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An Analysis of the Falk-Palocsay Algorithm

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AN ANALYSIS OF THE FALK-PALOCSAY ALGORITHM

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Abstract

The recently proposed algorithm by Falk and Palocsay for the sum-of-ratios fractional program is analyzed and contrasted with the method of Cambini, Martein and Schaible.

Introduction

Recently Falk and Palocsay [6], [7] suggested a new method, here denoted by FP, which solves the sum-of-ratios problem

$$\mathbb{P} : \max \{ \sum_{i=1}^{m} \frac{n_i(x)}{d_i(x)} : x \in S \}.$$

Here $S=\{x\in \mathbb{R}^n: Ax\leq b,\ x\geq 0\}$ is a bounded convex polytope, the numerator n_i and the denominator d_i of each ratio are affine functions and the denominator of each ratio is positive on S.

This method does not operate in the variable space S but uses the image

$$T = \{r = (r_1, r_2,\dots, r_m) : r_i = \frac{n_i(x)}{d_i(x)},\ i=1,..,m,\ x\in S \}.$$

The suggested approach is of particular interest since the method works for any number of ratios.

For the special case of two linear ratios a parametric algorithm (CMS) was suggested by Cambini, Martein and Schaible in [3], which is based on an

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earlier procedure by Martein [9] for a sum of a linear and linear fractional function. This method operates in the variable space S.

In the present paper we will give some suggestions for improving algorithm FP. We will also compare the algorithms FP and CMS from a theoretical point of view. For recent surveys of methods for the sum of ratios problem see [8], [10], [11], [12]. A comprehensive bibliography of fractional programming can be found in [11].

1. The algorithms FP and CMS

In order to contrast the algorithms FP and CMS, we will give a brief description of each one.

1.1 The algorithm by Falk and Palocsay

We outline the algorithm for m = 2. In order to determine the image $r^{opt} = (r_1^{opt}, r_2^{opt}) \in T$ of an optimal solution x^{opt} of problem IP lower bounds and upper bounds for $r_1^{opt} + r_2^{opt}$ are determined which are iteratively improved. Initially two linear fractional programs

$$P^i : r^i_{max} = \max \{ \frac{n_i(x)}{d_i(x)} : x \in S \} \quad i=1,2$$

are solved, for instance with help of the Charnes-Cooper transformation [4], yielding optimal solutions $x^{1,0}$, $x^{2,0}$, respectively. In addition to $u^0 = (r^1_{max}, r^2_{max})$ the points $r(x^{1,0})$, $r(x^{2,0})$ are determined. One of these last two points will yield a better lower bound for $r_1^{opt} + r_2^{opt}$; for example, $l^0 = r(x^{1,0})$ (see Fig.1).

The point v^0 is then determined as the point on $r_2 = u^0_2$ which is on the line r_1+r_2= constant through l^0. Obviously, an upper bound for $r_1^{opt}+r_2^{opt}$ is provided by u^0, namely $u^0_1+u^0_2$. An optimal solution r^{opt} is located within the triangle (l^0, v^0, u^0). It is a point in T which maximizes r_1+r_2.

The idea of the algorithm is to decrease the size of the triangle containing r^{opt}. This is accomplished by constructing points l^k, v^k, u^k which provide better lower and upper bounds for $r_1^{opt}+r_2^{opt}$. A sequence of linear fractional programs is to be solved that will reduce the size of the triangle (l^0, v^0, u^0) containing r^{opt}.

In the illustrated example (see Fig. 1) the fractional program consists of maximizing the second ratio over S while ensuring that the first ratio does not fall below the already achieved level v^0_1. With help of these fractional programs points v^k on the line between v^0 and l^0 are constructed that move from the outside towards the intersection of that line with the boundary of T. While the lower bound has not improved in this case, the upper bound has been reduced as new points u^k on the horizontal line through v^k and the line $r_1=u^0_1$ are determined.

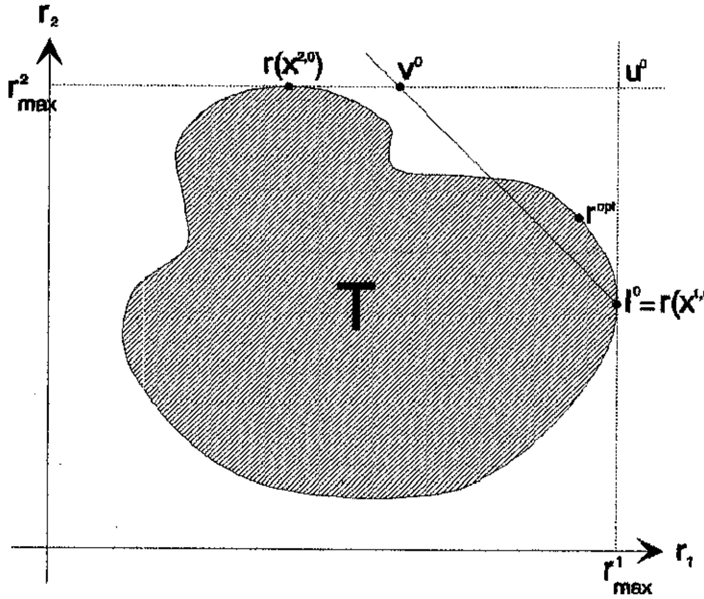

Fig. 1

Once v^k (and thus u^k) does not improve anymore (within a tolerance), the algorithm stalls since neither the lower nor the upper bound changes. In [6], [7] a procedure is suggested that overcomes the stalling problem. To the last triangle (l^k, v^k, u^k) the point (v^k_1 , l^k_2) is added. A square is constructed with help of these four points. This square is divided vertically into two equally sized rectangles. Then two cases can occur: either r_1+r_2 can be increased in at least one of these rectangles or not. In the first case a smaller triangle containing r^{opt} can be found by solving a linear fractional program. Both lower and upper bounds have improved and the algorithm is restared. Otherwise the algorithm will be applied separately to each rectangle. An obvious stopping criterion is $u^k= l^k$.

1.2 The algorithm by Cambini, Martein and Schaible

For solving problem IP a parametric algorithm is suggested by Cambini, Martein and Schaible in [3]. It is based on an earlier procedure by Martein [9] for the sum of a linear and linear fractional function. In fact, any problem IP with two linear ratios and linear constraints can be transformed with help of a generalized version of Charnes-Cooper's variable transformation [4] into the problem considered by Martein:

$$M = \max \{ h^T x + g^T x / f^T x : x \in X \} \qquad (1.1)$$

where h, g, f $\in \mathbb{R}^n$. The feasible region X is not necessarily bounded in [3].

The equivalent problem (1.1) of any linear two-ratio problem is solved in [9] by changing the one denominator $f^T x$, parametrically, i.e. the following parametric linear program is solved:

$$P(\xi): 1/\xi \ \sup \{\xi \ h^T x + g^T x : x \in X, \ f^T x = \xi\}. \qquad (1.2)$$

An optimal solution of $P(\xi)$ is called an optimal level solution. By raising the level $f^T x = \xi$, starting with the smallest value on X, a sequence of optimal level solutions is generated.

The sequential method proposed in [3] is shown to be finite. Only finitely many optimal level solutions are needed to calculate a local optimal solution for problem (1.1), and only finitely many local, non global optimal solutions exist. Hence, the procedure finds a global maximum in finitely many steps (assuming nondegeneracy in X) or it shows that the objective function in (1.1) is unbounded. Two variants of the algorithm are proposed in [3] which both find a global optimum in finitely many steps.

2. Analysis of the theoretical basis of the algorithm FP

In [1] Almogy and Levin introduce the following function

$$H(r) = \max \{ h(x,r) = \sum_{i=1}^{m} [n_i(x) - r_i \, d_i(x)] : x \in S\}, \ r \in R^m.$$

The goal is to extend Dinkelbach's results for m =1 in [5] to problem IP. In [6], [7] Falk and Palocsay showed that this is not possible. Nevertheless the function H(r) has some useful properties in the context of this algorithm.

The main properties of H(r) are summarized in the following

Proposition 2.1 [6]:

i) $H(r) \geq 0 \ \forall \ r \in T$;

ii) $H(r) > 0 \ \forall \ r \in T$ such that $r_i < \dfrac{n_i(x^{opt})}{d_i(x^{opt})}$, i=1,..,m where x^{opt} is an optimal solution of problem IP;

iii) $H(r) < 0$ for every r such that $r_i > \max \{ \dfrac{n_i(x)}{d_i(x)} : x \in S \}$, i=1,..,m;

iv) the function H(r) is convex over R^m.

Furthermore, working in the image space, Falk and Palocsay stated the following sufficient optimality condition in case of m = 2 where $r^{opt} = (r_1^{opt}, r_2^{opt})$ is the image of x^{opt} in T.

Optimality condition C: Let $r^1 \in T$ and $r^2, r^3 \in R^2$ such that r^1, r^2 and r^3 are the extreme points of a triangular region in R-space. If $H(r^1) = 0$, $H(r^2) < 0$ and $H(r^3) < 0$, then $r^{opt} = r^1$.

Unfortunately, the previous condition is not always sufficient as the following example from [9] shows:
Example 2.1: $P_1 : \max \{ (x_1 - x_2) + (2x_1 + 7x_2 + 6)/(x_1 + x_2 + 1) : x \in S_1 \}$ where $S_1 = \{x \in \mathbb{R}^2 : 0 \leq x_2 \leq 4, x_1 - x_2 \leq 4, x_1 \geq 0\}$.

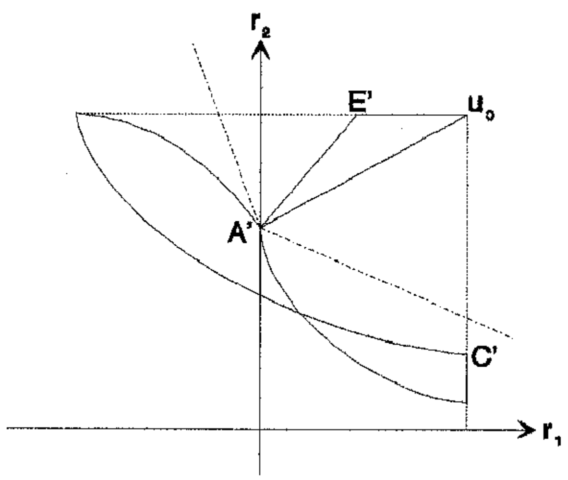

Fig.2

If we consider the triangular region defined by $r^1 = A' = (0,6)$, $r^2 = u^0 = (4,34/5)$ and $r^3 = E' = (2, 34/5)$, we have $H(r^1) = 0$, $H(r^2) < 0$ and $H(r^3) < 0$. However, the optimal solution is reached in C'=(4,50/13), and not in A'. A' is only a local, but not a global optimum.

A correct statement involving condition C is
Proposition 2.2: Let $r^1 \in T$ and $r^2, r^3 \in R^2$ such that r^1, r^2 and r^3 are the extreme points of a triangular region in R-space. If $H(r^1) = 0$, $H(r^2) < 0$ and $H(r^3) < 0$, then r^1 is the only point of the triangular region belonging to T.
Proof: The result follows from the convexity of H(r); see proof of Theorem 5 in [6]. ♦

Proposition 2.2 allows us to obtain the following proposition which may be used to improve the upper bound in the algorithm FP.

According to Proposition 2.1, there exist r^*_1, r^*_2, on lines $r_1=r^1_{max}$, $r_2=r^2_{max}$ respectively, such that $H(r^*_1, r^2_{max}) = 0$ and $H(r^1_{max}, r^*_2) = 0$. The following proposition shows that it is possible to find a better upper bound than $u^0_1+u^0_2$.

Proposition 2.3: Assume $(r^*_1, r^2_{max}), (r^1_{max}, r^*_2) \notin T$.
If $r^2_{max} - r^*_2 < r^1_{max} - r^*_1$, then $r_1^{opt} + r_2^{opt} < r^1_{max} + r^*_2$.
Proof: Set $A = (r^1_{max}, r^*_2)$, $B = (r^1_{max} + r^2_{max} - r^*_2, r^2_{max})$ and $C=(r^1_{max}, r^2_{max})$. Since $r^*_1 < r^1_{max} - r^2_{max} + r^*_2$, we have $H(C) <0$. Taking into account $H(A)=0$ and $H(B)<0$, convexity of H (see Proposition 2.1) implies

$$H(\alpha_1 A + \alpha_2 B + \alpha_3 C) \leq \alpha_1 H(A) + \alpha_2 H(B) + \alpha_3 H(C) <0$$

for $\forall \alpha_i \geq 0$ $i=1,2,3$ such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$ except for $\alpha_i = 1$. Hence A is the only point in the triangle ABC which could be in T. However by the assumption it is not. Hence the assertion follows by the convexity of the level curve $H(r)=0$. ♦

In Proposition 2.3, $r^1_{max} + r^*_2$ is a better upper bound than $u^0_1+ u^0_2$. If instead $r^2_{max} - r^*_2 \geq r^1_{max} - r^*_1$, then $r^*_1 + r^2_{max}$ can be used as a better upper bound than $u^0_1+ u^0_2$.

Now we will establish some properties of the function H which may be useful for improving the algorithm FP for $m = 2$.

Problem IP in the image space becomes

$$P' : \max \{r_1 + r_2 : r \in T\}$$

where $T = \{ r = (r_1, r_2) : r_1 = \frac{n_1(x)}{d_1(x)}, r_2 = \frac{n_2(x)}{d_2(x)} : x \in S\}$.

The following proposition characterizes the level curves $\Gamma_\lambda = \{r : H(r) = \lambda\}$ of the function H. Let $v^1, \ldots, v^s$ be the s vertices of S.

Proposition 2.4:
$r \in \Gamma_\lambda \Leftrightarrow$ r solves one of the following systems for $i = 1,2,..,s$

$$\begin{cases} h(r, v^i) = \lambda \\ h(r, v^j) \leq \lambda \ \forall \ j \neq i \end{cases}$$

Proof: For any fixed $r = (r_1, r_2)$, function h is linear in x; since S is a compact polyhedron, the maximum value H(r) is attained in a vertex of S. This implies the assertion. ♦

From Proposition 2.4 we see

Corollary 2.1:

i) Γ_λ are piecewise linear level curves;

ii) Γ_λ are decreasing to the origin.

The level curve Γ_0 is shown as a dotted line in the examples.

In the image space the level curves of problem P' are lines given by the equation $r_1 + r_2$ = constant. The line corresponding to the optimal solution is $r_1 + r_2 = r_1^{opt} + r_2^{opt}$ which supports T. However, in general, the level curve of H corresponding to the optimal solution r^{opt} (Γ_{opt}) does not support the region T, while Γ_0 supports T. This is equivalent to saying that problem $\mathbb{P}$ and problem min $\{H(r) : r \in T\}$ do not have the same optimal solution.

The following examples demonstrate, graphically, that the optimal solution of problem P' is not necessarily reached at the value r with H(r)=0 and that Γ_{opt} does not always support the region T.

In Example 2.2 the optimal solution is a vertex of S which corresponds to B'= (1,8) in T with H(B') = 0. In this case $\Gamma_{opt} = \Gamma_0$ supports T, but H(A')=0 and A'=(2,4) is not an optimal solution of P_2. In the previous example 2.1, the optimal solution belongs to a vertex of S which corresponds to C' in T with H(C')>0.

In Example 2.3 the optimal solution is a vertex of S which corresponds to E' = $(1-\sqrt{2}/2, 1-\sqrt{2}/2)$ in T with H(E')>0.

Example 2.2:

$P_2 : \max \{ 2 x_1 /(x_2+4) + (x_1 + x_2) : x \in S_2 \}$ where

$S_2 = \{ x \in \mathbb{R}^2 : 0 \leq x_1 \leq 4, 0 \leq x_2 \leq 4 \}$

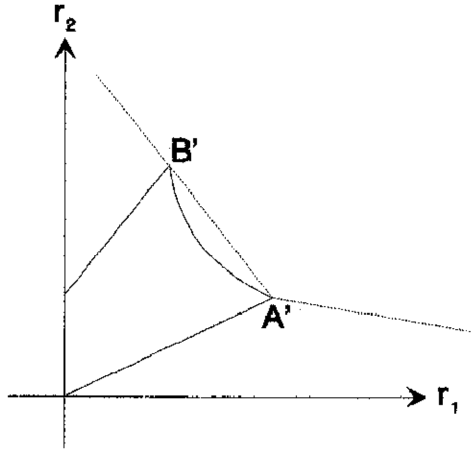

Fig. 3

Example 2.3:
P_3: max $\{ (x_1 + x_2)/ (x_1 +1) - (1/2\, x_1 + x_2 - 1/2) : x \in S_3\}$ where
$S_3 = \{ x \in \mathbb{R}^2 : 0 \leq x_1 \leq 2, 0 \leq x_2 \leq 2\}$.

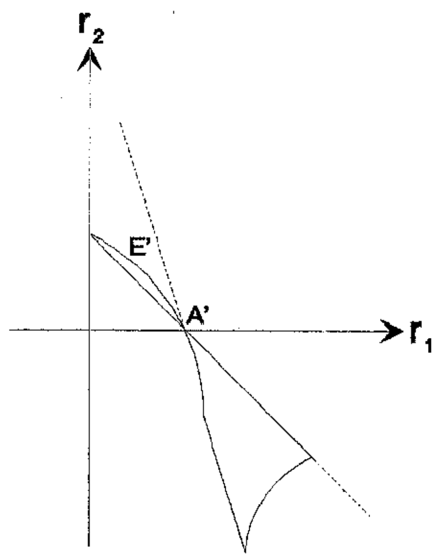

Fig. 4

Taking into account these examples, the following problem arises: find conditions under which $H(r) = 0$ implies that r is an optimal solution of problem P'.

With regards to this problem, let S^1_{max}, S^2_{max} be the sets of optimal solutions of P^1 and P^2, respectively. The following lemma states a necessary and sufficient condition for $H(r)=0$ at (r^1_{max}, r^2_{max}).

Lemma 2.1: $H(r^1_{max}, r^2_{max}) = 0$ if and only if $S^1_{max} \cap S^2_{max} \neq \emptyset$.
Proof: If $H(r^1_{max}, r^2_{max}) = 0$, then there exists a point $x_C \in S$ such that
$$h(r^1_{max}, r^2_{max}, x_C) = n_1(x_C) - r^1_{max}\, d_1(x_C) + n_2(x_C) - r^2_{max}\, d_2(x_C) = 0 \quad (2.1)$$
and $h(r^1_{max}, r^2_{max}, x) \leq 0\ \forall x \in S$. Taking into account that $\frac{n_1(x)}{d_1(x)} \leq r^1_{max}$ and $\frac{n_2(x)}{d_2(x)} \leq r^2_{max}\ \forall x \in S$, we see that x_C is an optimal solution for both P^1 and P^2 since otherwise condition (2.1) could not hold. Conversely, if there exists $x_C \in S^1_{max} \cap S^2_{max}$, then $h(r^1_{max}, r^2_{max}, x_C) = 0$, $h(r^1_{max}, r^2_{max}, x) \leq 0\ \forall x \in S$. Hence $H(r^1_{max}, r^2_{max}) = 0$. ♦

Set $\underline{r}^2 = \max \{ \frac{n_2(x)}{d_2(x)} : x \in S^1_{max} \}$. The following theorem states a sufficient optimality condition in the image space.

Proposition 2.5: If $H(r^1_{max}, \underline{r}^2) = 0$ and $H(r^1_{max} + r^2_{max} - \underline{r}^2, r^2_{max}) < 0$, then $(r^1_{max}, \underline{r}^2)$ is an optimal solution of problem P'.
Proof: Let us consider the level $r^1 + r^2 = r^1_{max} + \underline{r}^2$ and the triangular region with the vertices $A = (r^1_{max}, \underline{r}^2)$, $B = (r^1_{max} + r^2_{max} - \underline{r}^2, r^2_{max})$ and $C = (r^1_{max}, r^2_{max})$. The assumption implies that $H(A) = 0$ and $H(B) < 0$ while Proposition 2.1 implies $H(C) < 0$. From Proposition 2.2 we see that A is the unique point of the triangular region belonging to T, and as a consequence, the line $r_1 + r_2 = r^1_{max} + \underline{r}^2$ is a supporting hyperplane for T. ♦

Now we will find conditions in the decision space under which $H(r) = 0$ implies that r is an optimal solution for problem P'.

Proposition 2.6 : If $H(r^1_{max}, \underline{r}^2) = 0$ and $[d_2(x) - d_1(x)] \geq 0\ \forall x \in S$, then $(r^1_{max}, \underline{r}^2)$ is an optimal solution of problem P'.
Proof: We must prove that
$$(*) \quad \frac{n_1(x)}{d_1(x)} - r^1_{max} + \frac{n_2(x)}{d_2(x)} - \underline{r}^2 \leq 0 \quad \forall\, x \in S.$$
Set $L_1(x) = n_1(x) - r^1_{max}\, d_1(x)$ and $L_2(x) = n_2(x) - \underline{r}^2\, d_2(x)$. Then we have

$$\frac{n_1(x)}{d_1(x)} - r^1_{max} + \frac{n_2(x)}{d_2(x)} - \underline{r}^2 = \frac{L_1(x)d_2(x)+L_2(x)d_1(x)}{d_1(x)d_2(x)} =$$

$$= \frac{L_1(x)d_2(x)+L_2(x)d_1(x)+L_1(x)d_1(x)-L_1(x)d_1(x)}{d_1(x)d_2(x)} =$$

$$= \frac{[L_1(x)+L_2(x)]d_1(x)+L_1(x)[d_2(x)-d_1(x)]}{d_1(x)d_2(x)}.$$

The assertion follows since condition $H(r^1_{max}, \underline{r}^2) = 0$ implies $[L_1(x) + L_2(x)] \le 0\ \forall x \in S$, by definition of r^1_{max}, $L_1(x) \le 0\ \forall x \in S$ and $[d_2(x) - d_1(x)] \ge 0$, $d_1(x)>0\ \forall x \in S$. ♦

Remark 2.1: Let us note that $H(r^1_{max}, \underline{r}^2) = 0$ does not imply $[d_2(x) - d_1(x)] \ge 0$ $\forall x \in S$. This is shown by Example 2.2 where $H(A') = 0$, but $[d_2(x) - d_1(x)] < 0$ $\forall x \in S$.

Now we will prove that problem IP can always be tranformed into an equivalent problem P* satisfying the inequalitity $[d_2(x) - d_1(x)] \ge 0\ \forall \in S$. With this aim in mind, consider the problem

$$P^*: \max \left\{ \frac{n_1(x)}{d_1(x)} + \frac{n_2^*(x)}{d_2^*(x)} : x \in S \right\}$$

where $n^*_2(x) = kn_2(x)$, $d^*_2(x) = kd_2(x)$, $k = \frac{\max_{x\in S} d_1(x)}{\min_{x \in S} d_2(x)}$.

The following lemma holds

Lemma 2.2:
i) Problems P^* and IP have the same optimal solutions,
ii) $[d^*_2(x) - d_1(x)] \ge 0\ \forall\ x \in S$.

<u>Proof</u>: Since $\frac{d_1(x)}{d_2(x)} \le \frac{\max_{x\in S} d_1(x)}{\min_{x \in S} d_2(x)} = k$, we have

$[k\, d_2(x) - d_1(x)] \ge 0\ \forall x \in S$, that is $[d^*_2(x) - d_1(x)] \ge 0\ \forall x \in S$. ♦

Set $H^*(r) = \max \{ h^*(r,x) = n_1(x) - r_1 d_1(x) + k\,(n_2(x) - r_2\, d_2(x)) : x \in S \}$. Taking into account Proposition 2.6 and Lemma 2.2, we have the following corollary.

Corollary 2.2: If $H^*(r^1_{max}, \underline{r}^2) = 0$, then $(r^1_{max}, \underline{r}^2)$ is an optimal solution of problem P'.

3. A comparison of the algorithms of FP and CMS

From a theoretical point of view the two algorithms are quite different:

- In [3] optimality conditions are stated for problem $\mathbb{P}$ which allow to establish if a given point is a local optimal solution or not; in [6] no necessary optimality conditions are given and the sufficient condition stated is not correct, as pointed out in example 2.1;
- In [3] the convergence of the algorithm is finite which is not the case in [6];
- In [3] local optimal solutions are generated, the last of which is a global optimal solution. The algorithm of FP is not able to recognize if a point is a local optimal solution for problem $\mathbb{P}$. The reason of this may perhaps be found in the fact that the transformation used by Falk and Palocsay does not transform interior or boundary points of S into interior or boundary points of T, as can be seen from the following example.

Example 3.1:

$P_4 : \max \{ (x_1 + x_2 + x_3) - (x_1 + 2\, x_2) : x \in S_4\}$ where

$S_4 = \{x \in \mathbb{R}^3 : 0 \le x_1 \le 1,\ 0 \le x_2 \le 1,\ 0 \le x_3 \le 1\}$.

We conjecture that from a computational point of view, algorithm of CMS might be better than FP, in the sense that it may require fewer iterations.

Consider the following example in [6]

Example 3.2:

$P_5 : \max \{ (37x_1 + 73x_2 + 13)/ (13x_1 + 13x_2 + 13) +$

$(63x_1 - 18\, x_2 + 39)/ (13x_1 + 26x_2 + 13) : x \in S_5\}$ where

$S_5 = \{x \in \mathbb{R}^2 : 1.5 \le x_1 \le 3,\ 5x_1 - 3x_2 = 3,\ x_2 \ge 0\}$

FP find the optimal solution of problem P_5 after 20 iterations; while the algorithm CMS needs only one iteration to find a global optimal solution.

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