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Optimality and Duality in Nonlinear Programming Involving Semilocally Preinvex and Related Functions

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Abstract

For a nondifferentiable nonlinear programming problem, Fritz John and Kuhn-Tucker necessary optimality conditions and sufficient optimality conditions are given and duality results are stated for Wolfe's type and Mond-Weir type duals by the concepts of semilocally quasi-preinvex and semilocally pseudo-preinvex functions.

1. Introduction.

In [3], Ewing introduces the concept of semilocally convex functions and in [5], Kaul and Kaur have extended this concept to semilocally quasiconvex and semilocally pseudoconvex functions. For some properties of these functions see Kaur [8]. Also Kaul and Kaur [6], by using these concepts gave sufficient optimality conditions for a nonlinear programming problem. For necessary optimality conditions and duality see Kaul and Kaur [7]. Following the ideas of Bector [2] and Gupta and Bector [4], Suneja and Gupta [12] established other properties of the quotient and the product of the above mentioned functions. The necessary conditions on the assumption that the right differentials of these functions at some point are convex, are also given in [12]. They also study the duals of Wolfe and Mond-Weir types.

In this paper we obtain Fritz John and Kuhn-Tucker necessary conditions and sufficient optimality conditions as well as duality results for the Wolfe and Mond-Weir dualities. These results are obtained under weaker assumptions of convexity than those in Kaul and Kaur [7] and Suneja and Gupta [12]. Thus, the concepts of semilocally preinvex, semilocally quasi-preinvex and semilocally pseudo-preinvex functions are introduced as an extension of those given in Ewing [3], Kaul and Kaur [5], Suneja and Gupta [12] by replacing the concept of convexity with that of preinvexity, introduced by Ben Israel and Mond [1]. In a different way was it treated in Suneja and Gupta [12], using an alternative theorem of Weir and Mond [13] for preinvex functions, we get the necessary conditions under the assumption that the right differentials of these functions at some point are preinvex. On the other hand, we use in section 5 a general Mond-Weir dual for the nonlinear programming problem.

2. Definitions and Preliminaries.

Let $X^0 \subseteq \mathbf{R^n}$ be a set and $\eta : X^0 \times X^0 \longrightarrow \mathbf{R^n}$ be a vectorial application.

Definition 2.1. *We say that the set X^0 is η-vex at $\bar{x} \in X^0$ if $\bar{x} + \lambda\eta(x, \bar{x}) \in X^0$ for any $x \in X^0$ and $\lambda \in [0, 1]$.*
We say that the set X^0 is η-vex if X^0 is η-vex at any $x \in X^0$.

We remark that if $\eta(x, \bar{x}) = x - \bar{x}$ for any $x \in X^0$ then X^0 is η-vex at $\bar{x}$ iff X^0 is a convex set at $\bar{x}$.

Definition 2.2. *We say that the set $X^0 \subseteq \mathbf{R^n}$ is an η-locally starshaped set at $\bar{x}$, $\bar{x} \in X^0$, if for any $x \in X^0$ there exists $0 < a_\eta(x, \bar{x}) \leq 1$ such that $\bar{x} + \lambda\eta(x, \bar{x}) \in X^0$ for any $\lambda \in [0, a_\eta(x, \bar{x})]$.*

Definition 2.3. *Let $f : X^0 \longrightarrow \mathbf{R}$ be a function, where $X^0 \subseteq \mathbf{R^n}$ is an η-locally starshaped set at $\bar{x} \in X^0$. We say that f is:*

(i_1) semilocally preinvex (slpi) at $\bar{x}$ if corresponding to $\bar{x}$ and each $x \in X^0$, there exists a positive number $d_\eta(x,\bar{x}) \leq a_\eta(x,\bar{x})$ such that $f(\bar{x}+\lambda\eta(x,\bar{x})) \leq \lambda f(x) + (1-\lambda)f(\bar{x})$ for $0 < \lambda < d_\eta(x,\bar{x})$;

(i_2) semilocally quasi-preinvex (slqpi) at $\bar{x}$ if corresponding to $\bar{x}$ and each $x \in X^0$, there exists a positive number $d_\eta(x,\bar{x}) \leq a_\eta(x,\bar{x})$ such that $f(x) \leq f(\bar{x})$ and $0 < \lambda < d_\eta(x,\bar{x})$ implies $f(\bar{x}+\lambda\eta(x,\bar{x})) \leq f(\bar{x})$.

Definition 2.4. *Let $f : X^0 \longrightarrow \mathbf{R}$ be a function, where $X^0 \subseteq \mathbf{R^n}$ is an η-locally starshaped set at $\bar{x} \in X^0$. We say that f is η-semidifferentiable at $\bar{x}$ if $(df)^+(\bar{x},\eta(x,\bar{x}))$ exists for each $x \in X^0$, where $(df)^+(\bar{x},\eta(x,\bar{x})) = \lim\limits_{\lambda\to 0+} \frac{1}{\lambda}[f(\bar{x} + \lambda\eta(x,\bar{x})) - f(\bar{x})]$ (the right derivative at $\bar{x}$ along the direction $\eta(x,\bar{x})$).*
If f is η-semidifferentiable at any $\bar{x} \in X^0$, then f is said to be η-semidifferentiable on X^0.

Theorem 2.5. *Let $f : X^0 \longrightarrow \mathbf{R}$ be an η-semidifferentiable function at $\bar{x} \in X^0$. If f is slqpi at $\bar{x}$ and $f(x) \leq f(\bar{x})$ then $(df)^+(\bar{x},\eta(x,\bar{x})) \leq 0$.*

Definition 2.6. *We say that f is semilocally pseudo-preinvex (slppi) at $\bar{x}$ if for any $x \in X^0$, $(df)^+(\bar{x},\eta(x,\bar{x})) \geq 0 \Longrightarrow f(x) \geq f(\bar{x})$.*

If f is slppi at any $\bar{x} \in X^0$, then f is said to be slppi on X^0.

Definition 2.7. *Let X and Y be two subsets of X^0, and $\bar{y} \in Y$. We say that Y is η-starshaped at $\bar{y}$ with respect to X if for any $x \in X$ there exists $0 < a_\eta(x,\bar{y}) \leq 1$ such that $\bar{y} + \lambda\eta(x,\bar{y}) \in Y$ for any $0 \leq \lambda \leq a_\eta(x,\bar{y})$.*

Definition 2.8. *Let Y be η-starshaped at $\bar{y}$ with respect to X and f be an η-semidifferentiable function at $\bar{y}$. We say that f is:*

(i_1) slppi at $\bar{y} \in Y$ with respect to X, if for any $x \in X$, $(df)^+(\bar{y},\eta(x,\bar{y})) \geq 0 \Longrightarrow f(x) \geq f(\bar{y})$;

(i_2) strictly semilocally pseudo-preinvex (sslppi) at $\bar{y} \in Y$ with respect to X, if for each $x \in X$, $x \neq \bar{y}$, $(df)^+(\bar{y},\eta(x,\bar{y})) \geq 0 \Longrightarrow f(x) > f(\bar{y})$.

We say that f is slppi (sslppi) on Y with respect to X if f is slppi (sslppi) at any point of Y with respect to X.

Remark: In case of $X = Y = X^0$, the last definition, part (i_1), reduces to definition 2.6.

Now we give some properties of the above defined functions.

Theorem 2.9. *Let $f, g : X^0 \longrightarrow \mathbf{R}_+$ be such that f and $-g$ are slpi on X^0 and g is strictly positive and finite function on X^0. Then $h = \frac{f}{g}$ is a slqpi function on X^0.*

Proof: Let $x_1, x_2 \in X^0$ such that $h(x_2) \leq h(x_1)$. Since f and $-g$ are slpi on X^0, there exist $0 < d_{1\eta}(x_1, x_2) \leq a_\eta(x_1, x_2)$ and $0 < d_{2\eta}(x_1, x_2) \leq a_\eta(x_1, x_2)$ such that

$f(x_1 + \lambda\eta(x_2, x_1)) \leq (1-\lambda)f(x_1) + \lambda f(x_2)$, $0 < \lambda < d_{1\eta}(x_1, x_2)$, and
$g(x_1 + \lambda\eta(x_2, x_1)) \geq (1-\lambda)g(x_1) + \lambda g(x_2)$, $0 < \lambda < d_{2\eta}(x_1, x_2)$.
We take $d_{3\eta}(x_1, x_2) = \min\{d_{1\eta}(x_1, x_2), d_{2\eta}(x_1, x_2)\}$. Then

$$h(x_1 + \lambda\eta(x_2, x_1)) - h(x_1) = \frac{f(x_1 + \lambda\eta(x_2, x_1))}{g(x_1 + \lambda\eta(x_2, x_1))} - \frac{f(x_1)}{g(x_1)} \leq$$

$$\leq \frac{g(x_1)[(1-\lambda)f(x_1) + \lambda f(x_2)] - f(x_1)[(1-\lambda)g(x_1) + \lambda g(x_2)]}{g(x_1)g(x_1 + \lambda\eta(x_2, x_1))} =$$

$$= \frac{\lambda g(x_2)[h(x_2 - h(x_1)]}{g(x_1 + \lambda\eta(x_2, x_1))} \leq 0$$

for $0 < \lambda < d_{3\eta}(x_1, x_2)$ because $x_1 + \lambda\eta(x_2, x_1) \in X^0$ for $0 < \lambda < d_{3\eta}(x_1, x_2) \leq a_\eta(x_1, x_2)$. Thus we have $h(x_1 + \lambda\eta(x_2, x_1)) \leq h(x_1)$ for all $0 < \lambda < d_{3\eta}(x_1, x_2)$, i.e. h is slqpi on X^0.

Corollary 2.10. *If $-f$ is slpi and non-negative function on X^0 and g is slpi, strictly positive and finite function on X^0, then $-h$ is slqpi on X^0.*

Corollary 2.11. *If g is strictly positive and finite function on X^0, then $\frac{1}{g}$ is slqpi on X^0 if and only if $-g$ is slqpi on X^0.*

Theorem 2.12. *Let f be a slpi and non-negative function on X^0, $-g$ be a slpi, strictly negative and finite function on X^0. Then $u = \frac{f^2}{g}$ is slpi on X^0.*

Proof: Let $x_1, x_2 \in X^0$. By hypotheses on f and $-g$, there exist $0 < d_{1\eta}(x_1, x_2) \leq a_\eta(x_1, x_2)$ and $0 < d_{2\eta}(x_1, x_2) \leq a_\eta(x_1, x_2)$ such that $f(x_1 + \lambda\eta(x_2, x_1)) \leq (1-\lambda)f(x_1) + \lambda f(x_2)$, $0 < \lambda < d_{1\eta}(x_1, x_2)$, and $g(x_1 + \lambda\eta(x_2, x_1)) \geq (1-\lambda)g(x_1) + \lambda g(x_2)$, $0 < \lambda < d_{2\eta}(x_1, x_2)$. We take $d_{3\eta}(x_1, x_2) = \min\{d_{1\eta}(x_1, x_2), d_{2\eta}(x_1, x_2)\}$. Now we have:

$$u(x_1 + \lambda\eta(x_2, x_1)) - [(1-\lambda)u(x_1) + \lambda u(x_2)] =$$

$$= \frac{f^2(x_1+\lambda\eta(x_2,x_1))}{g(x_1+\lambda\eta(x_2,x_1))} - \left[(1-\lambda)\frac{f^2(x_1)}{g(x_1)} + \lambda\frac{f^2(x_2)}{g(x_2)}\right] \leq$$
$$\leq \frac{[(1-\lambda)f(x_1)+\lambda f(x_2)]^2}{(1-\lambda)g(x_1)+\lambda g(x_2)} - (1-\lambda)\frac{f^2(x_1)}{g(x_1)} - \lambda\frac{f^2(x_2)}{g(x_2)} =$$
$$= \frac{-\lambda(1-\lambda)[f(x_1)g(x_2)-f(x_2)g(x_1)]^2}{g(x_1)g(x_2)[(1-\lambda)g(x_1)+\lambda g(x_2)]} \leq 0$$

for $0 < \lambda < d_{3\eta}(x_1, x_2)$, because $g(x_1) > 0$ and $g(x_2) > 0$. Thus u is slpi on X^0.

Theorem 2.13. *Let $f, g : X^0 \subseteq \mathbf{R}^n \longrightarrow \mathbf{R}$ be functions such that f and $-g$ are slpi and non-negative on X^0. Then $v = -fg$ is slqpi on X^0.*

Proof: Let us consider $x_1, x_2 \in X^0$ such that $v(x_1) \leq v(x_2)$. We have to prove that $v(x_1) \leq v(x_1 + \lambda\eta(x_2, x_1))$ for all $0 < \lambda < d_\eta(x_1, x_2)$, where $0 < d_\eta(x_1, x_2) \leq a_\eta(x_1, x_2)$. Using the fact that $-f$ and $-g$ are slpi functions, there exist $0 < d_{1\eta}(x_1, x_2) \leq a_\eta(x_1, x_2)$ and $0 < d_{2\eta}(x_1, x_2) \leq a_\eta(x_1, x_2)$ such that $f((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)f(x_1) + \lambda f(x_2)$, for $0 < \lambda < d_{1\eta}(x_1, x_2)$, and $g((1-\lambda)x_1 + \lambda x_2)) \geq (1-\lambda)g(x_1) + \lambda g(x_2)$, for $0 < \lambda < d_{2\eta}(x_1, x_2)$. Now, for $0 < \lambda < d_\eta(x_1, x_2) = \min\{d_{1\eta}(x_1, x_2), d_{2\eta}(x_1, x_2)\}$, we have

$$v(x_1+\lambda\eta(x_2,x_1)) = f(x_1+\lambda\eta(x_2,x_1))g(x_1+\lambda\eta(x_2,x_1)) \geq$$
$$\geq [(1-\lambda)f(x_1)+\lambda f(x_2)][(1-\lambda)g(x_1)+\lambda g(x_2)] \geq$$
$$\geq v(x_1) + \lambda(1-\lambda)\Delta(f,g,x_1,x_2)$$

where $\Delta(f, g, x_1, x_2) = f(x_1)[g(x_2) - g(x_1)] + g(x_1)[f(x_2) - f(x_1)]$. To complete the proof we proceed as in Suneja and Gupta [12] Theorem 2.11.

3. Optimality Conditions.

In this section we consider the following nonlinear optimality problem:

$$(\mathrm{P}) \begin{cases} \text{minimize } f(x) \\ \text{subject to: } g(x) \leq 0 \ , \ x \in X^0 \end{cases}$$

where $X^0 \subseteq \mathbf{R^n}$ is a nonempty η-locally starshaped set at any $x \in X^0$.

Let $X = \{x \in X^0 \mid g(x) \leq 0\}$ be the set of all feasible solutions for (P). For $\bar{x} \in X$ we denote $B(\bar{x}) = \{i \mid 1 \leq i \leq m, g_i(\bar{x}) = 0\}$, $NB(\bar{x}) = \{1, ..., m\} \backslash B(\bar{x})$ and $g^0 = (g_i)_{i \in B(\bar{x})}$.

Definition 3.1. *We say that g satisfies the generalized Slater's constraint qualification (GSQ) at $\bar{x} \in X$, if g^0 is slppi at $\bar{x}$ and there exists an $\hat{x} \in X$ such that $g^0(\hat{x}) < 0$.*

The following Lemma can be proved without difficulty.

Lemma 3.2. *Let $\bar{x} \in X$ be a (local) minimum solution for (P). Futher we assume that g_i is continuous at $\bar{x}$ for any $i \in NB(\bar{x})$, and that f, g^0 are η-semidifferentiable at $\bar{x}$. Then the system*

$$\begin{cases} (df)^+(\bar{x}, \eta(x, \bar{x})) < 0 \\ (dg^0)^+(\bar{x}, \eta(x, \bar{x})) < 0 \end{cases} \tag{3.1}$$

has no solution $x \in X^0$.

Lemma 3.3. *(Weir and Mond [13], Theorem 2.1) Let S be a nonempty set in $\mathbf{R^n}$ and let $\varphi : S \times S \longrightarrow \mathbf{R^m}$ be a preinvex function on S, where S is a φ-vex set. Then either $\varphi(x) < 0$ has a solution $x \in S$, or there is some $\lambda \in \mathbf{R^m}$, $\lambda \geq \mathbf{0}$, $\lambda \neq \mathbf{0}$, such that $\lambda^\top \cdot \varphi(x) \geq 0$ for all $x \in S$, but both alternatives are never true.*

In the next Theorem we obtain an important result of a Fritz-John type necessary optimality criteria.

Theorem 3.4. *Let us suppose that g_i is continuous at $\bar{x}$ for $i \in NB(\bar{x})$, $(df)^+(\bar{x}, \eta(x, \bar{x}))$ and $(dg^0)^+(\bar{x}, \eta(x, \bar{x}))$ are η_1-preinvex functions of x on X^0 - a η_1-vex set at $\bar{x}$. If $\bar{x}$ is a (local) minimum solution for (P), then there exist $\bar{u}_0 \in \mathbf{R}, \bar{u} \in \mathbf{R^m}$ such that*

$$\bar{u}_0 (df)^+(\bar{x}, \eta(x, \bar{x})) + \bar{u}^\top \cdot (dg^0)^+(\bar{x}, \eta(x, \bar{x})) \geq 0 \text{ for all } x \in X^0 \tag{3.2}$$

$$\bar{u}^\top \cdot g(\bar{x}) = 0 \tag{3.3}$$

$$(\bar{u}_0, \bar{u}) \neq 0 \ , \ (\bar{u}_0, \bar{u}) \geq 0 \tag{3.4}$$

Proof: Since the conditions of Lemma 3.2 are satisfied, we get that the system (3.1) has no solution $x \in X^0$. But the assumptions of Lemma 3.3 also hold and since the system (3.1) has no solution $x \in X^0$ we obtain that there exists $\bar{u}_0 \in \mathbf{R}$ and $\bar{u}_i \in \mathbf{R}, (i \in B(\bar{x}))$, with $\bar{u}_0 \geq 0, \bar{u}_i \geq 0, (i \in B(\bar{x}))$ and $(\bar{u}_0, (\bar{u})_{i \in B(\bar{x})}) \neq 0$, such that

$$\bar{u}_0 (\mathrm{d}f)^+(\bar{x}, \eta(x, \bar{x})) + \sum_{i \in B(\bar{x})} \bar{u}_i (\mathrm{d}g_i)^+(\bar{x}, \eta(x, \bar{x})) \geq 0 \tag{3.5}$$

If we put $\bar{u}_i = 0$ for $i \in NB(\bar{x})$, by (3.5), we get (3.2). Finally, the relations (3.3) and (3.4) follow obviously and the proof is complete.

The next Theorem is a Kuhn-Tucker type necessary optimality criteria. In this theorem, the above defined generalized constraint qualification is very important.

Theorem 3.5. *Let $\bar{x} \in X$ be a (local) minimum solution for (P), let g_i be continuous at $\bar{x}$ for $i \in NB(\bar{x})$ and let $(df)^+(\bar{x}, \eta(x, \bar{x}))$ and $(dg^0)^+(\bar{x}, \eta(x, \bar{x}))$ are η_1-preinvex functions of x on X_0 - a η_1-vex set at $\bar{x}$. If g satisfies GSQ at $\bar{x}$, then there exists $\bar{u} \in \mathbf{R^m}$ such that*

$$\bar{u}_0 (df)^+(\bar{x}, \eta(x, \bar{x})) + \bar{u}^{\top} \cdot (dg)^+(\bar{x}, \eta(x, \bar{x})) \geq 0 \text{ for any } x \in X^0 \tag{3.6}$$

$$\bar{u}^{\top} \cdot g(\bar{x}) = 0 \tag{3.7}$$

$$g(\bar{x}) \leq 0 \tag{3.8}$$

$$\bar{u} \geq 0 \tag{3.9}$$

Proof: Since the hypotheses of Theorem 3.4 are satisfied, it follows that there exist $\bar{u}_0 \in \mathbf{R}$, $\bar{u} \in \mathbf{R^m}$ such that (3.2) - (3.4) hold. The proof is complete if we show that $\bar{u}_0 \neq 0$. For this we proceed by contradiction. Let us assume that $\bar{u}_0 = 0$. Then from (3.2) - (3.4) we get

$$\bar{u}^{\top} \cdot (\mathrm{d}g)^+(\bar{x}, \eta(x, \bar{x})) \geq 0 \text{ for all } x \in X^0 \tag{3.10}$$

$$\bar{u}^{\top} \cdot g(\bar{x}) = 0 \tag{3.11}$$

$$\bar{u} \geq 0 \tag{3.12}$$

By the generalized Slater's constraint qualification at $\bar{x}$ we have that g^0 is a slppi function at $\bar{x}$ and there exists an $\hat{x} \in X$ such that

$$g^0(\hat{x}) < 0 \tag{3.13}$$

Using (3.10) and (3.11) it follows

$$\sum_{i\in B(\bar{x})} \bar{u}_i(\mathrm{d}g_i)^+(\bar{x},\eta(x,\bar{x})) \geq 0 \tag{3.14}$$

Since $g^0(\bar{x}) = 0$, by (3.13) we obtain $g^0(\hat{x}) < g^0(\bar{x})$. Now by the slppi of g^0 at $\bar{x}$ it follows

$$(\mathrm{d}g^0)^+(\bar{x},\eta(\hat{x},\bar{x})) < 0 \tag{3.15}$$

Combining (3.12) with (3.15) we obtain

$$\sum_{i\in B(\bar{x})} \bar{u}_i(\mathrm{d}g_i)^+(\bar{x},\eta(\hat{x},\bar{x})) < 0$$

and this is in contradiction with (3.14). Thus the proof is complete.

The next theorem gives a sufficient optimality criteria.

Theorem 3.6. *Let $u \in \mathbf{R}^m$ and $f(\cdot) + u^\top \cdot g(\cdot)$ be slppi on X^0. We assume that at $\bar{x} \in \mathbf{R}^n$, f and g are η-semidifferentiable and the following conditions are satisfied*

$$(df)^+(\bar{x},\eta(x,\bar{x})) + u^\top \cdot (dg)^+(\bar{x},\eta(x,\bar{x})) \geq 0 \text{ for all } x \in X^0 \tag{3.16}$$

$$g(\bar{x}) \leq 0 \tag{3.17}$$

$$u \geq 0 \tag{3.18}$$

$$u^\top \cdot g(\bar{x}) = 0 \tag{3.19}$$

Then $\bar{x}$ is a minimum solution for the problem (P).

Proof: By (3.16), for any $x \in X^0$, we have $(\mathrm{d}(f + u^\top \cdot g))^+(\bar{x},\eta(x,\bar{x})) \geq 0$. Now, by assumption $f(\cdot) + u^\top \cdot g(\cdot)$ is slppi at $\bar{x}$, it follows

$$f(x) + u^\top \cdot g(x) \geq f(\bar{x}) + u^\top \cdot g(\bar{x}) \tag{3.20}$$

for all $x \in X^0$. But for $x \in X$, from (3.18) we get $u^\top \cdot g(x) \leq 0$. Hence (3.20) implies

$$f(x) \geq f(\bar{x}) + u^\top \cdot g(\bar{x}) \text{ , for any } x \in X \tag{3.21}$$

Now, from (3.21) and (3.19) follows $f(x) \geq f(\bar{x})$ for any $x \in X$, i.e. $\bar{x}$ is an optimal solution for (P) and the theorem is proved.

4. Wolfe Duality.

Relative to η and the problem (P), we consider the following Wolfe dual:

$$(\mathrm{D})\begin{cases} \text{maximize } \Psi(u,y) = f(u) + y^{\top}\cdot g(u) \\ \text{subject to: } (\mathrm{d}f)^{+}(u,\eta(x,u)) + y^{\top}\cdot(\mathrm{d}g)^{+}(u,\eta(x,u)) \geq 0 \text{ for any } x \in X \\ y \geq 0 \ ,\ y \in \mathbf{R}^{\mathbf{m}} \ ,\ u \in X^{0} \end{cases}$$

where X^0 is a nonempty η-locally starshaped set at any $x \in X^0$.

Remark: If $\eta(x,u) = x - u$ we obtain the Wolfe dual defined in Kaul and Kaur [7], Mahajan and Vartak [9] and Suneja and Gupta [12].

Let W be the set of all feasible solutions of problem (D). We define the following sets:

$$Y = \{y \in \mathbf{R}^{\mathbf{m}} \mid (u,y) \in W \text{ for some } u \in X^0\};$$
$$U(y) = \{u \in X^0 \mid (u,y) \in W\}, \text{ for } y \in Y \ ;$$
$$U = \bigcup_{y\in Y} U(y).$$

Using the same technique as in the previous section and following the proofs of Theorems 4.4, 4.6 and 4.7 in [12], we get the following

Theorem 4.1. *(Weak Duality) If $x \in X$ and $(u,y) \in W$ and the function $f(\cdot) + y^{\top}\cdot g(\cdot)$ is slppi on the set $U(y)$ with respect to X, then $\Psi(u,y) \leq f(x)$.*

Corollary 4.2. *Let $x \in X$ and $(u,y) \in W$ be such that $f(x) = \Psi(u,y)$ and the function $f(\cdot) + y^{\top}\cdot g(\cdot)$ is slppi on the set $U(y)$ with respect to X. Then x is an optimal solution for problem (P) and (u,y) is an optimal solution for problem (D).*

Theorem 4.3. *(Direct Duality) Let $\bar{x}$ be a (local) optimal solution for (P) such that:*

a_1) $(df)^{+}(\bar{x},\eta(x,\bar{x}))$ and $(dg^0)^{+}(\bar{x},\eta(x,\bar{x}))$ are η_1-preinvex functions of x on X^0 - a η_1-vex set at $\bar{x}$;

a_2) for $i \in NB(\bar{x})$, g_i is continuous at $\bar{x}$;

a_3) g satisfies the GSQ.

Then there exists $\bar{y} \in \mathbf{R}^{\mathbf{m}}$ such that $(\bar{x},\bar{y}) \in W$ and $f(\bar{x}) = \Psi(\bar{x},\bar{y})$.

Further on, if for any fixed $y \in Y$, the function $f(\cdot) + y^{\top}\cdot g(\cdot)$ is slppi on the set $U(y)$ with respect to X, then $\bar{x}$ is optimal for (P) and $(\bar{x},\bar{y})$ is optimal for (D).

Theorem 4.4. *(Strict Converse Duality) Let $\bar{x}$ be an optimal solution for (P) such that:*

a_1) $(df)^+(\bar{x}, \eta(x, \bar{x}))$ and $(dg^0)^+(\bar{x}, \eta(x, \bar{x}))$ are η_1-preinvex functions of x on X^0 - a η_1-vex set at $\bar{x}$;

a_2) for $i \in NB(\bar{x})$, g_i is continuous at $\bar{x}$;

a_3) g satisfies the GSQ;

a_4) for any fixed $y \in Y$, the function $f(\cdot) + y^\top \cdot g(\cdot)$ is slppi on the set $U(y)$ with respect to X.

If $(\bar{u}, \bar{y})$ is optimal for (D) and $f(\cdot) + y^\top \cdot g(\cdot)$ is sslppi at $\bar{u} \in U(y)$ with respect to X, then $\bar{u} = \bar{x}$, i.e. $\bar{u}$ is an optimal solution for (P) and $f(\overline{x}) = \Psi(\bar{u}, \bar{y})$.

5. Mond-Weir Duality.

For problem (P) we consider a general Mond-Weir dual problem [11]:

$$
(\text{MWD}) \begin{cases} \text{maximize } \Psi(u, y) = f(u) + y_{I_0}^\top \cdot g_{I_0}(u) \\ \text{subject to:} \\ \quad (\mathrm{d}f)^+(u, \eta(x, u)) + y^\top (\mathrm{d}g)^+(u, \eta(x, u)) \geq 0 \ , \ (\forall)\ x \in X \\ \quad y_{I_s}^\top \cdot g_{I_s}(u) \geq 0 \ , \ (s = 1, 2, ..., r) \\ \quad y \geq 0 \ , \ y \in \mathbf{R^m} \ , \ u \in X^0 \end{cases}
$$

where $r \geq 1$, $I_p \cap I_q = \emptyset$ for $p \neq q$, $\bigcup_{p=0}^{r} I_p = \{1, 2, ..., m\}$, and $y_{I_p} = (y_i)_{i \in I_p}$, $g_{I_p} = (g_i)_{i \in I_p}$. Also we suppose that X^0 is a η-locally starshaped set at any $x \in X^0$.

We proceed in a same way as in section 2 and following the steps in [12] for the generalized Mond-Weir duality, we get the following results about the duality between the problems (P) and (MWD).

We denote by $\tilde{W}$ the set of all feasible solutions of (MWD) and we define $\tilde{Y}, \tilde{U}(y)$ and $\tilde{U}$ as follows:

$$
\begin{aligned}
\tilde{Y} &= \{y \in \mathbf{R^m} \mid (u, y) \in \tilde{W} \text{ for some } u \in X^0\}; \\
\tilde{U}(y) &= \{u \in X^0 \mid (u, y) \in \tilde{W}\} \qquad \text{for } y \in \tilde{Y} \ ; \\
\tilde{U} &= \bigcup_{y \in \tilde{Y}} \tilde{U}(y).
\end{aligned}
$$

Theorem 5.1. *(Weak Duality) Let $x \in X, (u, y) \in \tilde{W}$ and f, g are η- semidifferentiable functions on X. We assume that the function $f(\cdot) + y_{I_0}^\top \cdot g_{I_0}(\cdot)$ is slppi, and for $1 \leq p \leq r$, $y_{I_0}^\top \cdot g_{I_0}(u)$ is slqpi on $\tilde{U}(y)$ with respect to X. Then $f(x) \geq \Psi(u, y)$.*

Corollary 5.2. *Let $x \in X$ and $(u, y) \in \tilde{W}$ be such that $f(x) = \Psi(u, y)$. We assume that the hypotheses of Theorem 5.1 are satisfied. Then x and (u, y) are optimal solutions for (P) and (MWD) respectively.*

Theorem 5.3. *(Direct Duality) Let $\bar{x} \in X$ be a (local) minimum solution for (P) at which GSQ is satisfied. We assume that $(df)^{+}(\bar{x}, \eta(x, \bar{x}))$ and $(dg^{0})^{+}(\bar{x}, \eta(x, \bar{x}))$ are η_1-preinvex functions of x on X^0 - a η_1-vex set at $\bar{x}$. If for $i \in NB(\bar{x})$, g_i is continuous at $\bar{x}$, then there exists $\bar{y} \in \mathbf{R}^{\mathbf{m}}$ such that $(\bar{x}, \bar{y}) \in \tilde{W}$ and $f(\bar{x}) = \Psi(\bar{x}, \bar{y})$. Further on, if the assumptions of Theorem 5.1 are satisfied for any $y \in \tilde{Y}$, then $(\bar{x}, \bar{y})$ is an otimal solution for (MWD).*

Theorem 5.4. *(Converse Duality) Let $(\bar{u}, \bar{y}) \in \tilde{W}$ and f, g be η-semidifferentiable functions at $\bar{u}$. We assume that $f(\cdot) + \bar{y}_{I_0}^{\top} \cdot g_{I_0}(\cdot)$ is slppi, and for $1 \le p \le r$, $\bar{y}_{I_0}^{\top} \cdot g_{I_0}(u)$ is slqpi on $\tilde{U}(\bar{y})$ with respect to X. If there exists $\bar{x} \in X$ such that $f(\bar{x}) = \Psi(\bar{u}, \bar{y})$, then $\bar{x}$ is an optimal solution for (P).*

Remark 1: If there exists p, $1 \le p \le r$, such that $I_p = \{1, 2, ..., m\}$ and $\eta(x, u) = x - u$, then we obtain the Mond-Weir dual for (P) defined in Suneja and Gupta [12].

Remark 2: It is also possible to state a strict converse theorem for (P) and (MWD) of a Mangasarian type [10].

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