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A New Approach to Second Order Optimality Conditions in Vector Optimization

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A NEW APPROACH TO SECOND ORDER OPTIMALITY CONDITIONS IN VECTOR OPTIMIZATION

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Abstract
The aim of this paper is to establish some second order necessary and sufficient optimality conditions for a multiobjective problem where optimality is studied with respect to arbitrary closed convex cones. The proposed approach extend the one recently given by the same authors.

1. Introduction

In these last years, the image space has been frequently used [5-8, 11, 12] as a general framework within which some topics related to optimization have been studied; recently, for a scalar problem, Cambini and Martein [9] have suggested a new approach for studying necessary second order optimality conditions based on a characterization of a suitable tangent cone in the image space [9]; such an approach has been deepened by R. Cambini in [10], where several second order necessary and sufficient optimality conditions are obtained for a multiobjective Pareto problem and where a new look inside second order constraints qualification is given.
The aim of this paper is to extend and to collect all of these recent results for a vector optimization problem where optimality is studied with respect to arbitrary closed convex cones.

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The paper has been discussed jointly by the authors and has been developed by R. Cambini.

2. Statement of the problem

In this paper we will consider the following multiobjective problem

P: U-max $\varphi(x)$, $x \in S = \{x \in X: g(x) \in V\}$

where $X \subset \mathbf{R}^n$ is an open set, $\varphi = (\varphi_1 \ldots \varphi_s): X \rightarrow \mathbf{R}^s$, $g = (g_1 \ldots g_m): X \rightarrow \mathbf{R}^m$, $s \geq 1, m \geq 1$, $\varphi_j, g_i, j=1,..,s, i=1,..m$ are twice differentiable functions, and $U \subset \mathbf{R}^s$, $V \subset \mathbf{R}^m$ are closed convex cones with vertices at the origin such that $\mathrm{int}U \neq \varnothing$, $\mathrm{int}V \neq \varnothing$.

A point $x_0 \in S$ is said to be a <u>local efficient point</u> for problem P if there is no feasible x belonging to a suitable neighbourhood of x_0 such that

$$\varphi(x) \in \varphi(x_0) + U^0 \tag{2.1}$$

where $U^0 = U \backslash \{0\}$.

We say that x_0 is an <u>efficient point</u> for P if (1.1) holds for every $x \in S$.

Let us note that when $s=1$, $U=\mathbf{R}_+$, $V=\mathbf{R}^m_+$, problem P reduces to a scalar optimization problem and (2.1) collapses to the ordinary definition of a local maximum point, while when $U=\mathbf{R}^s_+$, $V=\mathbf{R}^m_+$, (2.1) collapses to the ordinary definition of a local Pareto point.

Let x_0 be a feasible point and assume, without lost of generality, that $g(x_0)=0$. Set

$f(x) = \varphi(x) - \varphi(x_0)$, $F(x) = (f(x), g(x))$, $K = F(X)$, $H = U^0 \times V$

We will refer to $\mathbf{R}^n$ as the <u>decision space</u> and to $\mathbf{R}^{s+m}$ as the <u>image space</u>.

It is easy to prove that x_0 is an efficient point if and only if

$$K \cap H = \varnothing \tag{2.2}$$

In such a way, the efficiency of x_0 is reduced to study the disjunction of these two sets. Since K does not have in general particular properties, the idea underlying our suggested approach is to substitute K with the following tangent cone, which is a subcone of the Bouligand tangent cone to K at $F(x_0)$:

$$T_1 = \{ t: \exists \alpha_n \rightarrow +\infty \; x_n \rightarrow x_0 \text{ with } \alpha_n F(x_n) \rightarrow t \}$$

By means of T_1 the following optimality conditions can be proven [7] :

Theorem 2.1

i) Let x_0 be a local efficient point for problem P. Then $T_1 \cap \text{int}H = \emptyset$

ii) If $T_1 \cap \text{cl}H = \{0\}$, then x_0 is a local efficient point for P.

iii) The feasible point x_0 is a local efficient point for P if and only if the following condition holds:

$\forall\ t \in T_1 \cap \text{cl}H,\ t \neq 0$, and for any sequence $x_n \to x_0$ such that there exists $\alpha_n \to +\infty$ with $\alpha_n F(x_n) \to t$, we have $F(x_n) \notin H \ \ \forall n$.

Some necessary and sufficient first order optimality conditions have been established in [7] by means of a first order characterization of T_1.

The following second order characterization of T_1 has been suggested and utilized in [9] in establishing second order necessary optimality conditions for a scalar problem and in [10] in establishing second order necessary and sufficient optimality conditions for a vector Pareto problem:

$$T_1 = (K_L + \bar{K}_Q) \cup A_2 \tag{2.3}$$

with:

$K_L = F'(x_0)(\mathbf{R}^n)$, where $F'(x_0)$ is the Jacobian matrix of F at x_0,

$\bar{K}_Q = K_Q \cup \{0\}$, $K_Q = \{F''(x_0)(w,w) : w \in W\}$, $W = \{w \in \mathbf{R}^n : F'(x_0)(w) = 0\}$

$F''(x_0)(w,w) = (w^T H_{f_1(x_0)} w, .., w^T H_{f_s(x_0)} w,\ w^T H_{g_1(x_0)} w, .., w^T H_{g_m(x_0)} w)$,

where $H_{f_j(x_0)}$, $H_{g_i(x_0)}$, are the Hessian matrices of the functions f_j, g_i, $j=1,..,s$, $i=1,..m$ at the point x_0 respectively,

$$A_2 = \{t \in T_1 :\ t \neq 0,\ \exists\ x_n \to x_0,\ \exists\, \alpha_n \to +\infty, \text{ with } \alpha_n (F(x_n) - F(x_0)) \to t,\ \frac{F(x_n)-F(x_0)}{||x_n - x_0||^2} \to 0\}$$

The following Theorem [10] states a sufficient condition for A_2 to be empty:

Theorem 2.2 If $0 \notin (K_L + K_Q)$ then $A_2 = \emptyset$.

In the following sections we will extend the results given in [10] to arbitrary convex cones.

3. Second order necessary optimality conditions and regularity conditions

Taking into account (3.1) and i) of Theorem 2.1, we have the following second order necessary optimality condition for problem P

Theorem 3.1 Let x_0 be a local efficient point for problem P. Then

$$(K_L+\bar{K}_Q) \cap \text{int } H = \emptyset \tag{3.1}$$

Let us note that (3.1) does not involve either Lagrange multipliers or second order constraints qualification unlike the classical second order necessary optimality condition of a scalar problem (s=1, $U=\mathbf{R}_+$, $V= \mathbf{R}_+^m$); the suggested approach in the image space allows to give a new look inside this aspect.
In order to find second order optimality conditions involving Lagrange multipliers, the problem of separation between the two cones $K_L+\bar{K}_Q$ and int H arises, that is the problem of the existence of $\alpha \in H^+= \{\alpha : \alpha^T h \geq 0 \ \forall h \in H\}$ such that $K_L+\bar{K}_Q \subseteq \alpha^\perp_-$, where $\alpha^\perp = \{z \in \mathbf{R}^{s+m} : \alpha^T z = 0\}$, $\alpha^\perp_-=\{z: \alpha^T z \leq 0\}$.
With this regards we have the following Theorem holds, where co(•) denotes the convex hull of set (•):

Theorem 3.2 The following conditions

i) $\exists \alpha \in H^+$, $\alpha \neq 0$, such that $K_L \subseteq \alpha^\perp$ and $K_Q \subseteq \alpha^\perp_-$,

ii) $\text{Co}(K_L+\bar{K}_Q) \cap \text{Int}(H)=\emptyset$. (3.2)

are equivalent. Furthermore the two cones $K_L+\bar{K}_Q$ and int H are separate if and only if i) or ii) holds.

Proof i)⇒ii)

Suppose ab absurdo that $z \in \text{Co}(K_L+\bar{K}_Q) \cap \text{Int}H \neq \emptyset$; since $z \in \text{Co}(K_L+\bar{K}_Q)$, z is a convex combination of k elements of $(K_L+\bar{K}_Q)$, that is $\exists\ l_1,\ldots,l_k \in K_L$, $\exists q_1,\ldots,q_k \in K_Q$, $\exists\ \lambda_1,\ldots,\lambda_k > 0$ such that $\sum_{i=1}^{k}\lambda_i=1$ and $z=\sum_{i=1}^{k}\lambda_i(l_i+q_i)$; then we have

$\alpha^T z=\sum_{i=1}^{k}\lambda_i\alpha^T l_i+\sum_{i=1}^{k}\lambda_i\alpha^T q_i = \sum_{i=1}^{k}\lambda_i\alpha^T q_i \leq 0$, and this is absurd since $z \in \text{Int}H$, $\alpha \in H^+$, $\alpha \neq 0$, implies $\alpha^T z>0$.

ii)⇒i) For a known separation theorem, $\exists\alpha\in H^+$, $\alpha\neq 0$, such that $\alpha^T z\leq 0$ $\forall z\in Co(K_L+\bar{K}_Q)$; since $K_L\subseteq Co(K_L+\bar{K}_Q)$ it results $\alpha^T F'(x_0)v\leq 0$ $\forall v\in \mathbf{R}^n$ and this implies $\alpha^T F'(x_0)=0$, that is $K_L\subseteq\alpha^\perp$; as a consequence $\alpha^T(l+q)=\alpha^T l+\alpha^T q=\alpha^T q\leq 0$ $\forall q\in K_Q$ so that $K_Q\subseteq\alpha^\perp_-$.
The last statement of the Theorem is obvious. ♦

Since (3.2) is equivalent to the existence of a vector of "multipliers" $\alpha\in H^+$, any condition which ensure the validity of (3.2) can be interpreted as a second order regularity condition (let us note that we use the term "regularity conditions" instead of "constraints qualification" since in the image space both the objective functions and the constraints are involved).
In such a way the suggested approach allows us to define two different kinds of regularity conditions:

weak regularity Assume that $(K_L+\bar{K}_Q)\cap \text{Int}H=\emptyset$. We will say that a condition $\mathcal{R}$ is a weak second order regularity condition if it is implies
$Co(K_L+\bar{K}_Q)\cap \text{Int}H=\emptyset$

strong regularity Assume that $(K_L+\bar{K}_Q)\cap \text{Int}H=\emptyset$. We will say that a condition $\mathcal{R}$ is a strong second order regularity condition if it is implies
$K_Q\subseteq K_L\text{-Cl}H$.

Some second order regularity conditions are given in the following Theorems [10]:

Theorem 3.3 Consider problem P; then the following conditions are weak second order regularity conditions:
i) $K_L+\bar{K}_Q$ is a convex cone,
ii) $\bar{K}_Q$ is a convex cone,
iii) $\dim K_L = n-1$.

Theorem 3.4 Consider problem P; then the following conditions are strong second order regularity conditions:

i) $K_Q \subseteq K_L$,

ii) $K_Q = \emptyset$,

iii) $\dim K_L = n$,

iv) $\dim K_L = s+m-1$.

Remark 3.1 When s=1, the linearly independence of the gradients of the constraints binding at x_0 is equivalent to iv) of Theorem 3.4.
Another second order strong regularity condition in the decision space which generalizes, in the scalar case, the McCormick constraint qualification can be found in [9,10].

Let us note that strong regularity implies weak regularity but the converse is not true as is shown in the following example:

Example 3.1 Consider problem P where s=1, m=1, $\varphi(x,y)=x^2+y^3$, $g(x,y)=-x^2-y^2$ $U=\mathbf{R}_+$, $V=\mathbf{R}_+$, and the feasible point (0,0). Since (0,0) is the only feasible point then it is also an efficient point.
It is easy to verify that $K_L=\{(0,0)\}$, $K_Q=\{q: q=\lambda(1,-1)+\mu(0,-1), \lambda,\mu\geq 0, \lambda+\mu\neq 0\}$, so that $co(K_L+\bar{K}_Q)=(K_L+\bar{K}_Q)$ and the weak regularity condition i) of Theorem 3.3 holds; on the other hand the vector $\alpha=(2,1)^T$ is such that $K_L\subseteq\alpha^\perp$ but $K_Q\not\subseteq\alpha^\perp_-$ so that strong regularity cannot hold.

Now we are able to state some second order necessary optimality conditions in the image space and in the decision space.

Theorem 3.5

Let x_0 be a local efficient point for problem P. Then:

i) if a second order weak regularity condition $\mathcal{R}$ holds then:

$$\exists\ \alpha\in H^+, \alpha\neq 0, \text{ such that } K_L\subseteq\alpha^\perp \text{ and } K_Q\subseteq\alpha^\perp_-,$$

ii) if a second order strong regularity condition $\mathcal{R}$ holds then:

$$\forall\ \alpha\in H^+, \alpha\neq 0, \text{ such that } K_L\subseteq\alpha^\perp, \text{ we have } K_Q\subseteq\alpha^\perp_-.$$

Proof

It follows immediately from Theorems 3.1 and 3.2 and by the given definitions of regularity. ◆

Theorem 3.5 is equivalent to the following one stated in the decision space:

Theorem 3.6

Let x_0 be a local efficient point for problem P. Then:

i) if a second order weak regularity condition $\mathcal{R}$ holds then:

$$\exists\ \alpha=(\bar{\alpha}, \alpha^*)\in H^+,\ \bar{\alpha}\in \mathbf{R}^s,\ \alpha^*\in \mathbf{R}^m, \text{ such that } \sum_{i=1}^{s} \bar{\alpha}_i \nabla f_i(x_0) + \sum_{j=1}^{m} \alpha^*_j \nabla g_j(x_0) = 0$$

and

$$w^T\Big[\sum_{i=1}^{s} \bar{\alpha}_i H_{f_i}(x_0) + \sum_{j=1}^{m} \alpha^*_j H_{g_j}(x_0)\Big] w \le 0 \quad \forall w\in W$$

ii) if a second order strong regularity condition $\mathcal{R}$ holds then :

$$\forall\ \alpha=(\bar{\alpha}, \alpha^*)\in H^+,\ \bar{\alpha}\in \mathbf{R}^s,\ \alpha^*\in \mathbf{R}^m, \text{ such that } \sum_{i=1}^{s} \bar{\alpha}_i \nabla f_i(x_0) + \sum_{j=1}^{m} \alpha^*_j \nabla g_j(x_0) = 0$$

it results:

$$w^T\Big[\sum_{i=1}^{s} \bar{\alpha}_i H_{f_i}(x_0) + \sum_{j=1}^{m} \alpha^*_j H_{g_j}(x_0)\Big] w \le 0 \quad \forall w\in W$$

Remark 3.2 When s=1, ii) of Theorem 3.6 reduces to the well known and classic second order necessary optimality condition.

4. Second Order Sufficient Optimality conditions in the image space

In this section we will derive some second order sufficient optimality conditions by means of the second order characterization of the tangent cone T_1.

With this aim, set $\alpha^{\perp}_{=} = \{z\in \mathbf{R}^{s+m}: \alpha^T z<0\}$. As a direct consequence of ii) of Theorem 2.1, taking into account (2.3) and Theorem 2.2 we have :

Theorem 4.1 If (4.1) holds

$$(K_L+\bar{K}_Q)\cap \mathrm{Cl}H=\{0\} \text{ and } 0\notin (K_L+K_Q), \qquad (4.1)$$

then x_0 is a local efficient point for P.

Let us note that Theorem 4.1 is stated without use of multipliers; obviously any condition which ensures (4.1) becomes a second order sufficient optimality condition. In this order of ideas we have the following corollaries:

Corollary 4.1 If i) and ii) hold then $x_0 \in S$ is a local efficient point for problem P:

i) $0 \notin (K_L+K_Q)$,

ii) $\exists\ \alpha \in \text{int}H^+$ such that $K_L \subseteq \alpha^\perp$ and $K_Q \subseteq \alpha^\perp_-$.

Proof. We will prove that i) and ii) imply (4.1). Assume that $(K_L+\bar{K}_Q) \cap \text{Cl}H \neq \{0\}$; then there exist $l \in K_L$, $q \in K_Q$ such that $l+q \in \text{cl}H \setminus \{0\}$, so that $\alpha^T(l+q) > 0$; on the other hand for ii) we have $\alpha^T(l+q) = \alpha^T l + \alpha^T q = \alpha^T q \leq 0$ and this is absurd. ♦

Corollary 4.2 If the following condition (4.2) holds:

$$\exists\ \alpha \in \text{int}H^+ \text{ such that } K_L \subseteq \alpha^\perp \text{ and } K_Q \subseteq \alpha^\perp_=, \qquad (4.2)$$

then $x_0 \in S$ is a local efficient point for problem P.

Corollaries 4.1 and 4.2 are equivalent in the decision space to Theorems 4.2 and 4.3 , respectively:

Theorem 4.2 Consider problem P and assume that $x_0 \in S$ verifies the following conditions:

i) $\exists\ \alpha=(\bar{\alpha}, \alpha^*) \in \text{int}H^+$, such that $\sum_{i=1}^{s} \bar{\alpha}_i \nabla f_i(x_0) + \sum_{j=1}^{m} \alpha^*_j \nabla g_j(x_0) = 0$

ii) $w^T[\sum_{i=1}^{s} \bar{\alpha}_i H_{f_i}(x_0) + \sum_{j=1}^{m} \alpha^*_j H_{g_j}(x_0)]\, w \leq 0 \quad \forall w \in W$

iii) $0 \notin (K_L+K_Q)$

Then x_0 is a local efficient point for P.

Theorem 4.3 Consider problem P and assume that $x_0 \in S$ verifies the following conditions:

i) $\exists\ \alpha=(\bar{\alpha},\alpha^*) \in \text{int}H^+$, $\bar{\alpha} \in \mathbf{R}^s$, $\alpha^* \in \mathbf{R}^m$, such that $\sum_{i=1}^{s} \bar{\alpha}_i \nabla f_i(x_0) + \sum_{j=1}^{m} \alpha^*_j\, g_j(x_0)=0$

ii) $w^T[\sum_{i=1}^{s} \bar{\alpha}_i H_{f_i}(x_0) + \sum_{j=1}^{m} \alpha^*_j H_{g_j}(x_0)]\, w < 0 \quad \forall w \in W$

Then x_0 is a local efficient point for P.

Let us note that conditions i) and ii) of Theorem 4.3 imply i), ii), iii) of Theorem 4.2, but the converse is not true as is shown by the following example:

Example 4.1 Consider problem P where s=1, m=1, $\varphi(x)=x^2$, $g(x)= -x^2$, $U=\mathbf{R}_+$, $V=\mathbf{R}_+$, and consider the feasible point $x_0=0$.

It is easy to verify that $f'(x_0)+g'(x_0)=0$, $W=\mathbf{R}$. $K_L = \{0\}$, $K_Q = \{(x^2, -x^2), x \neq 0\}$, $0\notin (K_L+K_Q)$. Conditions i), ii), iii) of Theorem 4.2 are verified so that x_0 is a local efficient point for the problem, while ii) of Theorem 4.3 is not verified.

The previous sufficient optimality conditions have been obtained starting from the sufficient condition $T_1 \cap clH =\{0\}$. In order to study the case $T_1\cap clH\neq\emptyset$, we define the following sets:

$K_Q^*=\{q\in \mathbf{R}^{s+m}: q=w^T H_F(x_0)w, w\in W^*\}$, $W^*=\{w\in \mathbf{R}^n : F'(x_0)w\in Fr(H) , w\neq 0 \}$.

$K_\alpha^*=\{ q\in \mathbf{R}^{s+m}: q=v^T H_F(x_0)v, v\in W_\alpha \}$. $W_\alpha=\{w\in \mathbf{R}^n : F'(x_0)w\in \alpha^\perp, w\neq 0 \}$

The following Theorem holds:

Theorem 4.4 If the following condition (4.3) holds:

$$\exists\alpha\in H^+, \alpha\neq 0, \text{ such that } K_L\subseteq\alpha^\perp \text{ and } K_Q^*\subseteq \alpha_{\leq}^{\perp}, \qquad (4.3)$$

then $x_0\in S$ is a local efficient point for problem P.

Proof

Ab absurdo suppose that $x_0\in S$ is not a local efficient point; then there exists a sequence $\{x_k\} \subset S\setminus\{x_0\}$, $x_k\to x_0$, such that $F(x_k)\in H$ $\forall k$.

The first order Taylor expansion gives $F(x_k) - F(x_0) = F'(x_0) (x_k - x_0) + \sigma(x_k, x_0)$ with $\lim_{k\to+\infty} \frac{\sigma(x_k, x_0)}{||x_k - x_0||} = 0$, so that there exists a subsequence of $\{x_k\}$ (without loss of generality we can suppose to be the same sequence) such that $\lim_{k\to+\infty} \frac{F(x_k) - F(x_0)}{||x_k - x_0||} = F'(x_0)w$, with $w= \lim_{k\to+\infty} \frac{x_k - x_0}{||x_k - x_0||}$. Since $F(x_k) - F(x_0) \in H$ $\forall k$, it results $\frac{F(x_k) - F(x_0)}{||x_k - x_0||} \in H$ $\forall k$, so that $F'(x_0)w \in clH$ and consequently, taking into account that $K_L\subseteq\alpha^\perp$, $F'(x_0)w \in FrH$, that is $w \in W^*$.

Consider now the second order Taylor expansion

$F(x_k) - F(x_0) = F'(x_0) (x_k - x_0) + 1/2F''(x_0)((x_k - x_0),(x_k - x_0)) + \sigma(x_k, x_0)$ with $\lim_{k\to+\infty} \frac{\sigma(x_k, x_0)}{||x_k - x_0||^2} = 0$. Taking into account that $\alpha^T F'(x_0)=0$, we have:

$\alpha^T(F(x_k) - F(x_0)) = 1/2 (x_k - x_0)^T(\alpha^T F''(x_0))(x_k - x_0) +\alpha^T \sigma(x_k, x_0)$, so that $\lim_{k\to+\infty} \frac{\alpha^T[F(x_k)-F(x_0)]}{||x_k-x_0||^2} = 1/2\, w^T(\alpha^T F''(x_0))w\geq 0$; on the other hand $w \in W^*$ implies $w^T(\alpha^T F''(x_0))w < 0$ and this is absurd. ♦

By means of the previous theorem, taking into account that $K_Q^* \subseteq K_\alpha^*$ for each $\alpha\in H^+$, $\alpha\neq 0$, such that $K_L\subseteq\alpha^\perp$, we have the following corollary:

Corollary 4.3 If the following condition (4.4) holds:

$$\exists \alpha \in H^+,\ \alpha \neq 0, \text{ such that } K_L \subseteq \alpha^\perp \text{ and } K^*_\alpha \subseteq \alpha^\perp_=, \tag{4.4}$$

then $x_0 \in S$ is a local efficient point for problem P.

Theorem 4.4 and Corollary 4.3 are equivalent in the decision space to Theorem 4.5 and Theorem 4.6, respectively.

Theorem 4.5 Consider problem P and assume that $x_0 \in S$ verifies the following conditions:

i) $\exists\ \alpha=(\bar{\alpha},\alpha^*) \in H^+,\ \bar{\alpha} \in \mathbf{R}^s,\ \alpha^* \in \mathbf{R}^m$, such that $\sum_{i=1}^{s} \bar{\alpha}_i \nabla f_i(x_0) + \sum_{j=1}^{m} \alpha^*_j \nabla g_j(x_0) = 0$

ii) $w^T[\sum_{i=1}^{s} \bar{\alpha}_i H_{f_i}(x_0) + \sum_{j=1}^{m} \alpha^*_j H_{g_j}(x_0)]\, w < 0 \quad \forall w \in W^*$

Then x_0 is a local efficient point for P.

Theorem 4.6 Consider problem P and assume that $x_0 \in S$ verifies the following conditions:

i) $\exists\ \alpha=(\bar{\alpha}, \alpha^*)\alpha \in H^+,\ \bar{\alpha} \in \mathbf{R}^s,\ \alpha^* \in \mathbf{R}^m$, such that $\sum_{i=1}^{s} \bar{\alpha}_i \nabla f_i(x_0) + \sum_{j=1}^{m} \alpha^*_j g_j(x_0) = 0$

ii) $w^T[\sum_{i=1}^{s} \bar{\alpha}_i H_{f_i}(x_0) + \sum_{j=1}^{m} \alpha^*_j H_{g_j}(x_0)]\, w < 0 \quad \forall w \in W_\alpha$

Then x_0 is a local efficient point for P.

Let us note that in the scalar case condition ii) of Theorem 4.6 states the well known second order sufficient optimality condition; the following example points out that the optimality condition stated in Theorem 4.5 is more general than the previous one:

Example 4.2 Consider problem P where s=1, m=3, $U=\mathbf{R}_+$, $V=\mathbf{R}^3_+$, $\varphi(x,y,z)=x+y-x^2-y^2+z^2$, $g_1(x,y,z)=-x-y$, $g_2(x,y,z)=z$, $g_3(x,y,z)=-z$ and consider the feasible point $x_0=(0,0,0)$.

By means of simple calculations we obtain:

$\nabla f(x_0) + \nabla g_1(x_0) = 0$, so that i) of Theorem 4.5 and i) of Theorem 4.6 hold;

$W_\alpha = \{(w_1,-w_1,w_3), w_1, w_3)\in \mathbf{R}\}$, so that $w^T(Hf(x_0)+Hg_1(x_0))w = -2w_1^2+w_3 \in \mathbf{R}$ and this implies that ii) of Theorem 4.6 is not verified. On the other hand $W^* = \{(w_1, -w_1, 0), w_1 \in \mathbf{R}\}$ and it results $w^T(Hf(x_0) +Hg_1(x_0))w = -2w_1^2$; condition ii) of Theorem 4.5 is verified and the origin is a local efficient point for the problem.

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