) if F is directionally differentiable and locally lipschitzian at x_0 then (5.4) is a sufficient condition for x_0 to be a local U*-efficient point for P;

iii) if F is differentiable at x_0 then (5.5) is a sufficient condition for x_0 to be a local U*-efficient point for P

$$J_{F_{x_0}}(v) \notin U , \quad \forall \; v \in T(S,x_0) , \quad v \neq 0. \tag{5.5}$$

Remark 5.2

When x_0 is an interior point of S the Bouligand tangent cone at x_0 reduces to the all space so that (5.4) and (5.5) hold for every $v \in \mathbf{R}^n$, $v \neq 0$.

Under suitable assumptions of convexity relation (5.5) can be characterized in the form given in the following Theorem:

Theorem 5.5

Let us consider problem P where U is a closed convex and pointed cone, S is a convex closed set and F is differentiable at x_0.Then condition (5.5) is equivalent to the following condition (5.6):

$$\exists \ \alpha \in \text{int}U^+ \text{ such that } \alpha^t J_{F_{x_0}}(v) \leq 0 \qquad \forall\ v \in T(S,x_0),\ v \neq 0 \qquad (5.6a)$$

$$J_{F_{x_0}}(v) \neq 0 \qquad \forall\ v \in T(S,x_0),\ v \neq 0 \qquad (5.6b)$$

Proof

The convexity of S implies that $T(S,x_0)$ is a closed convex cone, so that (5.6a) follows from (5.5) by applying a known separation Theorem; (5.6b) follows directly from (5.5) by noting that $0 \in U$.

Assume now that (5.6) holds.If (5.5) does not hold there exist $v \in T(S,x_0)$, $v \neq 0$ such that $J_{F_{x_0}}(v) \in U$; from (5.6b) we have $J_{F_{x_0}}(v) \in U^0$ so that $\alpha^t J_{F_{x_0}}(v) > 0$ and this contradicts (5.6a). ♦

The proof given in the previous Theorem points out that relation (5.6) implies (5.5) without any requirement of convexity so that, taking into account Corollary 5.1, Remark 5.2 and that (5.6b) implies the injectivity of $J_{F_{x_0}}$ when $v \in \mathbf{R}^n$, $v \neq 0$, we obtain the following sufficient optimality conditions for any kind of efficient point:

Theorem 5.6

Let us consider problem P where U is a closed cone and F is differentiable at x_0.

i) if condition (5.6) holds then x_0 is a local U*-efficient point for P;

ii) if x_0 is an interior point, $J_{F_{x_0}}$ is injective and $\exists\ \alpha \in \text{int}U^+$ such that $\alpha^t J_{F_{x_0}} = 0$ then x_0 is a local U*-efficient point for P.

The following example points out that relation (5.6b) does not imply that the restriction of the Jacobian matrix $J_{F_{x_0}}$ to the tangent cone $T(S,x_0)$ is injective.

Example 5.1

Consider the function $F(x_1,x_2)=(-x_1+x_2, x_1^2)$, the cone $U=\mathbf{R}^2_+$, the point $x_0=(0,0)$ and the feasible region $S=\{(x_1,x_2): x_1\geq 0, x_2\leq 0\}$.

It results $T(S,x_0)=S$, $J_{F_{x_0}}=\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$, so that choosing $\alpha^t=(1,1)$ we have $\alpha^t J_{F_{x_0}}(v)=(-v_1+v_2,0)^t \;\; \forall\, v=(v_1,v_2): v_1\geq 0, v_2\leq 0$.

Consequently $\alpha^t J_{F_{x_0}}(v)<0 \;\forall\, v\in T(S,x_0)$, $v\neq 0$, relations (5.6a) and (5.6b) are verified but the restriction of the Jacobian matrix $J_{F_{x_0}}$ to the tangent cone $T(S,x_0)$ is not injective since $J_{F_{x_0}}(1,0)^t= J_{F_{x_0}}(0,-1)^t=(-1,0)^t$.

The following example shows that condition (5.5) does not imply (5.6) when the feasible rezgion S is not a convex set.

Example 5.2

Consider the function $F(x_1,x_2)=(x_1-x_2, -x_1+2x_2)$, the cone $U=\mathbf{R}^2_+$, the point $x_0=(0,0)$ and the non convex feasible region $S=\{(x_1,x_2): x_1\geq 0, x_2=0\}\cup\{(x_1,x_2): x_2\geq 0, x_1=0\}$.
It results $T(S,x_0)=S$ and it is easy to verify the validity of (5.5) and the non existence of $\alpha\in \text{int}\mathbf{R}^2_+$, verifying (5.6).

At last we point out the role of vector generalized concavity in stating sufficient optimality conditions of the Kuhn-Tucker type.

Consider the vector optimization problem P in the form

$$P:\; U\text{-max } F(x)\,,\;\; x\in S=\{x\in X: G(x)\in V\}$$

where $X\subset \mathbf{R}^n$ is an open set, $F: X\rightarrow \mathbf{R}^s$, $G: X\rightarrow \mathbf{R}^m$ are continuous and differentiables functions, $s\geq 1$, $m\geq 1$, and $U\subset \mathbf{R}^s$, $V\subset \mathbf{R}^m$ are closed, pointed, convex cones with verteces at the origin such that $\text{int}U\neq\emptyset$, $\text{int}V\neq\emptyset$.

Let x_0 be a feasible point and assume that $G(x_0)=0$ (when $V=\mathbf{R}^m_+$, $G(x_0)=0$ means that x_0 is binding at all the constraints so that such an assumption is not restrictive taking into account the continuity of F and G).

It is well known that the U*-efficiency of the point x_0 implies the validity of the following F. John conditions:

$$\exists\ (\alpha_F,\alpha_G)\neq 0\ ,\ \alpha_F\in U^+,\ \alpha_G\in V^+:\ \alpha_F^t\, J_{F_{x_0}}+\alpha_G^t\, J_{G_{x_0}}=0 \qquad (5.7)$$

The following Theorem is the analogous of Theorem 5.6.

Theorem 5.7

If (5.7) holds with $\alpha\in \text{int}U^+$ and $J_{F_{x_0}}$ is injective then x_0 is a local U*-efficient point for P.

Proof.

The non local U*-efficiency of x_0 implies the existence of a sequence $\{x_n\}\subset S$, with $x_n\to x_0$, $\frac{x_n-x_0}{\|x_n-x_0\|}\to v$ such that $F(x_n)-F(x_0)\in U^*$ and $G(x_n)-G(x_0)\in U^*$ since $G(x_0)=0$. Consequently Property 5.1 and the injectivity of $J_{F_{x_0}}$ implies $J_{F_{x_0}}(v)\in U^0$, $J_{G_{x_0}}(v)\in U^*$, so that $\alpha_F^t\, J_{F_{x_0}}(v)+\alpha_G^t\, J_{G_{x_0}}(v)>0$ and this contradicts (5.7). ♦

The following theorem generalizes a result given in [4].

Theorem 5.8

Let us consider the vector optimization problem P where S is a star shaped set at x_0 and F, G are differentiable at x_0.

i) if F is (U^*, U^0)-pcv at x_0, G is V-qcv at x_0 and (5.7) holds with $\alpha_F\in \text{int}U^+$, then x_0 is a local U*-efficient point for P.

ii) if F is (U*, intU)-pcv at x_0, G is V-qcv at x_0 and (5.7) holds with $\alpha_F\in U^+\setminus\{0\}$, then x_0 is a local U*-efficient point for P.

Proof.

i) Suppose that there exists $x^*\in S$ such that $F(x^*)\in F(x_0)+U^*$. Since F is (U^*, U^0)-pcv at x_0 we have $J_{F_{x_0}}(x^*-x_0)\in U^0$; on the other hand if G is V-qcv at x_0 it easy to prove that $J_{G_{x_0}}(x^*-x_0)\in V$ and thus $\alpha_F^t\, J_{F_{x_0}}(x^*-x_0)>0$, $\beta_F^t\, J_{G_{x_0}}(x^*-x_0)\geq 0$ since $\alpha_F\in \text{int}U^+$ and $\alpha_G\in V^+$. Consequently $\alpha_F^t\, J_{F_{x_0}}(x^*-x_0)+\alpha_G^t\, J_{G_{x_0}}(x^*-x_0)>0$ and this contradicts (5.7).

ii) similar to the one given in i). ♦

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