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A Comparison of Alternative Discrete Approximations Of The Cox -Ingersoll - Ross Model

Eugene M. Cleur

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EUGENE M. CLEUR
Facoltà di Economia, via Cosimo Ridolfi 10, 56124 Pisa ; email: cleur@ec.unipi.it

Abstract

In this paper two discrete time approximations to the Cox,Ingersoll,Ross (1985) model (CIR (1985) henceforth) for interest rates, one based on the Euler scheme and the other on a conditional non-central chi-square distribution, are compared. In particular, the use of these two approximations in an indirect estimation procedure is examined.

Keywords: Interest rates, stochastic differential equations, Euler scheme approximation, non-central chi-square, indirect estimation.

1 Introduction

The estimation of the well known Cox-Ingersol-Ross (1985) model specified using the Feller (1951) non-linear stochastic differential equation

$$dr(t) = k(a - r)dt + \sigma\sqrt{r}dw(t), \tag{1}$$

where $r(t)$ is the instantaneously maturing rate, a is the long term equilibrium value around which $r(t)$ fluctuates, k is a drift parameter and σ the diffusion coefficient was recently considered in two separate Monte Carlo studies by Bianchi, Cesari and Panattoni (1995)

(BCP (1995) henceforth) and Bianchi and Cleur (1995) (BC (1995) henceforth). Since a closed form solution to the CIR model does not exist, in these two papers, the continuous process was approximated, in discrete time, using the Euler scheme approximation (see Maruyama (1955)):

$$r_t = r_{t-\delta} + k(a - r_{t-\delta})\delta + \sigma\sqrt{r_{t-\delta}}\epsilon_t, \qquad (2)$$

where $\epsilon_t = [w(t\delta + \delta) - w(t\delta)] \sim N(0, \delta)$, with $\delta = 1/n$ (n being the number of realisations per unit of time).

An alternative approximation to the CIR model may be obtained by recalling the following result (see, for ex., BCP (1995)) :

$$2cr_{t+\delta} \mid r_t \quad \sim \quad \chi^2(2q, u) \qquad (3)$$

that is, the conditional distribuition of the interest rate at time t+δ, given that at time t, is a non-central chi-square with 2q degrees of freedom and non-centrality parameter u where

$$q = \frac{2k\theta}{\sigma^2} > 1 \qquad (4)$$

$$u = 2cr_t exp(-k\delta) > 0 \qquad (5)$$

$$c = \frac{2k}{\sigma^2(1 - exp(-k\delta))} \qquad (6)$$

There are several ways for generating random samples from a non-central chi-squared distribution (see, for ex., Johnson and Kotz (1970)). However, for reasons that will become evident in Section 2 below, in this paper, given the random observations $z_\delta, ..., z_T$ drawn from a N(0,1), the following Wilson-Hilferty approximation (see Johnson and Kotz (1970))

Table 1: Descriptive statistics for two typical series

Approximation	Euler	Chi-square
Sample Size	250	250
Average	0.108204	0.108105
Median	0.104216	0.104065
Mode	0.103936	0.103529
Geometric Mean	0.104521	0.104502
Variance	8.18324E-4	8.01519E-4
Minimum	0.049047	0.049773
Maximum	0.195462	0.194531
Lower Quartile	0.089143	0.088532
Upper Quartile	0.124523	0.123463
Skewness	0.581380	0.588410
Kurtosis	0.208467	0.211274

is used:

$$r_t = \frac{GF}{2c}\left[\frac{\sqrt{2}z_t}{\sqrt{9F}} + 1 - \frac{z_t}{9F}\right]^3 + \frac{B}{2c} \tag{7}$$

where

$$B = -\frac{u^2}{2q+3u} < 0 \tag{8}$$

$$G = \frac{2q+3u}{2q+2u} > 0 \tag{9}$$

$$F = \frac{(2q+3u)^3}{(2q+2u)^2} > 0 \tag{10}$$

and c, q and u are defined in (4) - (6) above.

A bi-variate plot of the two series, shown in Fig. 1, and the histograms of two typical series generated by the Euler scheme approximation and by the non-central chi square approximation, shown in Figures 2 and 3, indicate that the two series are very similar. Table 1 reports some decriptive statistics for the two series and these together with the following regression estimate

$$chisq_t = 0.0010 + 0.9895 Euler_t; \qquad R^2 = 0.9997 \tag{11}$$

FIG. 1

Plot of Euler and chi-sq. approximations

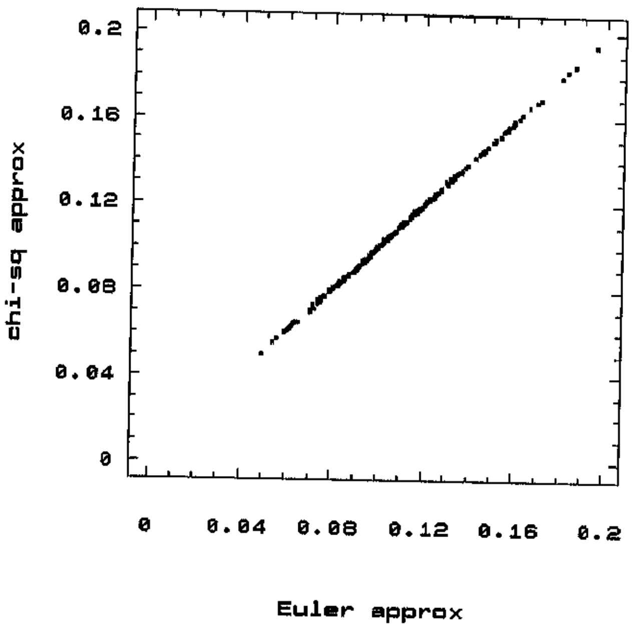


Fig. 2

Histogram of non-central chi-sq. approx.

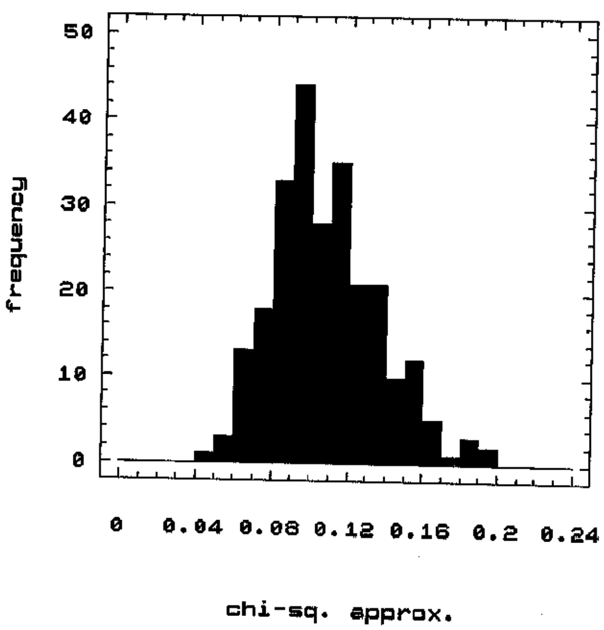
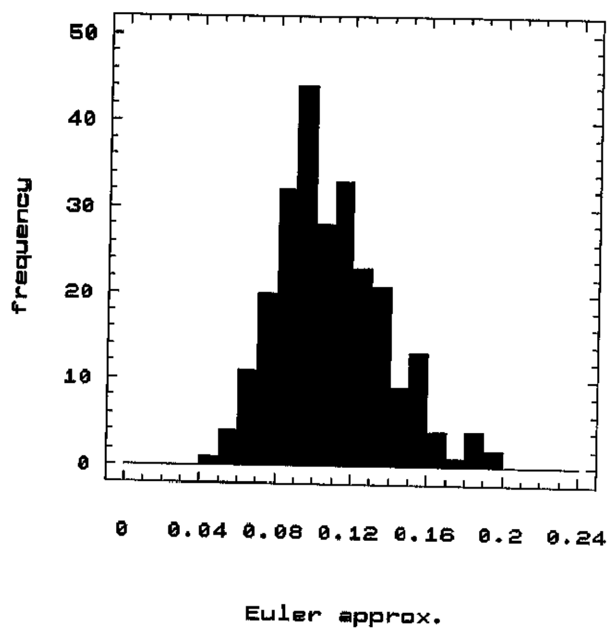


FIG. 3

Histogram of Euler approx. series



confirm that there is really no significant difference between the two approximations considered and it should therefore make no difference, as far as numerical results are concerned, as to which of the two we actually use.

A further comparison between the Euler scheme and the non-central χ^2 approximations will be carried out in the context of an indirect estimation of the parameters. In particular, one of the major problems encountered with the Euler scheme approximation used in BC (1995) is that unless bounds are placed on the three parameters during the minimisation phase of the indirect estimation procedure it becomes highly probabable that for a particular combination of values the interest rates generated become negative. In contrast, it may be argued that since $\chi^2(2q, u) > 0$, there should be no risk in generating such negative interest rates.

2 Indirect Estimates

For a detailed description of the indirect estimation procedure used in this paper see BC (1995). Briefly, this procedure may be described as follows:

We might have an observed series y = $(y_1, ..., y_T)$ generated by a computationally intractable stochastic model. This model could, however, be used to produce simulated observations $\tilde{y}(\theta, y_0) = (\tilde{y}_1, ..., \tilde{y}_T)$ for a given vector of parameters $\theta = (\theta_1, ..., \theta_p)$ and initial value y_0.

The problem is to choose θ so that the observed series, y, and the simulated series, $\tilde{y}(\theta, y_0)$, best agree on the basis of some statistical criterion.

The procedure applied in this paper is that proposed by Gouriéroux, Monfort and

Renault (1994) (GMR (1994) henceforth) and is based on a comparison of the estimates of a simplified model, called the auxilliary model, using the observed and the simulated data. In other words, given an auxilliary model with parameters $\beta = (\beta_1, ..., \beta_k)$, where $k \geq p$, the indirect estimation procedure reduces to solving the minimum distance problem

$$\min_{\theta \in \Theta} \left[\hat{\beta}_T - \tilde{\beta}_T(\theta, z_0)\right]' \hat{\Omega} \left[\hat{\beta}_T - \tilde{\beta}_T(\theta, z_0)\right], \tag{12}$$

where *hats* are for estimates using the observed series, y, *tildes* are for estimates using the simulated series, $\tilde{y}(\theta, y_0)$, and $\hat{\Omega}$ is a consistent estimate of a deterministic positive definite matrix Ω. In other words, we calibrate θ so that $\tilde{\beta}_T$ is close to $\hat{\beta}_T$.

An identical Monte Carlo experiment was performed on the unobservable instantaneously maturing rates $r(t)$, i.e. $\tau \to 0$ as in (2), and on the more realistic observable rates with non zero maturity $\tau = 0.25$ obtained by transforming the $r(t)$ (see BC (1995) or BCP (1995)). A time span T = 250 was used. The means, biases and standard errors of the estimates were calculated over 500 replications carried out on a Pentium 120 pc. In each replication, the *original* and *simulated* series, indicated rispectively by $r_\delta,\ r_{2\delta},\ \dots, r_{250}$ and $\tilde{r}_\delta,\ \tilde{r}_{2\delta},\ \dots,\ \tilde{r}_{250}$, are generated by (2) and by (7) with a *simulation* step $\delta = 0.10$ and starting value 0.10. In (7), the sequence of random observations $z_\delta, ..., z_T$ remains unchanged over iterations. The estimated auxilliary model, where Δ is the *observation* step, is given by

$$\frac{r_t - r_{t-\Delta}}{\sqrt{r_{t-\Delta}}} = \frac{\beta_0 \Delta}{\sqrt{r_{t-\Delta}}} + \beta_1 \Delta \sqrt{r_{t-\Delta}} + e_t, \tag{13}$$

where $E(e_t) = 0$ and $E(e_t^2) = \sigma^2 \Delta$. The *naive* estimates of the parameters of interest, using (13), are given by $\hat{k} = -\hat{\beta}_1/\Delta$, $\hat{a} = -\hat{\beta}_0/\hat{\beta}_1$ and $\hat{\sigma}^2 = Var(\hat{e})/\Delta$.

Table 2: CIR Model $\tau \to 0, \quad \Delta = 1$ $\tau = .25, \quad \Delta = 1$

	Parameter True Value	k 0.8	a 0.1	σ 0.06	k 0.8	a 0.1	σ 0.06
Naive Estimator (Euler scheme)	Mean	.5734	.1002	.0437	.5735	.1005	.0396
	Std.Err.	.0574	.0015	.0021	.0574	.0013	.0019
	Bias	-.2266	.0002	-.0163	-.2265	.0005	-.0204
Indirect Estimator (Euler scheme)	Mean	.8147	.1001	.0606	.8146	.1001	.0608
	Std.Err.	.1843	.0022	.0056	.1839	.0019	.0067
	Bias	.0147	.0001	.0006	.0146	.0001	.0008
Naive Estimator (non-central chi-square)	Mean	.5587	.1001	.0424	.5588	.1005	.0385
	Std.Err.	.0570	.0015	.0020	.0570	.0014	.0019
	Bias	-.2413	.0001	-.0176	-.2412	.0005	-.0215
Indirect Estimator (non-central chi-square)	Mean	.8170	.1001	.0607	.8167	.1001	.0609
	Std.Err.	.1935	.0022	.0060	.1931	.0020	.0072
	Bias	.0170	.0001	.0007	.0167	.0001	.0009

For the first 3 columns $\tau \to 0, \quad \Delta = 1$ and for the last three columns $\tau = 0.25, \quad \Delta = 1$

Since the original and the auxilliary models have the same number of parameters, Ω is set equal to the identity matrix.

The solution to the minimum distance problem (12) was subjected to the ergodicity condition $2k\theta > \sigma^2$ and to the conditions $k, \theta, \sigma > 0.01$. Without these lower bounds, both the Euler scheme approximation and the non-central chi-squared approximation often produced negative interest rates; this depends exclusively on the new parameter values chosen in the iterative solution to the minimum distance problem (12) carried out automatically by the DNCONG routine of the IMSL package.

The lower bounds specified in this paper are quite different from those in BC (1995) where, in addition, upper bounds were also defined for all three parameters. The results, however, are substantially the same; this may be seen by comparing Tables 2 and 3 of this paper with Table 3 in BC (1995). In other words, with the lower bounds specified in this paper, upper bounds become redundant. On the other hand, with the lower bounds

Table 3: CIR Model $\tau \to 0, \ \Delta = .1$ $\tau = .25, \ \Delta = .1$

	Parameter True Value	k 0.8	a 0.1	σ 0.06	k 0.8	a 0.1	σ 0.06
Naive Estimator	Mean	.8098	.1001	.0599	.8098	.1005	.0544
(Euler scheme)	Std.Err.	.0777	.0015	.0008	.0777	.0013	.0007
	Bias	.0098	.0001	-.0001	.0098	.0005	-.0056
Indirect Estimator	Mean	.7975	.1001	.0599	.7974	.1001	.0599
(Euler scheme)	Std.Err.	.1141	.0021	.0012	.1140	.0018	.0015
	Bias	-.0025	.0001	-.0001	-.0026	.0001	-.0001
Naive Estimator	Mean	.7788	.1001	.0576	.7788	.1005	.0523
(non-central	Std.Err.	.0763	.0015	.0008	.0762	.0013	.0007
chi-square)	Bias	-.0212	.0001	-.0024	-.0212	.0005	-.0077
Indirect Estimator	Mean	.7983	.1001	.0599	.7982	.1001	.0599
(non-central	Std.Err.	.1213	.0021	.0013	.1211	.0019	.0018
chi-square)	Bias	-.0017	.0001	-.0001	-.0018	.0001	-.0001

For the first 3 columns $\tau \to 0, \ \Delta = 0.1$ and for the last three columns $\tau = 0.25, \ \Delta = 0.1$

specified in BC (1995) the upper bounds become necessary in order to avoid the generation of negative interest rates during the solution to the minimum distance problem (12).

It should be noted that even the non-central chi-square approximation of the CIR model, if used in an indirect estimation procedure without limiting the sample space over which the optimisation algorithm searches for new values, could produce negative interest rates. The problem therefore lies, not with the approximation used, but with the iterative solution to the minimum distance problem (12) which is automatically controlled by the minimisation algorithm. An alternative to such an approach would be to search over a carefully predefined grid of values for each parameter.

The results of the Monte Carlo experiment carried out in this paper are summarised in Tables 2 and 3. In Table 2 the observation step $\Delta = 1.0$, and in Table 3 $\Delta = 0.1$; in other words, whereas in Table 2 every tenth value generated is retained in the subsequent calculations, in Table 3, all the values generated are used for estimating the parameters.

The naive estimates of the auxilliary model using data generated by both approximations are very similar which suggests that it probably makes no difference as to which of the two is used. This impression is confirmed by the indirect estimates. However, the indirect estimates obtained from the Euler scheme approximation necessitated much less computer time (this factor was not precisely calculated, but the difference was macroscopic) and this should tilt the balance in its favour.

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