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The Minimum-Risk Approach For Continuous Time Linear-Fractional Programming

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THE MINIMUM-RISK APPROACH FOR CONTINUOUS TIME LINEAR-FRACTIONAL PROGRAMMING

by

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1. Introduction

In this paper, the minimum-risk approach is applied to the stochastic continuous time linear-fractional problem. We note that the minimum-risk model was introduced in stochastic linear programming by Bereanu [2], [3] and Charnes and Cooper [5] (under the name of P-model). This approach was extended by Stancu-Minasian [10], Stancu-Minasian and Ţigan [12]-[16] to the stochastic programming with linear-fractional objective and by Ţigan [17] to the continuous time linear programming.

We consider two classes of continuous time fractional problems with a linear-fractional objective (see, section 2), respectively with an objective function having a linear-fractional kernel (see, section 4). In the case when the coefficients of the objective functions are simply randomized, we will show that, under some positivity conditions, the stochastic continuous time linear-fractional problem is equivalent with certain deterministic continuous time linear-fractional problem, while the stochastic continuous time fractional problem with an objective function having a linear-fractional kernel is equivalent with a deterministic continuous time nonlinear-fractional problem. Parametrical procedures are applied for solving these deterministic equivalent problems.

2. Problem formulation

The following programming problem which originated from Bellman's bottleneck problem [1] has received a great amount of attention in the last decades:

Find

$$(1) \qquad \sup \int_0^T a(t)\ z(t)\ dt$$

subject to

$$(2) \qquad B(t)\ z(t) \le c(t) + \int_0^t K(t,s)\ z(s)\ ds, \quad 0 \le t \le T,$$

$$(3) \qquad z(t) \ge 0, \quad 0 \le t \le T,$$

where $a: [0,T] \to \mathbb{R}^n$ and $c: [0,T] \to \mathbb{R}^m$ are vector-valued continuous known functions and $B: [0,T] \to \mathbb{R}^{n \times m}$, $K: [0,T] \times [0,T] \to \mathbb{R}^{n \times m}$ are matrix-valued continuous known mappings, while $z: [0,T] \to \mathbb{R}^n$ is a vector-valued continuous unknown function. Let denote by S the set of all vector-valued functions z satisfying constraints (2)-(3).

A partial reference to the earlier works on continuous time programming may be found in Farr and Hanson [7], Singh [9], Bodo and Hanson [4], Hanson and Mond [8], Tyndall [18] and Zalmai [19].

Next we consider a continuous time linear-fractional problem, which extends the linear continuous problem (1)-(3).

CFP. Find

$$(4)\qquad \sup_{z} \frac{\int_0^T a(t)\ z(t)\ dt}{\int_0^T b(t)\ z(t)\ dt}$$

subject to (2)-(3), where a, c, B, K are the same as in problem (1)-(3) and $b:[0,T]\to\mathbb{R}^n$ is a vector-valued continuous known function.

In (4) we make the assumption that:

$$(5)\qquad \int_0^T b(t)\ z(t)\ dt > 0, \quad \textit{for all } z \in S.$$

We assume that in the objective function of problem CFP the vector-valued function a(t) is simply randomized, that is:

$$(6)\qquad a(t;\omega) = a'(t) + \tau(\omega)\ a''(t), \quad t \in [0,T],$$

where $a', a'' : [0,T]\to\mathbb{R}^n$ are vector-valued continuous functions and $\tau(\omega)$ is a random variable on a probability space $(\Omega, \mathbf{K}, \mathbf{P})$ with a continuous and strictly increasing distribution function T^*.

Now we consider the following minimum-risk problem associated to the stochastic problem CFP:

CMR. Find

$$(7)\qquad \sup P\left\{ \omega \mid \frac{\int_0^T a(t;\omega)\ z(t)\ dt}{\int_0^T b(t)\ z(t)\ dt} \geq \beta \right\}$$

subject to

$$(8)\qquad z \in S,$$

where ß is a given number which reprezents a level for the objective function of stochastic problem CFP.

DEFINITION 1. A function $z^*\in S$, is said to be a minimum-risk solution of level ß for CMR problem, if for z^* is reached the supremum in (7).

3. Deterministic equivalent problem

In this section we show that for the minimum-risk problem CMR there exists a deterministic equivalent problem which is a continuous time linear-fractional programming problem.

THEOREM 1. If

$$(9)\quad \int_0^T a''(t)\ z(t)\ dt > 0,\quad \textit{for all } z \in S$$

and the probability distribution function T of the random variable $\tau(\omega)$ is continuous and strictly increasing, then every minimum-risk solution of the continuous time linear-fractional programming problem CMR can be found as an optimal solution of the following fractional optimization problem:

FD. Find

$$(10)\quad \textbf{inf}\ \frac{\int_0^T [\beta b(t) - a'(t)]\ z(t)\ dt}{\int_0^T a''(t)\ z(t)\ dt}$$

subject to (8).

PROOF. Obviously, by the assumptions (5), (6) and (9), we have

$$(11)\quad P\ \{\ \omega\ |\ \frac{\int_0^T [\ a'(t)\ +\ \tau(\omega)\ a''(t)\ ]z(t)\ dt}{\int_0^T b(t)\ z(t)\ dt} \geq \beta\ \}\ =$$

$$= P\{\omega \mid \int_0^T a'(t)z(t)\,dt + \tau(\omega)\int_0^T a''(t)z(t)\,dt \ge \beta\int_0^T b(t)z(t)\,dt\} =$$

$$= P\{\omega \mid \tau(\omega) \ge \frac{\int_0^T [\beta\, b(t) - a'(t)]\, z(t)\, dt}{\int_0^T a''(t)\, z(t)\, dt}\}.$$

Since T^* is strictly increasing and continuous, from (11), we have:

$$\sup_{z\in S} P\{\omega \mid \frac{\int_0^T a(t;\omega)\, z(t)\, dt}{\int_0^T b(t)\, z(t)\, dt} \ge \beta\} =$$

$$= 1 - T^*\left(\inf_{z\in S} \frac{\int_0^T [\beta b(t) - a'(t)]\, z(t)\, dt}{\int_0^T a''(t)\, z(t)\, dt}\right),$$

which concludes the theorem.

Next we give a sufficient condition which assure that assumption (9) from Theorem 1 holds.

PROPOSITION 1. If assumption (5) holds and

$$a''(t) > 0, \textit{ for all } t \in [0,T],$$

then the assumption (9) holds.

PROOF. Indeed, by assumption (5), it follows that feasible set S didn't contain the null mapping, that is, the vector-valued application

$$z(t) = 0, \text{ for all } t \in [0,T].$$

But, this fact together with the continuity of the functions z and a" implies that (9) holds.

4. The fractional objective kernel case

Next we consider a continuous time problem with fractional kernel of the objective function, that is :

FP. Find:

$$(12) \quad \sup_{z} \int_0^T \frac{a(t)\ z(t)}{b(t)\ z(t)}\, dt$$

subject to (2)-(3),
where a, b, c, B and K have the same significance as in the problem CFP.

Moreover, we suppose that:

(13) $b(t)\ z(t) > 0$, for all $t \in [0,T]$ and $z \in S$.

Next we assume that in the objective function of FP problem, the vector-valued mapping a(t) is simply randomized of the form (6).

We can state the following minimum-risk problem corresponding to the level ß associated to the stochastic problem FP:

FR. Find

$$(14) \quad \sup_{z} P\,\{\, \omega \mid \int_0^T \frac{a(t;\omega)\ z(t)}{b(t)\ z(t)}\, dt \geq \beta \,\}$$

subject to (8).

DEFINITION 2. A function $z^* \in S$, is said to be a minimum-risk solution of level ß for FR problem, if for z^* is reached the supremum in (14).

Next we show that for the minimum-risk problem FR there exists, under some supplementary assumption, a deterministic equivalent continuous time programming problem with a nonlinear fractional objective function.

THEOREM 2. If

$$(15)\qquad \int_0^T \frac{a''(t)\; z(t)}{b(t)\; z(t)}\, dt > 0, \quad \textit{for all } z \in S,$$

and the probability distribution function T^* of the random variable $\tau(\omega)$ is continuous and strictly increasing, then every minimum-risk solution of the continuous time fractional programming problem FR can be found as an optimal solution of the following deterministic continuous time fractional programming problem:

CFD. Find

$$(16)\qquad \inf_z \frac{\displaystyle\int_0^T \frac{[\frac{\beta}{T} b(t) - a'(t)]\; z(t)}{b(t)\; z(t)}\, dt}{\displaystyle\int_0^T \frac{a''(t)\; z(t)}{b(t)\; z(t)}\, dt}$$

subject to (8).

PROOF. Indeed, by (6) and (15), we have:

$$(17)\qquad P\{\, \omega \mid \int_0^T \frac{[\, a'(t) + \tau(\omega)\; a''(t)\,]\, z(t)}{b(t)\; z(t)}\, dt \geq \beta \,\} =$$

$$= P\{\, \omega \mid \int_0^T \frac{a'(t)\; z(t)}{b(t)\; z(t)}\, dt + \tau(\omega) \int_0^T \frac{a''(t)\; z(t)}{b(t)\; z(t)}\, dt \geq \beta \,\} =$$

$$= P\{\, \omega \mid \tau(\omega) \geq \frac{\displaystyle\int_0^T \frac{[\, \frac{\beta}{T}\, b(t) - a'(t)\,]\; z(t)}{b(t)\; z(t)}\, dt}{\displaystyle\int_0^T \frac{a''(t)\; z(t)}{b(t)\; z(t)}\, dt} \,\}.$$

Since T^{*} is strictly increasing and continuous, from (17) we have:

$$\sup_{z \in S} P\{\omega \mid \int_0^T \frac{a(t;\omega)\ z(t)}{b(t)\ z(t)}\, dt \geq \beta\} =$$

$$= 1 - T^{*}\left(\inf_{z \in S} \frac{\int_0^T \frac{[\frac{\beta}{T} b(t) - a'(t)]\ z(t)}{b(t)\ z(t)}\, dt}{\int_0^T \frac{a''(t)\ z(t)}{b(t)\ z(t)}\, dt} \right),$$

which concludes the theorem.

5. Algoritmic remarks

The deterministic continuous time programming problems FD and CFD have fractional objectives. For solving these classes of optimization problems the parametric procedure given by Ţigan [17] can be used.

We mention that this procedure generalizes the Dinkelbach method [6] for solving nonlinear fractional programming.

Next we present two particularisation of the parametrical procedure to the problems FP and CFD respectively.

Algorithm for the FD problem

Let δ be a given positive real number.

Step 1. Take k:=0 and find $z^0 \in S$.
Step 2. Compute

$$V_k = \frac{\int_0^T [\beta b(t) - a'(t)]\ z^k(t)\ dt}{\int_0^T a''(t)\ z^k(t)\ dt}.$$

Step 3. Find

$$(17) \quad Q_{k+1} = \inf_{z \in S} \int_0^T [\beta b(t) - a'(t) - V_k a''(t)]\ z(t)\ dt.$$

Let $z^{k+1} \in S$ the optimal solution for the linear continuous time problem (17).

Step 4. i) If $Q_{k+1} < -\delta$, then take k:=k+1 and go to Step 2.

ii) If $Q_{k+1} \geq -\delta$, then the algorithm stops. The optimal solution z^{k+1} of problem (17) is an approximation of the optimal solution of problem FD.

Algorithm for the problem CFD

Step 1. Take k:=0, and find a feasible solution z^0 of the problem CFD.

Step 2. Compute

$$V_k = \frac{\int_0^T \frac{[\frac{\beta}{T} b(t) - a'(t)]\ z^k(t)}{b(t)\ z^k(t)}\ dt}{\int_0^T \frac{a''(t)\ z^k(t)}{b(t)\ z^k(t)}\ dt}.$$

Step 3. Find

$$(18) \quad Q_{k+1} = \inf_{z \in S} \int_0^T \frac{[\frac{\beta}{T} b(t) - a'(t) - V_k a''(t)]\ z(t)}{b(t)\ z(t)}\ dt.$$

Let $z^{k+1} \in S$ the optimal solution for the linear-fractional continuous time problem (18).

Step 4. i) If $Q_{k+1} < -\delta$, then take k:=k+1 and go to Step 2.

ii) If $Q_{k+1} \geq -\delta$, then the algorithm stops. The optimal solution z^{k+1} of problem (18) is an approximation of the optimal solution of problem CFD.

For approximate solving the continuous time fractional problems (17) or (18) a discretization method can be used.

6. Conclusions

Two classes of stochastic continuous time fractional programming problems were considered. In the case when the denominator of the objective is simply randomized some deterministic equivalent continuous time fractional programming problem are obtained.

Similar results can be obtained in the case of complete randomization of the fractional objective (i.e. the nominator b of the fractional objective function is also random).

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