Report n.85

On Duality for Multiobjective Mathematical Programming of n-Set Functions

Vasile Preda and I.M. Stancu-Minasian

Pisa, November 1994

ON DUALITY FOR MULTIOBJECTIVE MATHEMATICAL PROGRAMMING OF n-SET FUNCTIONS

VASILE PREDA
Mathematics Faculty
University of Bucharest
14,Academiei Street
70109 Bucharest, ROMANIA

I.M.STANCU-MINASIAN
The Romanian Academy
Centre of Math.Statistics
Bd. Magheru 22
70158 Bucharest, ROMANIA

Abstract

For multiobjective mathematical programming of n-set functions, several Mond - Weir duality results under more general assumptions than convexity are established.

1. Introduction

Optimization problems containing set functions arise in many situations dealing with optimal constrained selection of measurable subsets.Some problems of this type have been encountered in statistics [13, 26] ,fluid flow [3] ,electrical insulator design [4] ,optimal plasma confinement [34] and regional design (districting, facility location, warehouse layout, urban planning) [11,12]. We note that for treating such problems ,in most cases, the theory and methods which have proposed are applicable only to some classes of problems. In [25], Morris had developed the first general theory for optimizing set functions. He defined the notions of local convexity, global convexity and differentiability for set functions. Also he established optimality conditions and Lagrangian duality relations for a general nonlinear programming problem involving set functions. Further Morris[24,25] stated some algorithms for numerical solution of these problems. These methods and results stated by Morris have been used and further extended in [7,8,10,16,18,19,32].

Thus in [10], Corley developed the general theory for n-set functions and gave the concepts of partial derivatives and the derivative of the n-set function.

Some optimality conditions are given for nonlinear programs with set functions in [6] and [18] and for nonlinear programs with n-set functions in [9], duality results for nonlinear programs with set functions are discussed in [8,18,19] and for nonlinear programs with n-set functions are established in [10,38]. For minmax programs containing n-set functions and Mond-Weir duality see [37,30].

In [19,22,27,29-31] different approaches to defining and characterizing the notion of convexity for set or n-set functions are followed, and also optimality and duality results based on these approaches are given. Recently, some Wolfe-type duality results involving properly efficient solutions were obtained in [35] for a multiobjective convex programs with point functions. Also see [5,14,15,28,36] for some results for multiobjective programs with point functions. For multiobjective programs involving set or n-set functions some optimality criteria and duality results are investigated in [6,16,20,32,38]. Thus recently in Zalmai [38] under convexity and generalized ρ-convexity assumptions, some sufficient optimality conditions and weak and strong duality results are stated.

In the present paper our aim is to extend some duality results stated by Weir [35] for multiobjective programs with point functions and Zalmai [38] for multiobjective programs involving n-set functions. Further we give two results relative to strict duality of Mangasarian type. The duality results are stated relative to a general Mond-Weir type dual [23, 35]. These results are stated under (ρ, b)-vexity and generalized (ρ, b)-vexity assumptions, where the concepts of (ρ, b)-vexity and generalized (ρ, b)-vexity defined in this paper are extensions of concepts of ρ-convexity, generalized ρ-convexity defined by Vial [33] and Jeyakumar [17] and b-vexity and generalized b-vexity recently defined by Bector et al. [1, 2].

Our results are stated relative to the following multiobjective programming problem involving n-set functions

$$(\mathrm{P}) \quad \begin{array}{l} \text{Minimize } F(S) = (F_1(S), \ldots\ldots, F_m(S)) \\ \text{subject to } G_j(S) \leqq 0, j \in J, S \in \Gamma^n \end{array}$$

where $J = \{1, 2, .., p\}, \Gamma^n$ is the n-fold product of a σ-algebra Γ of subsets of a given set X, and $F_1, \ldots, F_m, G_1, \ldots, G_p$ are real valued functions defined on Γ^n. We denote by $\mathcal{P}$ the set of all feasible solutions for problem (P).

2. Preliminaries

Let (X, Γ, μ) be a finite atomless measure space with $L_1(X, \Gamma, \mu)$ separable , and let $F_1, ..., F_m, G_1, ..., G_p$ be real valued n-set functions defined on Γ^n, the n-fold product of a σ-algebra Γ of subsets of X. Let (Γ^n, d) be a pseudometric space, where d is the pseudometric on Γ^n defined by

$$d(S,T) = \left\{ \sum_{k=1}^{n} \left(\mu\left(S_k \triangle T_k\right)\right)^2 \right\}^{1/2}$$

for $S = (S_1, ..., S_n)$, $T = (T_1, ..., T_n)$,where $\triangle$ denotes the symmetric difference. Each $\Omega \in \Gamma$ can be identified with its characteristic function $I_\Omega \in L_\infty(X, \Gamma, \mu) \subset L_1(X, \Gamma, \mu)$ and so that the σ-algebra Γ is identified as a subset $\{I_\Omega, \Omega \in \Gamma\}$ of $L_\infty(X, \Gamma, \mu)$. For $f \in L_1(X, \Gamma, \mu)$ and $\Omega \in \Gamma$, the integral $\int_\Omega f d\mu$ will be denoted by $\langle f, I_\Omega \rangle$.

Now we shall define the notion of differentiability for n-set functions . Morris [24] introduced the differentiability for set functions and Corley [10] defined this notion for n-set functions.

A set function $\varphi : \Gamma \to R$ is said to be differentiable at T if there exists $D\varphi_T \in L_1(X, \Gamma, \mu)$,called the derivative of φ at T ,such that for each $S \in \Gamma$,

$$\varphi(S) = \varphi(T) + \langle D\varphi_T, I_S - I_T \rangle + \psi(S,T) \ ,$$

where $\psi : \Gamma \times \Gamma \to R$ is such that $\psi(S,T)$ is 0(d(S,T)) that is, $\lim_{d(S,T) \to 0} \psi(S,T) / d(S,T) = 0$.

A function $h : \Gamma^n \to R$ is said to have a partial derivative at $S^0 = (S_1^0, ..., S_n^0)$ with respect to its k-th argument ($1 \leq k \leq n$), if the function $\varphi(S_k) = h(S_1^0, \ldots, S_{k-1}^0, S_k, S_{k+1}^0, \ldots, S_n^0)$ has derivative $D\varphi_{S_k^0}$, and define $D_k h(S^0) = D\varphi_{S_k^0}$. If there exists $D_k h(S^0)$, $1 \leq k \leq n$ we put $Dh(S^0) = (D_1 h(S^0), ..., D_k h(S^0))$. If $H : \Gamma^n \to R^m$, $H = (H_1, ..., H_m)$, we put $D_k H_{S^0} = (D_k H_{1S^0}, ..., D_k H_{mS^0})$.

The function $h : \Gamma^n \to R$ is differentiable at S^0 if there exist $Dh(S^0)$ and $\psi : \Gamma^n \times \Gamma^n \to R$ such that

$$h(S) = h\left(S^0\right) + \sum_{k=1}^{n} \left\langle D_k h(S^0), I_{S_k} - I_{S_k^0} \right\rangle + \psi\left(S, S^0\right)$$

where $\psi(S, S^0)$ is 0(d(S,T)) .

We shall next define the notions of (ρ, b)-vexity and generalized (ρ, b)-vexity for a n-set function.These notions are generalizations of those of ρ-convexity or generalized ρ-convexity (defined in Vial[33], Jeyakumar [17], Zalmai [37]) and b-vexity or generalized b-vexity (defined in Bector et al.[1,2]).

Let us consider $f : \Gamma^n \to R$ a differentiable function at S^0, $b : \Gamma^n \times \Gamma^n \to R_+$ and ρ a real number.

The function f is (ρ, b)-vex (strict (ρ, b)-vex) at S^0, if for all $S \in \Gamma^n$ $(S \neq S^0)$, we have

$$b(S, S^0)\left(f(S) - f(S^0)\right) \geqq (>) \sum_{k=1}^{n} \left\langle D_k f_{S^0}, I_{S_k} - I_{S_k^0} \right\rangle + \rho b(S, S^0) d^2(S, S^0)$$

The function f is said to be quasi (ρ, b)-vex (strict quasi (ρ, b)-vex) at S^0, if for all $S \in \Gamma^n$ $(S \neq S^0)$, such that $f(S) \leqq f(S^0)$ we have

$$b(S, S^0) \sum_{k=1}^{n} \left\langle D_k f_{S^0}, I_{S_k} - I_{S_k^0} \right\rangle \leqq (<) - \rho b(S, S^0) d^2(S, S^0)$$

If $\rho > 0$, $\rho = 0$, or $\rho < 0$, the function f is at S^0, strongly quasi b-vex, quasi b-vex, weakly quasi b-vex, respectively .

The function f is said to be pseudo (ρ, b)-vex (strict pseudo (ρ, b)-vex) at S^0, if for all $S \in \Gamma^n$ $(S \neq S^0)$, such that

$$\sum_{k=1}^{n} \left\langle D_k f_{S^0}, I_{S_k} - I_{S_k^0} \right\rangle \geqq -\rho d^2(S, S^0)$$

it results

$$b(S, S^0) f(S) \geqq (>) b(S, S^0) f(S^0).$$

We say that the function f is at S^0, strongly pseudo b-vex, pseudo b-vex, or weakly pseudo b-vex according as, $\rho > 0$, $\rho = 0$, or $\rho < 0$ respectively.

We remark that for b=1 we have the notions of convexity type defined by Zalmai [38] and Lin [20] for n-set functions and Vial [33] and Jeyakumar [17] for point functions.

In the last part of this section we consider some definitions and an optimality result for problem (P).

We say that a feasible solution S^0 of problem (P) is an efficient solution for (P) if there exists no other feasible solution S for (P) such that $F_i(S) \leqq F_i(S^0)$ for all $i \in I$, I={1,2,...,m}, with strict inequality for at least one $i \in I$.

The feasible solution S^0 of problem (P) is a weakly efficient solution for (P) if there exists no other feasible solution S for (P) such that $F_i(S) < F_i(S^0)$ for all $i \in I$.

Using these definitions we note that an efficient solution for (P) is also a weakly efficient solution for problem (P).

A feasible solution S^0 for problem (P) is a properly efficient solution (Geoffrion efficient solution) for problem (P) if it is an efficient solution for problem (P) and if there exists a positive real number M such that for each $i \in I$, and each $S \in \mathcal{P}$ satisfying $F_i(S) < F_i(S^0)$, there exists at least one $j \in I$ such that $F_j(S) < F_j(S^0)$ and

$$[F_i(S^0) - F_i(S)]/[F_j(S) - F_j(S^0)] \leqq M$$

A feasible solution S^0 for problem (P) is a regular feasible solution for problem (P) if there exists $T \in \Gamma^n$ such that for all $j \in J$ we have

$$G_j(S^0) + \sum_{k=1}^{n} \left\langle D_k G_{jS^0}, I_{T_k} - I_{S_k^0} \right\rangle < 0.$$

The following result will be needed in our discussion relative to duality for problem (P).

Lemma 2.1 (Zalmai [38],Theorem 3.2)

Let S^0 be a regular efficient (or weakly efficient) solution for problem (P) and F_i, $i \in I$, and G_j, $j \in J$, are differentiable at S^0. Then there exist $u^o \in R_+^m$, $\sum_{i=1}^m u_i^0 = 1$, $v^0 \in R_+^p$, such that

$$\left\langle u^{0T} D_k F_{S^0} + v^{0T} D_k G_{S^0}, I_{S_k} - I_{S_k^0} \right\rangle \geqq 0$$

for all $S_k \in \Gamma$, $1 \leqq k \leqq n$,

$$v_j^0 G_j(S^0) = 0, j \in J.$$

Now if $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_m)$ are two vectors from R^m, we put $x \leqq y$ iff $x_i \leqq y_i$ for any $i \in I$; $x \leq y$ iff $x \leqq y$ and $x \neq y$; $x < y$ iff $x_i < y_i$ for any $i \in I$. Also we put $x^T y$ for the inner product $\sum_{i=1}^m x_i y_i$ of x and y. We note that $x \in R_+^m$ iff $x \geqq 0$.

3. Mond-Weir duality results , I

In this section we shall formulate and discuss a general duality model for problem (P) and establish appropriate duality relations under (ρ, b) −vexity conditions.

Let $\{J_0, J_1, ..., J_r\}$ be a partition of J. Thus $J_k \subset J$, k=0,1,...,r, $J_{t_1} \cap J_{t_2} = \Phi$ for all $t_1, t_2 \in$ {0,1,2,...,r} with $t_1 \neq t_2$, and $\cup_{t=0}^{r} J_t = J$.

Now we consider the following problem

$(P^\bullet)$ Maximize $F^\bullet(T, u, v) = F(T) + v_{J_0}^T G_{J_0}(T) e$
subject to

$$\left\langle u^T D_k F_T + v^T D_k G_T, I_{S_k} - I_{T_k} \right\rangle \geqq 0 \tag{3.1}$$

for all $S_k \in \Gamma$, $k \in K$,

$$v_{J_t}^T G_{J_t}(T) \geqq 0, t \in L \tag{3.2}$$

$$T \in \Gamma^n, u \in R_+^m, \sum_{i=1}^{m} u_i = 1, v \in R_+^p \tag{3.3}$$

where $L = \{1, 2, ..., r\}$, $K = \{1, 2, ..., n\}$, $u^T D_k F_T = \sum_{i=1}^{m} u_i D_k F_{iT}$, $v_{J_t}^T G_{J_t}(T)$ $= \sum_{j \in J_t} v_j G_j(T)$, and e denotes the m-dimensional row vector (1,1,.......,1). We denote by $\mathcal{P}^*$ the set of all feasible solutions for $(P^\bullet)$.

In constrant to the case of single objective problems, different solution concepts can give rise to different notions of duality in multiobjective problems. Thus is necessary to clearly indicate the requirements to be met by a problem prior to declaring a dual problem $(P^\bullet)$. For this let b:$\Gamma^n \times \Gamma^n \to R_+$.

We put $F \underset{=b}{\geq} F^\bullet$ if there exist no feasible solutions $S' \in \mathcal{P}$ and $(T', u', v') \in \mathcal{P}^\bullet$ such that

$$b(S', T') F_i(S') < b(S', T') F_i^*(T', u', v'),$$

for any $i \in I$.

We shall call (P*) a dual problem for problem (P) if the following requirements are fulfilled:

(d1) If $\mathcal{P} \neq \Phi$ and $\mathcal{P}^* \neq \Phi$ then $F \underset{=b}{\geq} F^\bullet$;

(d2) If (P) has a properly efficient solution S^0, then (P*) has a weakly efficient solution (T^0, u^0, v^0) and $F(S^0) = F^*(T^0, u^0, v^0)$.

The Theorems 3.1, 3.2, 4.1 and 4.2 show that (P*) is a dual problem for (P). Further in Theorems 3.3 and 4.3 some strict converse duality results are given. In Theorems 3.1, 3.2 and 3.3 the assumptions are given relative to (ρ, b) −vexity.

Further in first two theorems these assumptions are required for each individual $F_i^*(.,u,v)$.

Theorem 3.1 (Weak Duality) *We suppose*
a1) For each $i \in I$, $F_i^*(T,u,v)$ is pseudo (ρ_i, b_i) −vex relative to variable T, for any u and v such that $(T,u,v) \in \mathcal{P}^*$;
a2) For each $t \in L$, $v_{J_t}^T G_{J_t}(T)$, is quasi (ρ'_t, b) −vex relative to variable T, for any v such that there exists u with $(T,u,v) \in \mathcal{P}^*$;
a3) $u^T\rho + \sum_{t\in L}\rho'_t \geqq 0$ for all u such that $(T,u,v) \in \mathcal{P}^*$ for some $T \in \Gamma^n$ and $v \in R_+^p$, where $\rho = (\rho_1, ..., \rho_m)$.
Then $F \geqq_b F^*$.

Proof. Assume that the contrary is true, i.e., there exist feasible solutions S' and (T',u',v') for (P) and (P*) respectively, such that
$b(S',T')\,F_i(S') < b(S',T')\,F_i^*(T',u',v')$ for all $i \in I$. Now, because $v'_jG_j(S') \leqq 0$, $j \in J$, and $b \geqq 0$ we obtain

$$\begin{aligned} b(S',T')\,[F_i(S') + v'^T_{J_0}G_{J_0}(S')] < \\ b(S',T')\,[F_i^*(T',u',v') + v'^T_{J_0}G_{J_0}(T')] \end{aligned} \tag{3.4}$$

for any $i \in I$, which implies, in view of (a1), that

$$\left\langle D_kF_{iT'} + v'^T_{J_0}D_kG_{J_0T'}, I_{S'_k} - I_{T'_k}\right\rangle < -\rho_i d^2(S',T') \tag{3.5}$$

for any $i \in I$. Now multiplying (3.5) by u'_i , $i \in I$, and summing, we obtain

$$\sum_{k=1}^{n}\left\langle u'^TD_kF_{T'} + v'^T_{J_0}D_kG_{J_0T'}, I_{S'_k} - I_{T'_k}\right\rangle < -u'^T\rho d^2(S',T') \tag{3.6}$$

Since $S' \in \mathcal{P}$ and $(T',u',v') \in \mathcal{P}^*$, it follows from (3.2) that for any $t \in L$,

$$v'^T_{J_t}G_{J_t}(S') \leqq 0 \leqq v'^T_{J_t}G_{J_t}(T')$$

which implies, by (a2), that for any $t \in L$

$$b(S',T')\sum_{k=1}^{n}\left\langle v'_{J_t}{}^TD_kG_{J_tT'}, I_{S'_k} - I_{T'_k}\right\rangle \leqq -\rho_t b(S',T')\,d^2(S',T') \tag{3.7}$$

Using (3.4) we get $b(S',T') > 0$. Thus by (3.7) we have

$$\sum_{k=1}^{n}\left\langle v'_{J_t}{}^TD_kG_{J_tT'}, I_{S'_k} - I_{T'_k}\right\rangle \leqq -\rho_t d^2(S',T') \tag{3.8}$$

for any $t \in L$. Combining the inequalities (3.8) and (3.6) we obtain

$$\sum_{k=1}^{n} \left\langle u'^T D_k F_{T'} + v'^T D_k G_{T'}, I_{S'_k} - I_{T'_k} \right\rangle < -(u'^T \rho + \sum_{t \in L} \rho'_t) d^2(S', T')$$

From the last inequality , together with the assumption (a3) , we obtain that $(T', u', v') \notin \mathcal{P}^\bullet$, which is a contradiction. Therefore $F \geqq_b F^\bullet$.

Theorem 3.2 (Strong Duality) *Let S^0 be a regular properly efficient solution for problem (P) and assume that*
(i1) For each $i \in I$, $F_i^*(T, u, v)$ is strict quasi $(\rho_i, b)-$vex relative to variable T, for any u and v such that $(T, u, v) \in \mathcal{P}^\bullet$;
(i2) For each $t \in L$, $v_{J_t}^T G_{J_t}(T)$, is quasi $(\rho'_t, b)-$vex relative to variable T, for any v such that there exists u with $(T, u, v) \in \mathcal{P}^\bullet$;
(i3) $u^T \rho + \sum_{t \in L} \rho'_t \geqq 0$ for all u such that $(T, u, v) \in \mathcal{P}^\bullet$ for some $T \in \Gamma^n$ and $v \in R_+^p$.

Then there exist u^0 and v^0 *such that (S^0, u^0, v^0) is a weakly efficient solution for problem (P*) and $F(S^0) = F^*(S^0, u^0, v^0)$.*

Proof. Using Lemma 2.1 we obtain that there exist u^0 and v^0 such that (S^0, u^0, v^0) $\in \mathcal{P}^*$ and further

$$v_j^0 G_j(S^0) = 0 \tag{3.9}$$

for any $j \in J$ .Thus we obtain

$$F(S^0) = F^*(S^0, u^0, v^0). \tag{3.10}$$

We suppose that (S^0, u^0, v^0) is not a weakly efficient solution for (P*). Then there exists $(T, u, v) \in \mathcal{P}^*$ such that

$$F_i(S^0) < F_i(T) + v_{J_0}^{0T} G_{J_0}(T) \tag{3.11}$$

for all $i \in I$. Using (3.9), (3.11), the feasibility of S^0 for (P) and $v \geqq 0$ we obtain

$$F_i(S^0) + v_{J_0}^{0T} G_{J_0}(S^0) < F_i(T) + v_{J_0}^{0T} G_{J_0}(T) \tag{3.12}$$

for all $i \in I$. Now using (i1) in (3.12) we get

$$\begin{aligned} b(S^0, T) \textstyle\sum_{k=1}^{n} \left\langle D_k F_{iT} + v_{J_0}^T D_k G_{J_0 T}, I_{S_k^0} - I_{T_k} \right\rangle < \\ -\rho_i b(S^0, T) d^2(S^0, T) \end{aligned} \tag{3.13}$$

for any $i \in I$. Because $u \geqq 0$ and $e^T u = 1$, multiplying (3.13) by u_i, $i \in I$, and summing we obtain

$$b(S^0,T)\sum_{k=1}^{n}\left\langle u^T D_k F_T + v_{J_0}^T D_k G_{J_0 T}, I_{S_k^0} - I_{T_k}\right\rangle < -(u^T\rho)b(S^0,T)d^2(S^0,T) \qquad (3.14)$$

By (3.13) we note that we must to have $b(S^0,T) > 0$. On the other hand, as in first part of the proof of Theorem 3.1 we have

$$b(S^0,T)\sum_{k=1}^{n}\left\langle v_{J_t}^T D_k G_{J_t T}, I_{S_k^0} - I_{T_k}\right\rangle \leqq \rho_t b(S^0,T)d^2(S^0,T)$$

for any $t \in L$. Summing these relations for $t \in L$ and having in view (3.14) and the fact that J_0,J_1,.......,J_r is a partition for J , we get

$$\sum_{k=1}^{n}\left\langle u^T D_k F_T + v^T D_k G_T, I_{S_k^0} - I_{T_k}\right\rangle < -(u^T\rho + \textstyle\sum_{t\in L}\rho_t)d^2(S^0,T) \qquad (3.15)$$

because $b(S^0,T) > 0$. Using (i3) in (3.15) we obtain

$$\sum_{k=1}^{n}\left\langle u^T D_k F_T + v^T D_k G_T, I_{S_k^0} - I_{T_k}\right\rangle < 0$$

which, according to (3.1) say that $(T',u',v') \notin \mathcal{P}^*$. Thus we have obtained a contradiction. Hence (S^0,u^0,v^0) is a weakly efficient solution for (P*). Having in view (3.10), the proof of this theorem is complete.

The following theorem is a Mangasarian-type [21] strict converse theorem for (P) and (P*).

Theorem 3.3 (Strict Converse Duality) *Let S^* and (S^0,u^0,v^0) be efficient solutions for (P) and (P*) respectively. Also we assume*
a1) $u^{0T}F(S^0) + v_{J_0}^{0T}G_{J_0}(S^0) \geqq u^{0T}F(S') + v_{J_0}^{0T}G_{J_0}(S')$;
a2) for any $i \in I$, $\omega_i(.) = F_i(.) + v_{J_0}^{0T}G_{J_0}(.)$ *is* $(\rho_i, b)-vex$ *and there exists* $i_0 \in I$ *such that* $u_{i_0}^0 > 0$ *and* $\omega_{i_0}(.)$ *is strict* $(\rho_{i_0}, b)-vex$;
a3) for any $j \in J_t$, $t \in L$, $G_j(.)$ *is* $\left(\rho_j', b\right)-vex$;
a4) $u^{0T}\rho + \sum_{t=1}^{r}\sum_{j\in J_t}\rho_j' \geqq 0$.
Then $S^* = S^0$.

Proof. We proceed by contradicting, i.e., we suppose $S^* \neq S^0$. By (a2) we have

$$\begin{gathered} b(S^*, S^0)\{[F_i(S^*) + v_{J_0}^{0T} G_{J_0}(S^*)] - [F_i(S^0) + v_{J_0}^{0T} G_{J_0}(S^0)]\} \geqq \\ \sum_{k=1}^n \left\langle D_k F_{iS^0} + v_{J_0}^{0T} D_k G_{J_0 S^0}, I_{S_k^*} - I_{S_k^0} \right\rangle + \rho_j b(S^*, S^0) d^2(S^*, S^0) \end{gathered}$$

for any $i \in I$ but with a strict inequality for $i = i_0$.

Now, because $u^0 \geq 0, e^T u^0 = 1, u_{i_0}^0 > 0$, we obtain

$$\begin{gathered} b(S^*, S^0)\{\sum_{i=1}^m u_i^0 [F_i(S^*) - F_i(S^0)] + v_{J_0}^{0T} [G_{J_0}(S^*) - G_{J_0}(S^0)]\} > \\ \sum_{k=1}^n \left\langle \sum_{i=1}^m u_i^0 D_k F_{iS^0} + v_{J_0}^{0T} D_k G_{J_0 S^0}, I_{S_k^*} - I_{S_k^0} \right\rangle + \\ (u^{0T}\rho) b(S^*, S^0) d^2(S^*, S^0). \end{gathered}$$

Using this inequality and the assumption (a1) we get

$$\begin{gathered} \sum_{k=1}^n \left\langle \sum_{i=1}^m u_i^0 D_k F_{iS^0} + v_{J_0}^{0T} D_k G_{J_0 S^0}, I_{S_k^*} - I_{S_k^0} \right\rangle < \\ -(u^{0T}\rho) b(S^*, S^0) d^2(S^*, S^0). \end{gathered} \tag{3.16}$$

According to (a3), for any $j \in J_t$, $t \in L$, we have

$$\begin{gathered} b(S^*, S^0)[G_j(S^*) - G_j(S^0)] \geqq \sum_{k=1}^n \left\langle D_k G_{jS^0}, I_{S_k^*} - I_{S_k^0} \right\rangle + \\ \rho'_j b(S^*, S^0) d^2(S^*, S^0). \end{gathered}$$

Now using $v^0 \geqq 0$ we obtain

$$\begin{gathered} b(S^*, S^0) \sum_{t=1}^r \sum_{j \in J_t} v_j^0 [G_j(S^*) - G_j(S^0)] \geqq \\ \sum_{k=1}^n \left\langle \sum_{t=1}^r \sum_{j \in J_t} v_j^0 D_k G_{jS^0}, I_{S_k^*} - I_{S_k^0} \right\rangle + \\ (\sum_{t=1}^r \sum_{j \in J_t} v_j^0 \rho'_j) b(S^*, S^0) d^2(S^*, S^0) \end{gathered} \tag{3.17}$$

Since $S^* \in \mathcal{P}$, $(S^0, u^0, v^0) \in \mathcal{P}^*$, $v^0 \geqq 0$ and $b \geqq 0$ we have

$$b(S^*, S^0) \sum_{t=1}^r \sum_{j \in J_t} v_j^0 [G_j(S^*) - G_j(S^0)] \leqq 0. \tag{3.18}$$

Combining (3.17) and (3.18) we get

$$\begin{gathered} \sum_{k=1}^n \left\langle \sum_{t=1}^r \sum_{j \in J_t} v_j^0 D_k G_{jS^0}, I_{S_k^*} - I_{S_k^0} \right\rangle \leqq \\ -(\sum_{t=1}^r \sum_{j \in J_t} v_j^0 \rho'_j) b(S^*, S^0) d^2(S^*, S^0) \end{gathered} \tag{3.19}$$

Because $J_0, J_1, \ldots.., J_r$ is a partition for J, by (3.1) we obtain

$$\begin{array}{c}\sum_{k=1}^{n}\left\langle \sum_{t=1}^{r}\sum_{j\in J_t} v_j^0 D_k G_{jS^0}, I_{S_k^*} - I_{S_k^0}\right\rangle \geqq \\ -\sum_{k=1}^{n}\left\langle \sum_{i=1}^{m} u_i^0 D_k F_{iS^0} + v_{J_0}^{0T} D_k G_{J_0 S^0}, I_{S_k^*} - I_{S_k^0}\right\rangle\end{array} \tag{3.20}$$

Using the inequalities (3.19) and (3.20) we get

$$\begin{array}{c}\sum_{k=1}^{n}\left\langle \sum_{i=1}^{m} u_i^0 D_k F_{iS^0} + v_{J_0}^{0T} D_k G_{J_0 S^0}, I_{S_k^*} - I_{S_k^0}\right\rangle \geqq \\ (\sum_{t=1}^{r}\sum_{j\in J_t} v_j^0 \rho_j')b(S^*, S^0)d^2(S^*, S^0)\end{array} \tag{3.21}$$

Now by (3.16), (3.21) and (a5) we obtain a contradiction. Thus the theorem is proved.

4. Mond-Weir duality results , II

Now we shall modify the assumptions concerning F^*used in Theorems 3.1 and 3.2. The new assumptions are relative to a scalarization of components of F^*. Here the duality results are stated under generalized (ρ, b)-vexity assumptions.

Theorem 4.1 (Weak Duality) *We assume*
a1') For each $t \in L$ *,the function* $\sum_{i=1}^{m} u_i F_i(T) + v_{J_0}^{0T} G_{J_0}(T)$ *is pseudo* (ρ, b)*-vex relative to* T *for any* $(T, u, v) \in \mathcal{P}^*$*;*
a2') For each $t \in L$*, the function* $\sum_{j\in J_t} v_j G_j(T)$ *is quasi* (ρ_t, b)*-vex relative to* T *for any* $(T, u, v) \in \mathcal{P}^*$*;*
a3') $\rho + \sum_{t=1}^{r} \rho_t \geqq 0$.
Then $F \geqq_b F^\bullet$.

Proof. Let $S \in \mathcal{P}$ and $(T, u, v) \in \mathcal{P}^*$. It follows that

$$\sum_{j\in J_t} v_j G_j(S) \leqq 0 \leqq \sum_{j\in J_t} v_j G_j(T) \tag{4.1}$$

for each $t \in L$. By (4.1) and (a2') we get

$$b(S,T)\sum_{k=1}^{n}\left\langle \sum_{j\in J_t} v_j D_k G_{jT}, I_{S_k} - I_{T_k}\right\rangle \leqq -\rho_t b(S,T)d^2(S,T) \tag{4.2}$$

for all $t \in L$.

Now we suppose the contrary to $F \geqq_b F^\bullet$. Hence, as in the proof of the theorem 3.1, we obtain

$$\begin{array}{c} b(S',T')[F_i(S') + \sum_{j\in J_0} v'_j G_j(S')] < \\ b(S',T')[F_i(T') + \sum_{j\in J_0} v'_j G_j(T')] \end{array} \tag{4.3}$$

for some $S' \in \mathcal{P}$ and $(T', u', v') \in \mathcal{P}^*$. Multiplying (4.3) by u'_i, $i \in I$, and summing, we get

$$\begin{array}{c} b(S',T')[\sum_{i=1}^m u'_i F_i(S') + \sum_{j\in J_0} v'_j G_j(S')] < \\ b(S',T')[\sum_{i=1}^m u'_i F_i(T') + \sum_{j\in J_0} v'_j G_j(T')] \end{array} \tag{4.4}$$

Using (4.4) and assumption (a1') we have

$$\begin{array}{c} \sum_{k=1}^n \left\langle \sum_{i=1}^m u'_i D_k F_{iT'} + \sum_{j\in J_t} v'_j D_k G_{jT'}, I_{S'_k} - I_{T'_k} \right\rangle < \\ -\rho d^2(S',T') \end{array} \tag{4.5}$$

Also using (3.1) and the fact that $J_0, J_1, \ldots.., J_r$ is a partition for J we obtain

$$\begin{array}{c} \sum_{k=1}^n \left\langle \sum_{i=1}^m u'_i D_k F_{iT'} + \sum_{j\in J_0} v'_j D_k G_{jT'}, I_{S'_k} - I_{T'_k} \right\rangle \geqq \\ -\sum_{k=1}^n \left\langle \sum_{t=1}^r \sum_{j\in J_t} v'_j D_k G_{jT'}, I_{S'_k} - I_{T'_k} \right\rangle. \end{array} \tag{4.6}$$

By (4.4) we have $b(S',T') > 0$ and then by (4.2) (for $S = S'$, $T = T'$) and (4.6) we get

$$\begin{array}{c} \sum_{k=1}^n \left\langle \sum_{i=1}^m u'_i D_k F_{iT'} + \sum_{j\in J_0} v'_j D_k G_{jT'}, I_{S'_k} - I_{T'_k} \right\rangle \geqq \\ (\sum_{t=1}^r \rho_t) d^2(S',T'). \end{array} \tag{4.7}$$

Combining (4.5), (4.7) and (a3') we obtain a contradiction. Hence $F \geqq_b F^\bullet$.

Theorem 4.2 (Strong Duality).*Let S^0 be a regular properly efficient solution for problem (P) and also we assume*
i1') the function $\sum_{i=1}^m u_i F_i(T) + v_{J_0}^{0T} G_{J_0}(T)$ is strict quasi (ρ, b)-vex relative to T for any $(T,u,v) \in \mathcal{P}^*$;
i2') For any $t \in L$, the function $\sum_{j\in J_t} v_j G_j(T)$ is quasi (ρ_t, b)-vex relative to T for any $(T,u,v) \in \mathcal{P}^*$;
i3') $\rho + \sum_{t=1}^r \rho_t \geqq 0$.

Then there exist u^0 and v^0 such that $(S^0, u^0, v^0) \in \mathcal{P}^*$ is a weakly efficient solution for (P) and $F(S^0) = F^*(S^0, u^0, v^0)$.*

Proof. Using Lemma 2.1 we have that there exist u^0 and v^0 such that $(S^0, u^0, v^0) \in \mathcal{P}^*$ and

$$v_j^0 G_j(S^0) = 0, \tag{4.8}$$

for any $j \in J$. Also, by (4.8) we obtain $F(S^0) = F^*(S^0, u^0, v^0)$. Now we proceed by contradicting. Hence if (S^0, u^0, v^0) is not an efficient solution for problem (P*), then there exists $(T, u, v) \in \mathcal{P}^*$ such that

$$F_i(S^0) < F_i(T) + \sum_{j \in J_0} v_j G_j(T) \tag{4.9}$$

for all $i \in I$. In view of (4.8) and feasibility of S^0, from (4.9) we get

$$F_i(S^0) + \sum_{j \in J_0} v_j G_j(S^0) < F_i(T) + \sum_{j \in J_0} v_j G_j(T) \tag{4.10}$$

for all $i \in I$. Multiplying (4.10) by u_i, $i \in I$, and summing we get

$$\sum_{i=1}^{m} u_i F_i(S^0) + \sum_{j \in J_0} v_j G_j(S^0) < \sum_{i=1}^{m} u_i F_i(T) + \sum_{j \in J_0} v_j G_j(T).$$

Now, using (i1') we obtain

$$b(S^0, T) \sum_{k=1}^{n} \left\langle \sum_{i=1}^{m} u_i D_k F_{iT} + \sum_{j \in J_0} v_j D_k G_{jT}, I_{S_k^0} - I_{T_k} \right\rangle < -\rho b(S^0, T) d^2(S^0, T)$$

which implies $b(S^0, T) > 0$. Hence

$$\sum_{k=1}^{n} \left\langle \sum_{i=1}^{m} u_i D_k F_{iT} + \sum_{j \in J_0} v_j D_k G_{jT}, I_{S_k^0} - I_{T_k} \right\rangle < -\rho d^2(S^0, T).$$

Now we proceed as in theorem 3.2 and we shall obtain $(T, u, v) \notin \mathcal{P}^*$. Thus we have a contradiction and the theorem is proved .

Theorem 4.3 (Strict Converse Duality) *Let S^* and (S^0, u^0, v^0) be efficient solutions for (P) and (P*) respectively with*
k1) $\sum_{i=1}^{m} u_i^0 F_i(S^0) + \sum_{j \in J_0} v_j^0 G_j(S^0) \geqq \sum_{i=1}^{m} u_i^0 F_i(S^*) + \sum_{j \in J_0} v_j^0 G_j(S^*)$;
k2) $\sum_{i=1}^{m} u_i^0 F_i(.) + \sum_{j \in J_0} v_j^0 G_j(.)$ *is strictly quasi (ρ, b)-vex ;*
k3) $\sum_{j \in J_t} v_j^0 G_j(.)$ *is quasi (ρ_t, b) -vex for any $t \in L$;*
k4) $\rho + \sum_{t=1}^{r} \rho_t \geqq 0$.

Then $S^0 = S^*$ and S^* is an efficient solution for problem (P).

Proof. We suppose on the contrary, i.e., $S^* \neq S^0$. Since S^* and (S^0, u^0, v^0) are feasible solutions for (P) and (P*) respectively , it follows that

$$\sum_{j \in J_t} v_j^0 G_j(S^*) \leqq 0 \leqq \sum_{j \in J_t} v_j^0 G_j(S^0)$$

for all $t \in L$. Now, from the quasi (ρ_t, b) -vexity of $\sum_{j\in J_t} v_j^0 G_j(.)$, we obtain

$$b(S^*, S^0)\sum_{k=1}^n \left\langle \sum_{j\in J_t} v_j^0 D_k G_{jS^0}, I_{S_k^*} - I_{S_k^0}\right\rangle \leqq -\rho_t b(S^*, S^0) d^2(S^*, S^0) \tag{4.11}$$

for all $t \in L$. Using $b(S^*, S^0) \geqq 0$, (4.11) and (3.1) we get

$$b(S^*, S^0)\sum_{k=1}^n \left\langle \sum_{i=1}^m u_i^0 D_k F_{iS^0} + \sum_{j\in J_0} v_j^0 D_k G_{jS^0}, I_{S_k^*} - I_{S_k^0}\right\rangle \geqq (\sum_{t=1}^r \rho_t) b(S^*, S^0) d^2(S^*, S^0). \tag{4.12}$$

But from (k4) we have $\rho + \sum_{t=1}^r \rho_t \geqq 0$ and then from (4.12) we get

$$b(S^*, S^0)\sum_{k=1}^n \left\langle \sum_{i=1}^m u_i^0 D_k F_{iS^0} + \sum_{j\in J_0} v_j^0 D_k G_{jS^0}, I_{S_k^*} - I_{S_k^0}\right\rangle \geqq -\rho b(S^*, S^0) d^2(S^*, S^0). \tag{4.13}$$

From (4.13) and the strict quasi (ρ, b)-vexity of $\sum_{i=1}^m u_i^0 F_i(.) + \sum_{j\in J_0} v_j^0 G_j(.)$ given by (k2), we obtain

$$\sum_{i=1}^m u_i^0 F_i(S^0) + \sum_{j\in J_0} v_j^0 G_j(S^0) < \sum_{i=1}^m u_i^0 F_i(S^*) + \sum_{j\in J_0} v_j^0 G_j(S^*)$$

which contradicts assumption (k1).

Remark 4.1 Also as in [27] and [37] the dual (P*) contains as special cases a number of interesting dual problems for problem (P). These duals can be identified by appropriately specializing the partitioning sets J_0,J_1,........,J_r and by altering the assumptions in Theorems 3.1 and 3.2 or Theorems 4.1 and 4.2 accordingly.

Remark 4.2 The problem (P*) is a Mond-Weir type dual [14, 15, 27, 36, 37, 38].

References

[1] C.R.BECTOR and C.SINGH, B-vex functions, J. Optim. Theory Appl., 71(1991)2, 237-253.

[2] C.R.BECTOR, S.K.SUNEJA and C.S.LALITHA, Generalized B-vex functions and generalized B-vex programming, J.Optim. Theory Appl., 76(1993), 561-576.

[3] D.BEGIS and R.GLOWINSKI, Application de la méthode des éléments fini a l'approximation d'une probléme de domaine optimal. Méthodes de résolution de problémes approchés, Applied Math. Optim., 2(1975), 130-169.

[4] J.CEA, A.GIOAN and J.MICHEL, Quelque résultats sur l'identification de domaines, Calcolo, 10(1973), 133-145.

[5] V.CHANKONG and Y.Y.HAIMES, Multiobjective Decision Making: Theory and Methodology, North - Holland, New York, 1983.

[6] J.H.CHOU, W.S.HSIA and T.Y.LEE, On multiple objective programming problems with set functions, J. Math. Anal. Appl., 105(1985)2, 383-394.

[7] .H.CHOU, W.S.HSIA and T.Y.LEE, Second order optimality conditions for mathematical programming with set functions, J. Austral. Math. Soc. Ser.B, 26(1985), 284-292.

[8] J.H.CHOU, W.S.HSIA and T.Y.LEE, Epigraphs of convex set functions, J. Math. Anal. Appl., 118(1986)1, 247-254.

[9] J.H.CHOU, W.S.HSIA and T.Y.LEE, On multiple objective programming problems with set functions, J. Math. Anal. Appl., 15(1985), 383-394.

[10] H.W.CORLEY, Optimization theory for n-set functions, J. Math. Anal. Appl., 127(1987)1, 193-205.

[11] H.W.CORLEY and S.D.ROBERTS, Duality relationships for a partitioning problem, SIAM J. Appl. Math., 23 (1972), 490-494.

[12] H.W.CORLEY and S.D.ROBERTS, A partitioning problem with applications in regional design, Operations Res., 20(1982), 1010-1019.

[13] G.DANTZIG and A.WALD, On the fundamental lemma of NEYMAN and PEARSON, Ann. Math. Stat., 22(1951), 87-93.

[14] R.R.EGUDO, Efficiency and generalized convex duality for multiobjective programs, J. Math. Anal. Appl., 138 (1989)1, 84 - 94.

[15] R.R.EGUDO, T.WEIR and B.MOND, Duality without constraint qualification for multiobjective programming, J. Austral. Math. Soc. (Series B), 33 (1992), 531 - 544.

[16] W.S.HSIA and T.Y.LEE, Proper D-solutions of multiobjective programming problems with set functions, J. Optim. Theory Appl., 53(1987)2, 247-258.

[17] V.JEYAKUMAR, ρ-Convexity and second order duality, Utilitas Math., 29(1986), 71-85.

[18] H.C.LAI and S.S.YANG, Saddlepoint and duality in the optimization theory of convex set functions, J. Austral. Math. Soc. Ser. B, 24(1982), 130-137.

[19] H.C.LAI , S.S.YANG and G.R.HWANG, Duality in mathematical programming of set functions: on Fenchel duality theorem, J. Math. Anal. Appl., 95(1983)1, 223-234.

[20] L.J.LIN, On optimality of differentiable nonconvex n-set functions, J. Math. Anal. Appl., 168(1992), 351-366.

[21] O.L.MANGASARIAN, Nonlinear Programming, McGraw-Hill, New York, 1969.

[22] P.MAZZOLENI, On constrained optimization for convex set functions. Survey of Mathematical Programming, Vol.1, Edited by A.PREKOPA, North-Holland, Amsterdam, 1979, pp.273-290.

[23] B.MOND and T.WEIR, Generalized concavity and duality, in "Generalized Concavity in Optimization and Economics" (S.SCHAIBLE and W. T. ZIEMBA, Eds.), pp. 263 - 279, Academic Press, San Diego, 1981.

[24] R.J.T.MORRIS, Optimization Problem Involving Set Functions, Ph. D. dissertation, Universiry of California, Los Angeles, 1978.

[25] R.J.T.MORRIS, Optimal constrained selection of a measurable subset, J. Math . Anal. Appl. 70(1979)2, 546-562.

[26] NEYMANN and E.S.PEARSON, On the problem of the most efficient tests of statistical hypotheses. Philos. Trans. Roy. Soc. London, Ser.A, 231(1933), 289-337.

[27] V.PREDA, On minmax programming problems containing n-set functions, Optimization, 22(1991)4, 527-537.

[28] V.PREDA, On efficiency and duality for multiobjective programs, J. Math. Anal. Appl., 166 (1992), 365 - 377.

[29] J.ROSENMULLER, Some properties of convex set functions, Arch. Math., 22 (1971), 420-430.

[30] J.ROSENMULLER and H.G.WEIDNER, A class of extreme convex set functions with finite carrier, Adv. Math., 10(1973), 1-38.

[31] J.ROSENMULLER and H.G.WEIDNER, Extreme convex set functions with finite carrier: General theory, Discrete Math, 10(1974), 343-382.

[32] K.TANAKA and Y.MARUYAMA, The multiobjective optimization problem of set function, J. Inform. Optim. Sci., 5(1984), 293-306.

[33] J.P.VIAL, Strong and weak convexity of sets and functions, Math. Oper. Res., 8(1983)2, 231-259.

[34] P.K.C.WANG, On a class of optimization problems involving domain variations. Lectures Notes in Control and Information Sciences, Vol.2. Springer-Verlag, Berlin, 1977.

[35] T.WEIR, Proper efficiency and duality for vector valued optimization problems, J. Austral. Math. Soc. (Series A), 43 (1987), 21 - 34.

[36] T.WEIR and B.MOND, Multiple objective programming duality without a constraint qualification, Utilitas Math. 39 (1991), 41 - 55.

[37] G.J.ZALMAI, Optimality conditions and duality for constrained measurable subset selection problems with minmax objective functions, Optimization, 20(1989)4, 377-395.

[38] G.J.ZALMAI, Optimality conditions and duality for multiobjective measurable subset selection problems, Optimization, 22(1991)2, 221-238.