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Fractional Programming with Sums of Ratios

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ABSTRACT

One of the most difficult multi-ratio fractional programs is the sum-of-ratios problem. The paper surveys applications, theoretical results and various algorithmic approaches for this nonconvex problem. The presentation begins with a detailed survey of single-ratio fractional programming which provides the necessary background for applications and algorithms of the sum-of-ratios problem.

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1. Introduction

Numerous decision problems in management science and problems in economic theory give rise to constrained optimization of linear or nonlinear functions. If in the nonlinear case the objective function is a ratio of two functions or involves several such ratios, then the optimization problem is called a *fractional program.*

In a comprehensive bibliography of fractional programming in [142] the author listed well over one thousand contributions. In his previous bibliography [138] in 1982 less than half as many articles had appeared. Meanwhile, fractional programming has found its place in International Abstracts in Operations Research, Mathematical Reviews and Zentralblatt für Mathematik as a separate field in optimization, like quadratic programming or convex programming for example.

Apart from isolated earlier results, most of the work in fractional programming was done since about 1960. In their classical paper in 1962, Charnes and Cooper [34] showed that a single-ratio linear fractional program can be transformed into a linear program.

The analysis of fractional programs with only one ratio has largely dominated the literature until about 1980. Whereas the first monograph [135] in fractional programming in 1978 does not deal with multi-ratio fractional programs, the only other monograph, published by Craven [42] in 1988, discusses some of the earlier results on problems involving more than one ratio. Since the first international conference with an emphasis on fractional programming, the NATO Advanced Study Institute on "Generalized Concavity in Optimization and Economics" [147] in 1980, several similar conferences were held that indicate a shift of interest from the single to the multi-ratio case [149], [28], [111], [92]. However, in recent years this trend has been somewhat reversed.

It is interesting to note that some of the earliest publications in fractional programming, though not under this name, von Neumann's classical paper on a model of a general economic equilibrium [162], [163] in 1937, analyzes a multi-ratio fractional program. Even a duality theory was proposed for this nonconcave program, and this at a time when linear programming hardly existed. However, this early paper was followed almost exclusively by articles in single-ratio fractional programming until the early 1980s.

The present article will survey results on a particular type of multi-ratio fractional program, the sum-of-ratios problem. However, to ensure the necessary background, a rather detailed survey of the single-ratio problem will be given. The article is organized as follows. Following notation and definitions throughout the remainder of this section, Section 2 will present applications as well as theoretical and algorithmic results of single-ratio fractional programming*. In Section 3 we turn to the sum-of-ratios program and survey major applications, theoretical results and algorithmic approaches.

At the end of this introduction we present the notation and definitions as they are used in the article.

Let f, g, h_k $(k = 1,\dots,m)$ denote real-valued functions which are defined on the set C of the n-dimensional Euclidean space $\Re^n$. We consider

$$q(x) = \frac{f(x)}{g(x)} \tag{1}$$

* This section of the present paper largely follows the presentation of single-ratio fractional programming in [142].

over the set

$$S = \{x \in C: \quad h_k(x) \le 0, \quad k = 1,\ldots,m\}. \tag{2}$$

Here we assume that $g(x)$ is positive on C. For negative $g(x)$, $q(x) = (-f(x))/(-g(x))$ may be used instead. The nonlinear program

$$(P) \qquad \sup\{q(x): \quad x \in S\} \tag{3}$$

is called a (single-ratio) *fractional program.*

In some applications more than one ratio appears in the objective function. Examples are

$$\sup\left\{\min_{1 \le i \le p} q_i(x): \quad x \in S\right\} \tag{4}$$

and

$$\sup\left\{\sum_{i=1}^{p} q_i(x): \quad x \in S\right\} \tag{5}$$

where $q_i(x) = f_i(x) / g_i(x), \quad g_i(x) > 0$. Problem (4) is sometimes referred to as a *generalized fractional program* [44]. Problems (4) and (5) are both related to the *multi-objective fractional program*

$$\max\left\{\left(q_1(x),\ldots,q_p(x)\right): \quad x \in S\right\}. \tag{6}$$

The present survey will focus on problem (5) out of these three types of multi-ratio programs.

So far we did not specify the functions in the numerator and denominator. If f and g are affine (linear plus a constant) and S is a convex polyhedron, then (P) is called a *linear fractional program*. It is of the following form:

$$\sup\left\{\frac{c^T x+\alpha}{d^T x+\beta}: \quad Ax \le b, \quad x \ge 0\right\} \tag{7}$$

where $c, d \in \Re^n$, $\alpha, \beta \in \Re$, the superscript T denotes the transpose, A is an $m \times n$ matrix and $b \in \Re^m$.

In generalization of a linear fractional program, we call (P) a *quadratic fractional program* if f and g are quadratic and S is a convex polyhedron.

Problem (P) is said to be a *concave fractional program* if the numerator f is concave on C and g, h_k are convex on C, where C is a convex set. In addition it is assumed that $f(x)$ is nonnegative on S if g is not affine. Note that the objective function of a concave fractional program (3) is generally not a concave function, instead it is composed of a concave and a convex function. Even under these restrictive concavity/convexity assumptions, fractional programs are generally nonconcave programs.

2. Single-Ratio Fractional Programs

We consider the problem

$$(P) \qquad \sup\{q(x): \quad x \in S\} \tag{8}$$

where $q(x) = f(x)/g(x)$.

2.1 Applications

Fractional programs arise in economic planning as well as outside of it. In addition, they sometimes occur indirectly in modeling where initially no ratio is involved.

The following summary is meant to demonstrate the diversity of problems that can be cast into the form of a single-ratio fractional program. Completeness is not our aim here. For a more comprehensive coverage, the reader is referred to surveys that were attempted earlier, e.g. [42], [71], [107], [135], [143], [153]. We point out that the term 'application' is used here mainly in the sense of potential application. This survey, apart from being important in itself, will provide a better understanding of the use of sum-of-ratios problems discussed below.

2.1.1 Economic Applications

The efficiency of a system is sometimes characterized by a ratio of technical and/or economical terms. Maximizing the efficiency then leads to a fractional program. We mention a few examples.

Maximization of Productivity [73]

Gilmore and Gomory discuss a stock cutting problem in the paper industry [67] for which they show that under the given circumstances it is more appropriate to minimize the ratio of wasted and used amount of raw material rather than just minimizing the amount of wasted material. This stock cutting problem is formulated as a linear fractional program.

In the case study [81], Hoskins and Blom use fractional programming to optimize the allocation of warehouse personnel. The objective is to minimize the ratio of labor cost to the volume entering and leaving the distribution center.

Maximization of Return on Investment [76]

In some resource allocation problems the ratio profit/capital or profit/revenue is to be maximized. A related objective is return per cost maximization. Resource allocation problems with this objective are discussed in more detail by Mjelde in [114], [115]. In these models the term 'cost' may either be related to actual expenditures or may stand, for example, for the amount of pollution or the probability of a disaster in nuclear energy production. Depending on the nature of the functions describing return, profit, cost, or capital different types of fractional programs are encountered. If, for example, the price per unit depends linearly on the output and cost and capital are affine functions, then maximization of the return on investment gives rise to a concave quadratic fractional program (assuming linear constraints) [119], [120].

Maximization of Return/Risk

A concave nonquadratic fractional program arises in a portfolio selection problem by Ziemba, Parkan and Brooks-Hill [168]. Related concave fractional programs are obtained by Ziemba in [167] where the normal distribution is replaced by other stable distributions for the return; see also [89].

We mention some other models in portfolio theory. In [118] Ohlson and Ziemba obtain the fractional program

$$\max\left\{\frac{c^T x}{\left(x^T C x\right)^{\gamma}}: \quad x \in S\right\} \tag{9}$$

where $c \in \Re^n$ is positive, C is a positive definite $n \times n$ matrix and $\gamma \in (0, \frac{1}{2})$. Here c as well as C are determined by both the vector of expected returns e and the variance-covariance matrix V. Note that (9) is not a concave fractional program since the denominator is not convex for any $\gamma \in (0, \frac{1}{2})$. For an additional analysis of this model see [145], [146], [46], [105].

We mention that Mao [104] and Faaland and Jacobs [57] use a linear fractional program in portfolio selection. The authors maximize the ratio of the expected return to the beta-index in order to quantify the relationship between the extent of diversification and the value of the portfolio.

Another application of fractional programming in financial planning was recently suggested by Uberti [159], [160] for a problem in leasing. We also refer to a study by Konno and Inori [93] who optimize the average maturity or average desired yield in bond trading using fractional programming.

Minimization of Cost/Time

Several models are known where the cost-to-time ratio is to be minimized. Dantzig, Blattner and Rao [48] discuss a routing problem for ships or planes where a cycle in the network is to be determined that minimizes the cost-to-time ratio. The same ratio appears in [64], [99], [100] dealing with related problems.

In a cargo-loading problem considered by Kydland [97] profit per unit time is to be maximized. For a similar model see [17]. Both loading cost and loading time depend on the cargo chosen. In case of linear functions for revenue as well as linear functions for loading cost and time, one obtains a linear fractional program [97]. If loading cost and time are convex and quadratic, then a concave quadratic fractional program is met.

Stochastic processes also give rise to the minimization of cost per unit time as demonstrated in a paper by Derman [49]. An example of such a stochastic process is discussed by Klein [91] who formulates a maintenance problem as a Markov decision process that then leads to a linear fractional program. Here the ratio of the expected cost for inspection, maintenance and replacement and the expected time between two inspections is to be minimized. An inventory problem, where again the expected cost per unit is to be minimized, is discussed by Barlow and Proschan in [9, p. 115]. Sobel [152] studies the maximization of the ratio of mean and standard deviation in connection with Markov decision processes. Fox [63] considers a Markov renewal programming problem where the objective is the minimization of the expected cost-to-time ratio.

Maximization of Output/Input

Charnes, Cooper and Rhodes [36] use a linear fractional program to evaluate the activities of not-for-profit entities participating in public programs. Given a collection of decision making units, the efficiency of any unit is obtained from the maximization of a ratio of weighted outputs and weighted inputs subject to the condition that similar ratios for every decision making unit be less than or equal to unity. The variable weights are then the efficiency of each member relative to that of the others. A linear fractional program has to be solved to determine these efficiencies. For more results as well as implementations of Data Envelopment Analysis (DEA) see [37], [38], [39] and the literature referenced therein. In this context, we also refer to a recent study by Falk et al. [61] in which the effectiveness of medical institutions is computed with fractional programming. Here DEA is *not* used.

Another example of maximizing the output/input ratio is given by Stancu-Minasian in [153]. In his macroeconomic model he maximizes the growthrate of national income over investment and consumption.

So far we have looked at fractional programs that arise when the efficiency of a system is to be optimized. There has been some theoretical work by Eichhorn [52], [53] showing that, based on certain axioms, the terms technical or economical 'effectiveness' generally will be ratios of two functions. Only under more restrictive assumptions these terms are expressed as differences of two functions. Hence, it is not surprising that the optimization of ratios comes up when the effectiveness (or efficiency) of a system is to be maximized.

In the management science literature there recently has been an increasing interest in optimizing relative terms such as relative profit. No longer are these terms merely used to monitor past economic behavior. Instead the optimization of rates is getting more attention in decision making processes for future projects (see [74], [75], [76], [90]).

2.1.2 Non-Economic Applications

In information theory the capacity of a communication channel can be defined as the maximal transmission rate over all probabilities. This turns out to be a concave nonquadratic fractional program [113].

The eigenvalue problem in numerical analysis [117] can be reduced to the maximization of the Rayleigh quotient, and hence gives rise to a quadratic fractional program which is generally not concave.

An example of a fractional program in physics is given by Falk [58]. He maximizes the signal-to-noise ratio of a spectral filter and obtains a concave quadratic fractional program.

2.1.3 Indirect Applications

There are a number of operations research problems that indirectly give rise to a fractional program. This comes up as a surrogate problem or a subproblem.

A concave quadratic fractional program arises in location theory as the dual of a certain minimax location problem [56].

A rather rich source of fractional programs is large-scale mathematical programming. Often at least a part of the constraints in such a large model has a special structure and many coefficients are zero. Examples of some basic structures are the multidivisional or the multitime period problems [77], [98]. Applying a certain decomposition principle to a large-scale linear program, this can be reduced to a finite number of smaller problems with linear fractional objective functions. A decomposition method of that type is suggested for instance by Abadie and Williams [1]. The ratio in these fractional programs originates in the minimum-ratio-rule of the simplex method. The feasible region in the sequence of linear fractional programs is the same. It is determined by the specially structured constraints of the large-scale linear program. Hence, a sequence of linear fractional programs with specially structured constraints is obtained. For details see [98], [135].

Fractional programs are also met indirectly in stochastic programming as first shown by Charnes and Cooper [35] and Bereanu [13]. We want to illustrate this by two models. For other stochastic programs that lead to a fractional program of one type or another see [135].

Consider the following stochastic mathematical program

$$\max\left\{a^T x:\quad x \in S\right\} \tag{10}$$

where the coefficient vector a has a multivariate normal distribution and S is a (deterministic) convex feasible region. Let us assume that the decision maker replaces (10) by the deterministic problem.

$$\max\left\{P(a^T x \geq k):\quad x \in S\right\},$$

i.e., he wants to maximize the probability that $a^T x$ attains at least a prescribed level k. Then (10) reduces to

$$\max\left\{\frac{e^T x - k}{(x^T V x)^{1/2}}:\quad x \in S\right\} \tag{11}$$

where e is the mean-vector of a and V its variance-covariance matrix [13], [35]. Hence the maximum probability model of the concave program (10) gives rise to a concave fractional program. Note that the same type of nonquadratic fractional program is met also in a model in portfolio theory in [168] mentioned before. If in (10) the linear objective function is replaced by different types of nonlinear functions, then the maximum probability model leads to various other concave fractional programs as demonstrated in [135], [156].

In addition to (10), we consider a second stochastic concave program

$$\max\{f_0(x)+\theta f_1(x): \quad x \in S\} \tag{12}$$

where f_0, f_1 are concave functions on the convex feasible region S, $f_1 > 0$ and θ is a (continuous) random variable. Then the maximum probability model for (12) gives rise to the fractional program

$$\max\left\{\frac{f_0(x)-k}{f_1(x)}: \quad x \in S\right\}. \tag{13}$$

For a linear program (12) the deterministic equivalent (13) becomes a linear fractional program [13]. If f_0 is concave and f_1 linear, then (13) is still a concave fractional program. However, if f_1 is also a (nonlinear) concave function, then (13) is no longer a concave fractional program. Obviously, a quadratic program (12) reduces to a quadratic fractional program. For more details on (12), (13) see [135], [156].

Stochastic programs (10) and (12) are met in a wide variety of planning problems. We see that whenever the maximum probability model as the deterministic equivalent is applied, such decision problems lead to fractional programs of one type or another. Hence, fractional programs are encountered indirectly in many different applications of mathematical programming although initially the objective function is not a ratio.

With the recent advent of various interior-point methods for linear programming, fractional programming has been given more attention as well [5], [6], [68], [161], [165]. For example, the maximization of the potential function [47] gives rise to a fractional program.

Gaudioso and Monaco [66] use quadratic fractional programs as subproblems in an algorithm for convex nondifferentiable programs. They arise as duals of search direction subproblems. For another use of fractional programming in nonsmooth optimization see [18]. Sideri [148] approximates (locally) general pseudoconcave programs by linear fractional programs.

We now turn to theoretical and algorithmic results for single-ratio fractional programs.

2.2 Theoretical and Algorithmic Results

Most of the algorithms known so far for single-ratio fractional programs solve linear, or more generally, concave fractional programs (8). To a much lesser degree solution methods are available for nonconcave fractional programs (8). Examples of such methods can be found in [11], [12], [23], [31], [54], [133]. The discussion in the present article is restricted to algorithms in concave fractional programming.
We find at least four different strategies in the literature that can be used to solve a concave fractional program. These approaches will be presented below.

2.2.1 Direct solution of the quasiconcave program (*P*)

Concave fractional programs share some important properties with concave programs due to the generalized concavity of the objective function $q(x) = f(x)/g(x)$; for details see [8], [10], [103], [110]:

1. a local maximum is always a global maximum;
2. a maximum is unique if either the numerator is strictly concave or the denominator is strictly convex;
3. a solution of the Karush-Kuhn-Tucker optimality conditions is a maximum, assuming f, g, h_i are differentiable on the open set C;
4. a maximum is attained at an extreme point of the convex polyhedron S of a linear fractional program (provided an optimal solution exists).

Because of properties 1 and 3, it is possible to solve concave fractional programs by many of the standard concave programming algorithms. Indeed it was shown that several concave programming methods can be applied to programs with a quasiconcave objective function [7], [41], [110]. For example, the method by Frank and Wolfe [77], [110] can be used to solve such problems. In this algorithm the objective function is linearized at each iteration. In case of a concave fractional program either the ratio $q(x)$ as a whole is linearized [102] or the numerator and denominator separately [113]. Then, given linear constraints in (8), a sequence of linear or linear fractional programs is to be solved, depending on the linearization of $q(x)$ chosen. In both cases the solutions of the subproblems converge to a global maximum of (P).

If (P) is a linear fractional program, then property 4 can be used to calculate a maximum $\bar{x}$ by determining a sequence of extreme points x_i of S with increasing value $q(x_i)$. A simplex-like procedure of this type was suggested by Martos [109] and Swarup [157]. Under mild additional assumptions the method converges and is finite. For details see [110].

2.2.2. Solution of an equivalent concave program (P')

Some of the concave programming algorithms are not suitable for generalized concave programs [110]. Thus the choice of concave programming algorithms to solve concave fractional programs directly is limited. However it can be shown that every concave fractional program is transformable into a concave program: the variable transformation

$$y = \frac{1}{g(x)}x, \quad t = \frac{1}{g(x)} \tag{14}$$

reduces (P) to

$$(P') \qquad \sup\left\{ tf\left(\frac{y}{t}\right) : \; th_k\left(\frac{y}{t}\right) \le 0, \; k = 1,\ldots,m, \; tg\left(\frac{y}{t}\right) \le 1, \; \frac{y}{t} \in C, \; t > 0 \right\} \tag{15}$$

which is a concave program [126], [131]. If $\begin{pmatrix}\bar{y} \\ \bar{t}\end{pmatrix}$ is an optimal solution of (P'), then $\bar{x} = \frac{\bar{y}}{\bar{t}}$ is an optimal solution of (P). Such a transformation was initially suggested by Charnes and Cooper [34] who showed that with help of (14) a linear fractional program can be reduced to a linear program.

Because of the transformability into a concave program concave fractional programs (P) can indirectly be solved by *any* concave programming method applying the algorithm to the equivalent program (P'). Hence through transformation (14) we gain access to *all* concave programming algorithms.

To solve (P') rather than (P) may be particularly appropriate when the numerator f and the denominator g have a certain algebraic structure. For example, the maximum probability model (11) of the stochastic concave program (10) or the portfolio selection model in [168] have an affine numerator, and the denominator is the square root of a convex quadratic form. In this case (P') reduces to a concave quadratic program, and hence (P) can directly be solved by any of the standard quadratic programming techniques. For examples see [135], [137], [139].

In the special case of a linear fractional program (7) transformation (14) yields the linear program

$$\sup\left\{c^T y + \alpha t:\ \ Ay - bt \leq 0,\ d^T y + \beta t = 1,\ y \geq 0,\ t > 0\right\}. \tag{16}$$

Hence (7) can be solved by the simplex method for example [135], [77]. It is shown in [164] that the resulting algorithm generates the same sequence of extreme points as the method by Martos [109] mentioned in Section 2.2.1 if S is compact. However, computationally Martos' method is superior as demonstrated by Bitran [15]. For unbounded S Martos' algorithm may fail to work. However, a modified version of it by Cambini and Martein [24] does converge to an optimal solution in the unbounded case. It uses so-called optimal level solutions obtained by parametrically changing the denominator. Very recently, Ellero and Tomasin [55] showed its computational superiority over solving (16) with the simplex method. For different linear fractional programming algorithms as well as comparisons of these, the reader is referred to [14], [25], [26], [27], [116], [154], [155], [164].

We conclude this section on transformations of concave fractional programs with a remark on integer variable problems. As in the continuous variable case, (P') instead of (P) could be solved. However, the difficulty with doing so is that in the equivalent concave program (P') the variables are restricted to discrete values which are not integers in general. Another transformation of (P) into a concave program that does not change the feasible region may therefore be preferable. Such a transformation is suggested in [126], [135], [139]. It does however apply only to a limited class of concave fractional programs, namely to those where not only g but g^{ε} is convex for some $\varepsilon \in (0,1)$. For more on algorithms in integer fractional programming we refer to [16], [19], [32], [33], [62], [69], [70], [72], [86], [87], [122], [124], [136].

2.2.3 Solution of a Dual Program (D)

One of the disadvantages of solving (P) directly is that duality concepts of concave programming cannot be used since basic duality relations are no longer valid for these nonconcave programs [130]. However, transformation (14) enables us to gain access to concave programming duality. Thus a dual fractional program can be defined as one of the (classical) duals of the equivalent concave program (P') [127], [128], [130], [131]. With this the Lagrangean dual of (P') gives rise to the dual fractional program

$$(D) \qquad \inf\left\{\sup_{x\in C}\frac{f(x)-u^T h(x)}{g(x)}: \quad u\geq 0\right\}. \tag{17}$$

Here $h = (h_1,\ldots,h_m)^T$. In the differentiable case the Wolfe dual of (P') yields

$$(D_w) \qquad \begin{aligned} &\lambda \rightarrow \inf \\ &-\nabla f(x)+\left(\nabla h(x)\right)^T u+\lambda\nabla g(x)=0 \\ &-f(x)+\left(h(x)\right)^T u+\lambda g(x)\geq 0 \\ &x\in C,\ u\in\Re^m,\ u\geq 0,\ \lambda\geq 0 \ (\lambda \text{ not signrestricted if } g \text{ is affine}). \end{aligned} \tag{18}$$

As in concave programming [103], weak, strong and converse duality relations as well as other duality results can be established between (P) and (D) ((P) and (D_w)), see [130], [131].

The same or other duals have been suggested using different approaches. They are surveyed and related to each other in [2], [128], [130, [135] in case of linear fractional programs, and in [130], [131], [135] for nonlinear fractional programs. An updated comparison appeared in [42], [139]. Although several duals have been suggested and duality relations proved, little effort has been devoted to algorithmically using duality. Most of the work is theoretical in nature. The computational usefulness of the different duals still remains to be shown, and with this their relative superiority.

In order to make the dual (D) (or (D_W)) a computationally attractive alternative to (P) or (P'), the fractional program (P) should have a certain amount of algebraic structure in f, g and h_k. Otherwise it may well be easier to solve (P) rather than a dual of (P). If (P) is a concave quadratic fractional program with an affine denominator, then the dual (D_W) can be written as a linear program with one additional concave quadratic constraint [130]. Specialized solution methods are available for this type of a nonlinear program [110].

One advantage of a dual method is that in addition to an optimal solution of (P) also the sensitivity of the maximal value of $q(x)$ with regard to right-hand-side changes can be calculated [129], [135], [137]. The dual variables $\bar{u}_i$ in an optimal solution of (D_W) turn out to be proportional to the marginal values of $q(x)$ [135].

Sensitivity analysis for fractional programming has been extensively discussed in [16], [42], [43], [129], [135], [158]; see also the references there and in [30]. Very recent results are found in [54]. Parametric linear fractional programming with an unbounded feasible region has been studied recently by Cambini, Schaible and Sodini [30].

2.2.4 Solution of a Parametric Problem (P_q)

There is a rich class of algorithms based on the following parametric problem associated with (P):

$$(P_q) \qquad \max\{f(x) - qg(x): \quad x \in S\} \tag{19}$$

where $q \in \Re$ is a parameter. (P_q) is sometimes numerically more tractable than (P) because of its simpler structure of the objective function. For example, (P_q) is a parametric quadratic (linear) program if (P) is a quadratic (linear) fractional program, and (P_q) is a parametric concave program if (P) is a concave fractional program.

In the following we asume that S is compact and f, g are continuous on S. We denote the optimal value of the objective function of (P_q) by $F(q)$. Let $\bar{x}$ be an optimal solution of (P) and $\bar{q} = f(\bar{x})/g(\bar{x})$. It is easy to see [50], [88] that

$$F(q) > 0 \text{ iff } q < \bar{q},$$
$$F(q) = 0 \text{ iff } q = \bar{q},$$
$$F(q) < 0 \text{ iff } q > \bar{q}.$$

Furthermore, an optimal solution of $\left(P_{\bar{q}}\right)$ is also an optimal solution of (P). Thus solving (P) is essentially equivalent to finding the root of the nonlinear equation $F(q) = 0$. For this purpose, $F(q)$ has nice characteristics. It is continuous, convex, strictly decreasing, $F(q) \to \infty$ as $q \to -\infty$ and $F(q) \to -\infty$ as $q \to \infty$. In particular, $F(q) = 0$ has the unique solution $q = \bar{q}$.

In considering algorithms along this line, it should be emphasized that, although evaluating $F(q)$ for a given q could be rather time-consuming since it amounts to solving (P_q), the following extra information is also gained. For an optimal solution x′ of $\left(P_{q'}\right)$, the line given by

$$y = f(x') - qg(x') \tag{20}$$

is tangent to $F(q)$ at $q = q'$, implying that $-g(x')$ is a subgradient of $F(q)$ at q' (which is equal to the derivative if $F(q)$ is differentiable at q'). It is also easy to see that (20) crosses the q-axis at $q = f(x')/g(x')$.

With these properties in mind, three classical methods can be applied to solve the nonlinear equation $F(q) = 0$: Newton's method (Newton-Raphson method), the method of regula falsi and the binary search method (bisection method) [143].

The application of Newton's method to linear fractional programs was first discussed by Isbell and Marlow [85] and then generalized to nonlinear fractional programs by Dinkelbach [50]. It is often called Dinkelbach's method. Interestingly enough, the relationship of these methods and Newton's method was discovered only later [82]. As a general characteristic of it, the sequence of iterates q_i converges to $\bar{q}$ from below with a superlinear convergence rate. In practice the iteration is cut off when the i-th test point q_i comes sufficiently close to $\bar{q}$, i.e., $|F(q_i)| \leq \delta$ holds for a given nonnegative constant δ. For other stopping rules see [132]. A very efficient version of Newton's method was recently suggested by Pardalos and Phillips [121].

The use of the above three classical methods for finding the root of $F(q) = 0$ is discussed in [83], [143] where also computational results are reported.

Apart from these iterative methods, parametric programming procedures to determine F_q can be applied as well. Some limited computational experience in case of quadratic fractional programs is reported in [84]. In a comparison of parametric quadratic programming with Dinkelbach's method neither method shows a significant advantage over the other.

We mention that Megiddo [112] and Sniedovich [150], [151] use (P_q) as well. Megiddo suggests a method for combinatorial fractional programming in which the parametric program (P_q) is used differently than in Dinkelbach's method. For a discussion see [122], [124]. Sniedovich analyzes the relationship between (P_q) and classical optimization techniques applied to (P).

Very recently several combinatorial optimization problems with a linear fractional objective function have been studied by Radzig [122], [123], [124]. Complexity results for Dinkelbach's algorithm and Megiddo's algorithm have been derived and existing bounds on the number of iterations have been improved.

In conclusion, we state that most of the computational work in single-ratio fractional programming tests and compares algorithms that use the parametric program (P_q). Much more work is needed to compare computationally the various approaches in Sections 2.2.1, 2.2.2, 2.2.3 and 2.2.4 with each other and with the very recent polynomial-time interior-point method by Freund and Jarre [65]. Also new methods need to be developed for nonconcave fractional programs arising in applications.

3. Maximization of a Sum of Ratios

In this section we consider the multi-ratio fractional program

$$\sup\left\{\sum_{i=1}^{p}\frac{f_i(x)}{g_i(x)}:\quad x\in S\right\} \tag{21}$$

where $S\subseteq\Re^n$ is a convex feasible region.

3.1 Applications

Model (21) arises naturally in decision making when several rates are to be optimized simultaneously and a compromise is sought that optimizes a weighted sum of these rates. In light of the applications in Section 2, numerators and denominators may be representing profit, cost, capital, risk or time, for example. A multitude of applications of (21) can be envisioned this way. Model (21) does include the case where some ratios are not proper quotients, i.e., $g_i(x) = 1$. This describes situations where a compromise is sought between absolute and relative terms like profit and return on investment (profit/capital) or return and return/risk, for example [140].

Almogy and Levin [3] analyze a multistage stochastic shipping problem. A deterministic equivalent of this stochastic problem is formulated in the form of (21). For another presentation of this application see also [59].

Colantoni, Manes and Whinston [40] introduce a modification of classical profit-maximization by including fixed cost. These are assigned to each activity according to the ratio of variable cost of the activity to the total variable cost of all activities.

Without these fixed cost portions the total profit is an affine function whereas the inclusion of these gives rise to a sum of ratios.

Rao [125] discusses various models in cluster analysis. The problem of optimal partitioning of a given set of entities into a number of mutually exclusive and exhaustive groups (clusters) gives rise to various mathematical programming problems depending on which optimality criterion is used. If the objective is to minimize the sum of the average squared distances within groups, then a minimum of a sum of ratios is to be determined.

The minimization of the mean response time in queueing location problems gives rise to (21) as well, as shown by Drezner, Schaible and Simchi-Levi [51]; see also [166].

We also mention an inventory model in [144] which is designed to determine simultaneously optimal lot sizes and an optimal storage allocation in a warhouse [80]. The total cost to be minimized is

$$K(x)=\sum_{j=1}^{n}\left(\frac{\alpha_j}{x_j}+\beta_j x_j\right)+\sum_{j=1}^{n}\gamma_j\left[\left(\sum_{m=1}^{j}x_m\right)^{3/2}-\left(\sum_{m=1}^{j-1}x_m\right)^{3/2}\right]\Big/x_j. \tag{22}$$

Here the first term is the fixed cost per unit, the second one the storage cost per unit and the last one the material handling cost per unit.

3.2 Theoretical Results

As we saw in Section 2, the case of ratios of concave and convex functions is of particular interest in applications. Fortunately, it lends itself to an easy analysis of the single-ratio model (8). A local maximum is a global one, duality relations can be established and several solution techniques are available.

Unfortunately, for the sum-of-ratios problem none of this is true any longer if in the sum all ratios $f_i(x)/g_i(x)$ are quotients of concave and convex functions. In particular, a local maximum may not be a global one, even if a simple function like the sum of a linear and a linear fractional function is considered [78], [134]. Bykadorov [20], [21], [22] has studied generalized concavity properties of sums of linear ratios and, more generally, of sums of ratios of polynomials.

Only some limited results are known for the sum of concave ratios. Craven [42] shows that a maximum or minimum of such a function is attained at the boundary of the feasible region.

Cambini, Martein and Schaible [29] have shown that a sum of concave ratios can always be reduced to a sum of linear ratios by moving nonlinearities into the constraints. Furthermore, in case of linear ratios, one of these can be transformed into a linear function using a generalized Charnes-Cooper transformation of variables [135]. In the special case of two linear ratios this gives rise to the maximization of a linear and linear fractional function. For such problems Martein [106] shows that an optimal solution is located on an edge of the polyhedral feasible region.

Some results are also known for the maximization of the sum of certain nonconcave ratios. In [140], generalizing results in [134], the maximization of the sum of relative and absolute terms is analyzed:

$$\sup_{x \in S} \frac{f(x)}{g(x)} + \lambda f(x) \quad (\lambda \neq 0), \tag{23}$$

$$\sup_{x \in S} \frac{f(x)}{g(x)} + \mu g(x) \quad (\mu \neq 0). \tag{24}$$

It turns out that the sum is often still quasiconcave or quasiconvex [8] if $f_i(x)/g_i(x)$ are *not* concave ratios, but satisfy other kinds of concavity/convexity assumptions. In these cases a local optimum is a global one or an optimum is attained at a vertex [8]. The results in [140] are refined by Hirche in [79].

Another example of model (21) which is tractable and where the classical assumptions on f_i/g_i are *not* satisfied is the problem in (22). In the last expression of (22) the ratios are not convex since the numerators are not convex. In fact, the ratios are not even quasiconvex. However, it can be shown that there exists a one-to-one continuous transformation of variables such that the transformed function of $K(x)$ is (strictly) convex. This proves that a stationary point $\bar{x}$ $(\nabla K(\bar{x})=0)$ is a (unique) global minimum of $K(x)$. Hence the minimum of $K(x)$ can be calculated in a straightforward manner.

We see from this application that a sum-of-ratios problem (21) may well be tractable if the ratios are *not* concave. Of course, the major challenge remains to derive properties and, with help of these, methods for (21) when the ratios *are* concave.

3.3 Algorithmic Results

Compared with the other two types of multi-ratio fractional programs (4) and (6), the least is known about the sum-of-ratios problem (21). This is true in terms of theoretical properties as well as solution methods.

Nethertheless some progress has been made. In the following several algorithmic approaches will be summarized.

3.3.1 The approaches by Almogy and Levin

Almogy and Levin [4] proposed a generalization of Dinkelbach's parametric method in 2.2.4 to the sum-of-ratios problem. In case of a single ratio the idea of decloupling numerator and denominator with help of a parameter turns out to provide one of the most efficient algorithms. The same is true in case of the multi-ratio problem (4), the maximization of the smallest of several ratios [45]. Unfortunately, for the sum-of ratios problem (21) there does not exist such a straightforward generalization of Dinkelbach's parametric approach, in spite of the claim by Almogy and Levin in [4]. Given (15) for the single-ratio case, consider the parametric problem

$$\max \left\{ \sum_{i=1}^{p} [f_i(x) - q_i g_i(x)] : x \in S \right\} \tag{25}$$

where $q = (q_i, \ldots, q_p) \in R^p$ is a parameter. We assume that f_i, g_i are continuous on the compact feasible region S. Let $H(q)$ denote the optimal value in (25). It was recently shown by Falk and Palocsay [59] that the close relationship between (21) and (25) claimed by Almogy and Lavin is erroneous. A numerical example for $p = 2$ and linear ratios in [59] demonstrates that it is not true that at an optimal solution $\bar{x}$ of (21) $H(\bar{q})$ with $\bar{q} = f_i(\bar{x}) / g_i(\bar{x})$ is necessarily zero as in the single-ratio case; it may will be positive. This takes the basis away from Almogy and Levin's method in [4]. Finding a solution of $H(q) = 0$ is not necessarily solving the sum-of-ratios problem.

3.3.2 The Algorithm by Cambini, Martein and Schaible

For the special case of two linear ratios a parametric algorithm was suggested in [29] by Cambini, Martein and Schaible. It does not make use of H(q). Instead it is based on an earlier procedure by Martein [106] for the sum of a linear and linear fractional function.

As mentioned above in 3.2, any problem (21) with two linear ratios and linear constraints can be transformed with help of a generalized version of Charnes and Cooper's variable transformation [29] into the problem considered by Martein [106].

$$\sup \left\{ h^T x + \frac{c^T x + \alpha}{\alpha^T x + \beta} : x \in S \right\} \tag{26}$$

where

$$S = \left\{ x \in R^n : A\, x \leq b, x \geq 0 \right\}. \tag{27}$$

Here $h \in R^n$, and the remaining notation is as in section 1.

The feasible region S is not necessarily bounded.

The equivalent problem (26) of any linear two-ratio problem (21) is solved in [29], [106] by changing the one denominator $d^T x + \beta$, parametrically, i.e. the following parametric linear program is solved :

$$P(\xi) \quad : \frac{1}{\xi} \sup \left\{ \xi\, h^T x + c^T x + \alpha : x \in S,\ d^T x + \beta = \xi \right\} \tag{28}$$

An optimal solution of $P(\xi)$ is called an optimal level solution. By raising the level $\xi = d^T x + \beta$, starting with smallest value on S, a sequence of optimal level solutions is generated.

The sequential method proposed in [29] is obtained by combining algorithms in [24] and [106]. It is shown that with help of finitely many optimal level solutions for increasing values of ξ a local optimal solution for problem (26) can be calculated.

Since the set of local, nonglobal maxima of (26) is finite [106], the procedure finds a global maximum in finitely many steps (assuming non degeneracy in S) or it shows that the objective function in (26) is unbounded. Two variants of the algorithm are proposed in [29] with both find a global optimum in finitely many steps.

3.3.3 The Algorithm by Konno, Kuno and Yajima

In [94], [95] and [96] another approach for the sum-of-ratios problem was recently suggested. It was derived in the context of multiplicative programming.
In [94] Konno and Kuno first study the generalized linear moltiplicative program

$$\min \left\{ m(x) = g(x) + (c^T x + \alpha)\,(d^T x + \beta) : x \in S \right\} \tag{29}$$

where $S \subseteq R^n$ is a compact convex polyhedron and g is a convex function on S. Various sign-combinations for the affine functions $c^T x + \alpha$, $d^T x + \beta$, on S need to be considered. To select one, assume nonnegativity of both functions on S.

Konno and Kuno embed (29) into an (n+1) dimensional problem

$$\min \left\{ M(x,\xi) = g(x) + \xi \frac{(c^T x + \alpha)^2}{2} + \frac{1}{\xi} \frac{(d^T x + \beta)^2}{2} : x \in S, \xi > 0 \right\}$$

This socalled master problem is equivalent to (29) since for any given $\hat{x} \in S$

$$\min \left\{ M(\hat{x}, \xi) : \xi > 0 \right\} = m(\hat{x}). \tag{31}$$

The new objective function $M(x, \xi)$ is not convex in (x, ξ), but it is convex in x for given $\xi>0$. The authors of [94] suggest to solve the following parametrized convex subproblem

$$\min \{M(x, \xi) : x \in S\} \qquad \xi > 0 \text{ fixed.} \tag{32}$$

Consider $h(\xi) = M(\bar{x}(\xi), \xi)$ where $\bar{x}(\xi)$ denotes an optimal solution of (32). The search for a global minimum $\bar{\xi}$ of h can be restricted to a bounded interval $[\xi min, \xi max)]$. Then $\bar{x} = \bar{x}(\bar{\xi})$ is an optimal solution of (29). Several methods are suggested in [94], [96] for calculating the global optimum of h.

In [94] a similar approach is proposed for the related sum-of-ratios problem

$$\min \{f(x) = g(x) + \frac{c^T x + \alpha}{d^T x + \beta} : x \in S\} \tag{33}$$

Assume $c^T x + \alpha \geq 0$, $d^T x + \beta > 0$ on S. Konno and Kuno introduce the master problem

$$\min \{F(x,\xi) = g(x) + \xi \frac{(c^T x + \alpha)^2}{2} + \frac{1}{\xi} \frac{1}{2(d^T x + \beta)^2} : x \in S, \ \xi > 0\} \tag{34}$$

If $(\bar{x}, \bar{\xi})$ is an aptimal solution of (34), then $\bar{x}$ is an optimal solution of (33). This is so since for any $\hat{x} \in S$

$$\min \{F(\hat{x}, \xi) : \xi > 0\} = f(\hat{x}) \tag{35}$$

The ratio, like the product, has been written as a sum of functions which is convex in x for fixed ξ. Similarly to the above, a global optimum has to be determined for a function corresponding to h.

In [95] an extension of these results is proposed by Konno, Kuno and Yajima. Not only nonlinear functions are introduced, but more than one product or ratio is admitted. In case of multiplicative programming the following extension of (29) is studied

$$\min \left\{ m(x) = g(x) + \sum_{i=1}^{p} f_i(x)\, g_i(x) : x \in S \right\} \tag{36}$$

where f_i, g_i, are positive convex functions on the compact convex feasible region S, and g is convex as before. The model in [36] represents a large class of nonconvex programs including all quadratic programs.

The master problem of (36) is

$$\min\left\{ M(x,\xi,\eta)=g(x)+\sum_{i=1}^{p} (\xi_i \frac{(f_i(x))^2}{2})+\eta_i \frac{(g_i(x))^2}{2} : x\in S \right\} \xi_i\eta_i\geq 1,\ \xi_i>0,\ \eta_i>0 \quad i=1, .,p$$

(37)

An optimal solution of (37) $(\bar{x}, \bar{\xi}, \bar{\eta})$ fields an optimal solution $\bar{x}$ of (36). The following subproblem need to be solved

$$\min \left\{ M(x, \xi, \eta) : x \in S \right\} \quad , \quad (\xi, \eta) \text{ fixed} \tag{38}$$

A global optimum of the following problem is to be determined

$$\min \left\{ h(\xi, \eta) = M(\bar{x}(\xi, \eta), \xi, \eta) : \xi_i \eta_i \; 1, \xi_i > 0, \eta_i > 0 \quad i = 1, \ldots, p \right\} \tag{39}$$

The function h is concave and non decreasing. An other approximation algorithm for (39) is suggested in [95].

For computational experience for p= 1, 2, 3, 4 see [95] as well.

In [95] the authors also study the sum-of-ratios problem

$$\min \left\{ f(x) = g(x) + \sum_{i=1}^{p} \frac{c_i^T x + \alpha_i}{d_i^T x + \beta_i} : x \in S \right\} \tag{40}$$

assuming positivity of the affine functions $c_i^T x + \alpha_i$, $d_i^T x + \beta_i$ on S.
Since $g_i(x) = \frac{1}{d_i^T x + \beta_i}$ is convex, (40) is a special case of (36). Thus it can be solved by the algorithm for multiplicative programs. We mention that (40) satisfies the classical convexity assumptions.

As pointed out in [14], the linear ratios can be replaced by quotients of general positive convex and concave functions, and the algorithm still workes.

3.3.4 The Algorithm by Falk and Palocsay

Very recently a method for solving the sum-of-ratios problem was proposed by Falk and Palocsay [59] which does not operate in the variable space S, but in its image

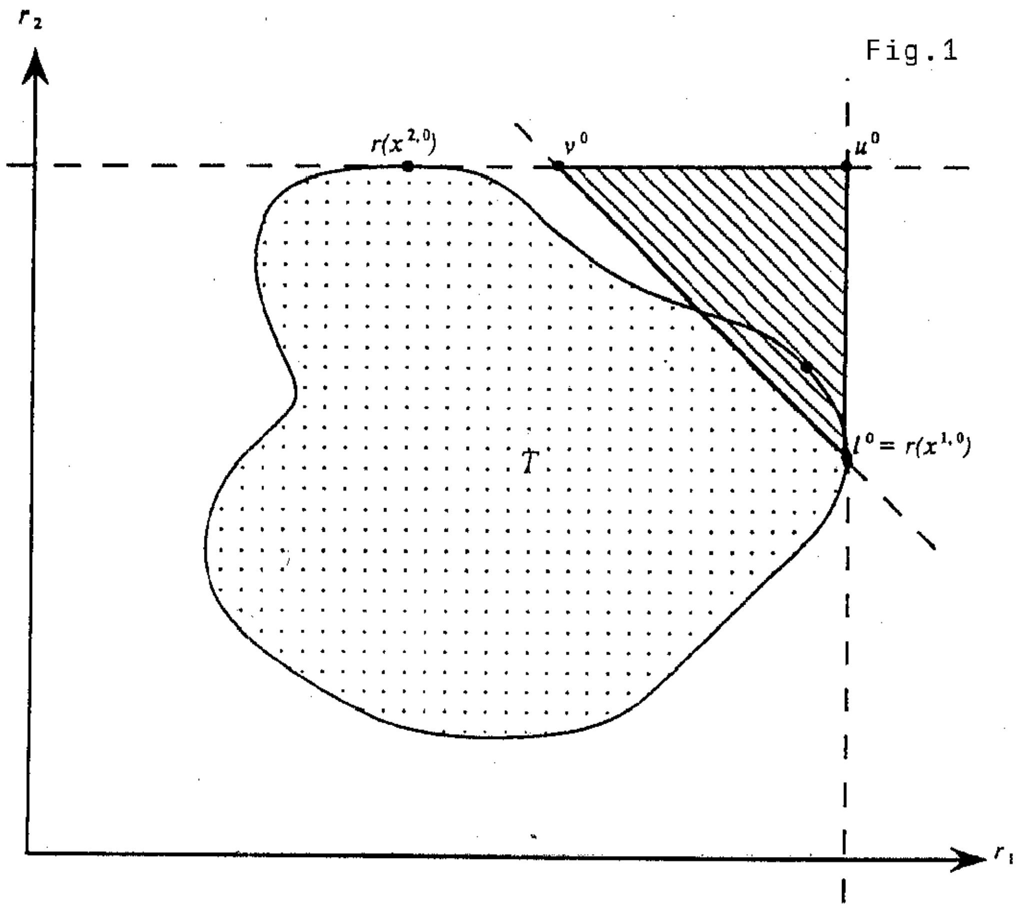
$$T = \{ r = (r_1, ..., r_p) \in R^p \; : \; r_i \; = \frac{f_i(x)}{g_i(x)} \; i = 1, ..., p \quad \text{for some } x \in S \} \tag{41}$$

The method was worked out in its details for $p = 2$ where $T \subseteq R^2$, but can be extended appropriately to $p > 2$; see [59]. Linear ratios and constraints are considered, and S is assumed to be bounded.

In order to determine the image $\bar{r} \in T$ of the optimal soluton $\bar{x}$ of

$$\max \left\{ \frac{c_1^T x + \alpha_1}{d_1^T x + \beta_1} + \frac{c_2^T x + \alpha_2}{d_2^T x + \beta_2} : x \in S \right\}, \tag{42}$$

lower and upper bounded for $\overline{r_1} + \overline{r_2}$ are determined which are iteratively improved. Figure 1 illustrates the start of the algorithm.



Initially two linear fractional programs are solved, for instance with help of the Charnes-Cooper transformation,

$$u_i^0 = \max \left\{ \frac{c_i^T x + \alpha}{d_i^T x + \beta} : x \in S \right\} \quad i=1,2 \tag{43}$$

yielding the optimal solution $x^{1,0}$, $x^{2,0}$ respectively. In addition to $u^0 = (u_1^0 , u_2^0)$ the points $r(x^{1,0})$, $r(x^{2,0})$ are determined.

One of these last two points will yield a better lower bound for $\overline{r_1} + \overline{r_2}$ In the example it is $l^0 = r(x^{1,0})$.

The point v^0 is then determined as the point on $r_2 = u_2^0$ which is on the line $r_1 + r_2 =$ constant through l^0. Obviously, an upper bound for $\overline{r_1} + \overline{r_2}$ is provided by u^0, namely $u_1^0 + u_2^0$. An optimal solution $\bar{r}$ is located within the triangle (l^0, v^0, u^0). It is that point in T which maximizes $r_1 + r_2$.

The idea of the algorithm is to decrease the size of the triangle containing $\bar{r}$. This is accomplished by constructing points l^k, v^k, u^k which provide better lower and upper bounds for $\overline{r_1} + \overline{r_2}$. A sequence of linear fractional programs is to be solved that will reduce the size of the triangle (l^0, v^0, u^0) containing $\bar{r}$. In the illustrated example the fractional program consists of maximizing the second ratio over S while ensuring that the first ratio does not fall below the already achieved level of v_1^0. With help of these fractional programs points v^k on the line between v^0 and l^0 are constructed that move from the outside towards the intersection of that line with the boundary of T. While the lower bound has not improved in this example, the upper bound has been reduced as new points u^k on the horizontal line through v^k and the line $r_1 = u_1^0$ are determined.

Once v^k (and thus u^k) does not improve anymore (within a tolerance), the algorithm stalls since neither lower now upper bounds change. In [59] a procedure is suggested that overcomes the stalling problem. To the last triangle (l^k, v^k, u^k) the point (v_1^k, l_2^k) is added. A square is constructed with help of these low points. This square is divided vertically into two equally sized rectangles. Then two case can occur either $r_1 + r_2$ can be increased in these rectangles or not. In the first case a smaller triangle containing $\bar{r}$ can be found by solving a linear fractional program. Both lower and upper bounds have improved and the algoritm is restarted. Otherwise the algorithm will be applied separately to each rectangle.

An obvious stopping criterion is $u^k = l^k$. In addition to that the authors suggest a sufficient optimality condition which however is not necessary. It involves the function

$$H(v) = \max \{ \sum_{i=1}^{2} (f_i(x) - v_i\, g_i(x) \; : x \in S \} \tag{44}$$

(see (25)). As mentioned in 3.3.1, H(v) is not necessarily zero at an optimal solution $r = \bar{r}$ as shown in [59]. However, if in a triangle (l^k, v^k, u^k) we have $H(l^k)=0$, $H(v^k)< 0$ and $H(u^k)<0$ then $r=l^k$ is an optimal solution. Similarly, v^k is optimal, if $H(v^k)=0$, $H(l^k)< 0$ and $H(u^k)<0$. Since this is only a sufficient, but not necessary optimality condition, an optimal solution may be identified as such only through additional iterations of the algorithm.

Several numerical examples illustrate the algorithm in [59], [101}. We mention an example where the method stalls after twelve iterations and is then restarted. Later it stalls again and finally an optimal solution is obtained at $\bar{r} = (-0.67, -0.96)$, i.e. $\bar{x} = (0, 0.28)$. This example is illustrated in Fig.2 [101].

The algorithm has been implemented and experiments with problems of two ratios involving up to twenty variables and constraints have been performed. An extension to more than two ratios is presented as well in [59]. In the follow-up pubblication [60] the authors use ideas of the algorithm to solve multiplicative programs.

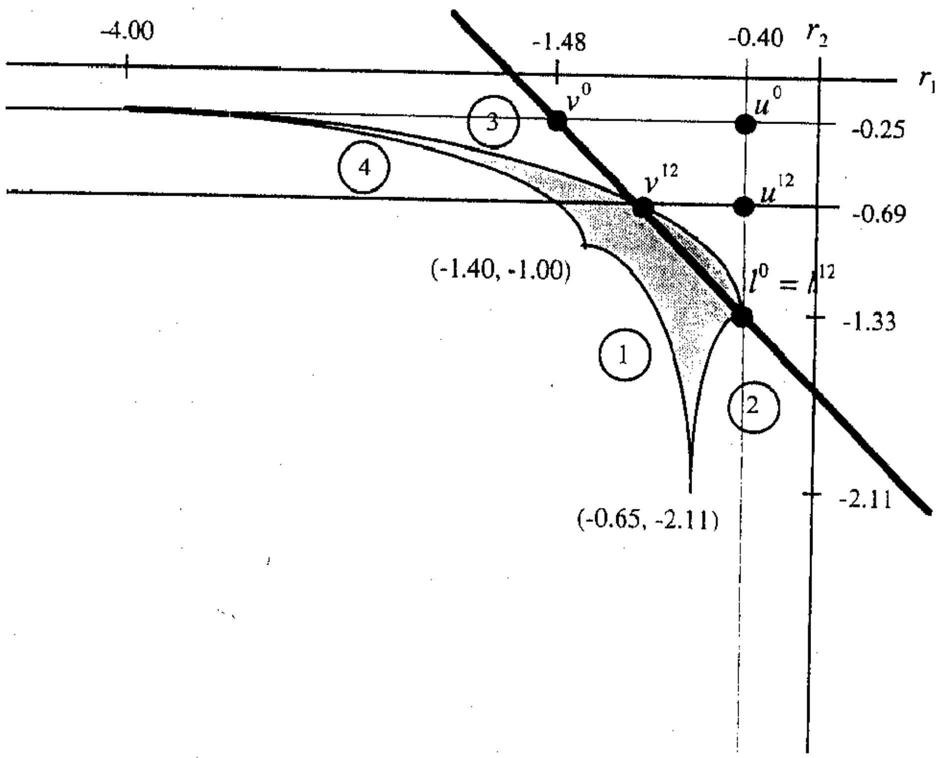


Figure 2

4. Conclusion

The paper outlines the major algorithmic approaches to the sum-of-ratios problem in fractional programming. A survey of single-ratio fractional programming precedes the presentation. By contrast, it becomes clear how limited the progress has been for the sum-of-ratios problem so far. Two of the three algorithms can solve problems with more than two ratios. They both have been tested computationally on small problems as well. In contrast to the third algorithm by Cambini, Martein and Schaible they are not finite. Neither of the three methods has been compared with any of the other methods computationally so far.

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