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Innovation and Capital Accumulation in a Vintage Capital Model: an Infinite Dimensional Control Approach

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Abstract

This paper is concerned with a model of capital accumulation and technology innovation in a vintage capital framework. The model is an infinite horizon/infinite dimensional optimal control model, the firm employs a continuum of technologies (a continuum of heterogeneous capital goods). Technology is specific to capital goods and is related to their vintage. The entrepreneur maximizes the profits obtained by employing the continuum of technologies under the assumption of constant returns to scale and bearing adjustment costs for investments. Existence of an optimal investment policy and the diffusion of a new technology are proved. Two specific models are considered: quality depreciation without technology innovation and exogenous technology innovation.

Key words: Capital Accumulation, Investments, Heterogeneous Capital Goods, Vintage, Innovation.

Classifications: (JEL) C61, C62, E22; (AMS) 49-02, 49J20, 49K20, 49N05, 90A11, 90A16.

1 Introduction

In this paper we present a model of capital accumulation and technology innovation in a vintage capital setting. The model is an infinite horizon/infinite dimensional capital accumulation model. The firm employs a continuum of technologies/ heterogeneous capital goods differentiated by their vintage and their productivity. The technology is specific to a capital good and is related both to its vintage and to technological progress, i.e. the vintage of a capital good will represent a technology label; for an early analysis of vintage in the capital accumulation process see [Solow, 1959, Solow et al., 1966].

To the classical scalar model of capital accumulation driven by a differential equation, where time is the unique dimension, we add a second dimension: vintage/technology status. The evolution of the stock of capital as a function of time and vintage is described by a *Partial Differential Equation* (PDE). This allows us to introduce heterogeneity in the capital accumulation process and to establish a relationship between the going on of the time and the technology status of the capital goods employed in the firm. The evolution of the stock of capital of the firm is characterized by the classical constant capital decay rate and by the changing of its productivity: as time goes, capital goods are going to be characterized by more and more mature technologies. Different phenomena take place inside the firm as time goes: the stock of capital decreases from a quantitative point of view and changes from a qualitative point of view because of experience, invention of new techniques and new goods, learning by doing and qualitative depreciation.

The dynamics of capital goods with respect to the technology is driven by factors *internal* and *external* to the firm. Internal factors are mainly qualitative depreciation, learning by doing and spillover effects: goods employed in a firm become less and less productive because they are employed in producing goods, but on the other side the productivity may rise because of experience effects, complementarities with new goods, spillover effects, externalities, etc. External factors are represented by innovation: as time goes new techniques and new goods are made available exogenously or by means of *R&D*. The model proposed in this paper incorporates both *process innovation* and *goods innovation*: capital goods differentiation occurs because more productive techniques or more productive goods are discovered or because new processes to reduce the cost of producing existing products are available; the vintage label can either mark a new capital good or a new technique.

We assume technology irreversibility: investments are locked up in a plant at a given date and then their productivity is determined as time goes by their vintage, i.e. capital is specific to a particular technology. We allow for a partial effect of innovation on the productivity of vintage capital goods. Firm may invest at any time in all the technologies, there is a market for vintage capital goods so that firm investments and the stock of capital are described by functions of the vintage. The firm bears adjustment costs in the capital accumulation process. Irreversibility concerns the technology of a capital good but not investments, investments may be negative.

In recent years, a large literature has grown up on growth, innovation, and vintage capital. Among the explanations investigated in the literature for a connection between the vintage of a stock of capital locked in a plant and technology, we can remember:

learning by doing, [Arrow, 1962, Lucas, 1991, Stokey, 1988, Young, 1993a], input substitution and complementarity, [Young, 1993b], qualitative depreciation of the capital goods as they are employed in producing goods, [Solow, 1959], exogenous technological progress, [Solow, 1959, Chari and Hopehhayn, 1991], *R&D* for process, quality and goods innovation, [Judd, 1985, Grossman and Helpman, 1991b, Romer, 1990, Segerstrom, 1991], etc.. These features of the firm life have not been analyzed in a complete dynamic/optimization setting; in the papers quoted above the economic decisions about capital accumulation, consumption, invention and investments are not analyzed in a complete dynamic/optimization framework because goods are not durable and can not be accumulated. Because of this feature, the intertemporal dynamic dimension of the economic decisions reduces to an intertemporal budget allocation problem, see [Grossman and Helpman, 1991a, Segerstrom, 1991, Young, 1993a], or to a pure static allocation problem, see [Lucas, 1991, Romer, 1990, Stokey, 1988, Stokey, 1991]; in many cases capital is absent as input factor and we have only labor, see [Chari and Hopehhayn, 1991].

In what follows we build a complete dynamic framework for the analysis of the firm accumulation problem with heterogeneous/vintage capital goods. Being the evolution of the stock of capital described by a partial differential equation, the firm maximization problem over the infinite horizon becomes an infinite dimensional optimal control problem: the firm maximizes the integral of the profits obtained by employing a continuum of technologies/capital goods. The entrepreneur faces adjustment costs installing capital goods and an innovation cost installing new capital goods, the innovation cost can be interpreted as a royalty to be paid to the inventor of the new technique. All the techniques/capital goods are characterized by constant returns to scale, there are no complementarity/substitution effects among capital goods. The coefficients describing the productivity of capital goods and adjustment costs are described by functions of the vintage. The entrepreneur invests at any time in every technology, not only in the newest; this allows us to study technology innovation and its diffusion, i.e. the scatter of the stock of capital and investments along vintage capital goods/technologies. We establish the existence of an optimal capital accumulation policy and we find out the long run stationary equilibrium.

The control problem analyzed in this paper is solved by adapting to this case techniques of control theory in finite and infinite dimension. More precisely the tools we use are: the theory of semigroups (see e.g. [Pazy, 1983, Bensoussan et al., 1992]), the framework of boundary control problems (see e.g. [Fattorini, 1968, Bensoussan et al., 1992, Cannarsa et al., 1993, Cannarsa and Tessitore, 1996], optimality conditions for optimal control problems (see e.g. [Tessitore, 1995, Carlson et al., 1991] for the case of finite dimension/infinite horizon and e.g. [Barron and Jensen, 1986, Cannarsa and Frankowska, 1992] for the case of infinite dimensions/finite horizon).

We consider two specific models. A model where vintage capital goods are differentiated because of quality depreciation, learning by doing, and a model incorporating both exogenous innovation and quality depreciation.

The diffusion of a new technology, i.e. the fact that the stock of capital and investments are not concentrated only on the newest technology but are *single-peaked* functions of the vintage, is a well established empirical result about firm's capital accumulation which seems to be difficult to be explained from a theoretical point of view. Many different approaches to explain this fact have been developed in the literature; among them, the more recent one is based on the presence of positive spillover effects, input complementarities, externalities, see for example [Chari and Hopehhayn, 1991, Jovanovic and Lach, 1989] and [Jovanovic and MacDonald, 1994]. In this paper we provide an explanation of the diffusion of a new technology by abstracting from these facts and restricting our attention to innovation costs, adjustment costs, exogenous innovation and quality depreciation of vintage capital goods; we look for an explanation of the diffusion of a new technology purely in terms of the capital accumulation process with adjustment costs: the firm may decide not to invest heavily in the newest technology because it is too expensive to install.

In a quality depreciation setting we show that the stationary stock of capital obtained as solution of the optimal investment problem can be either a decreasing function of the vintage or a *single-peaked* function with a maximum. *Capital deepening* results type are proved in this context. The stationary stock of capital turns out to be *single peaked* if a new technology is highly profitable, the innovation costs is high and the rates of quality and quantity depreciation are not too high.

The model with exogenous exponential innovation is characterized by perpetual growth. The growth rate in the long run is determined by the rate of technological improvement deflated either by the rate of growth of innovation costs or by the rate of growth of adjustment costs (the smaller one deflates the rate of technological improvement). In the limit the shape of the optimal stock of capital can be either a function of the vintage strictly decreasing, if the rate of growth of innovation costs are smaller than the rate of growth of installation costs, or *single peaked*, if the opposite holds.

The paper is organized as follows. In Section 2 we present the model; in Section 3 we describe the state equation by means of semigroups, in Section 4 we define the value function associated with the optimal control problem and we find out the optimal solution. Sections 5 and 6 are devoted to the analyzis of the optimal solution and of the long run equilibrium of the model. In Section 7 we study the model under investments irreversibility. In Section 8 we analyze a specific model with exponential quality depreciation. In Section 9 we analyze the model with exogenous technology innovation.

2 The Model

The capital accumulation/technology evolution is described by the following controlled dynamical system

$$\begin{cases} \dfrac{\partial k(t,s)}{\partial t} + \lambda \dfrac{\partial k(t,s)}{\partial s} + \mu k(t,s) = u(t,s) & t \in (0,+\infty),\ s \in [0,\overline{s}] \\ k(t,0) = u(t,0) & t \in (0,+\infty) \\ k(0,s) = k_0(s) & s \in [0,\overline{s}] \end{cases} \tag{1}$$

where $\mu, \lambda > 0$ and $\overline{s} \in (0,+\infty]$. The index t stands as the time ($t \geq 0$), s as the vintage ($s \geq 0$). $s' > s$ means that capital goods indexed by s' are older than capital goods indexed by s. The maximal value of the vintage considered is $\bar{s}$, that can also be infinite. $k(t,s)$ is the amount of capital goods of vintage s accumulated at time t; $u(t,s)$ represents gross investment at time t in capital goods of vintage s. $u(t,0)$ represents investments in new capital goods at time t and represents also the boundary condition for the evolution of the stock of capital k.

The partial differential equation (1) generalizes the classical dynamical system describing the firm capital accumulation, $\dot{k}(t) = u(t) - \mu k(t)$. With respect to this equation, we have that the law of motion for the capital (1) relates the flow of time to the vintage of the stock of capital. At time t the firm stock of capital is described by a function of s, $k(t,s)$. Leaving aside the capital decay rate component and the control in (1), we have that with $\lambda = 1$ the stock of capital at time t of vintage s becomes after the time period δt the stock of capital of vintage $s + \delta t$, i.e. $k(t,s) = k(t + \delta t, s + \delta t)$. The parameter λ describes the connection between time and vintage of the capital goods, if $\lambda = 1$ we have a one to one correspondence between the going on of the time and of the vintage, a λ larger than 1 means that the capital goods of the plant quickly go out of date; more generally $k(t,s) = k(t + \delta t, s + \lambda \delta t)$.

The technology is characterized by constant returns to scale. The coefficient of production $\alpha(t,s)$, $\alpha(t,s) > 0$, is constant for every stock of capital of vintage s employed at time t. No assumption a-priori is done on $\alpha(t,s)$, we only require for the moment that an age $\bar{s}$ exists such that $\alpha(t,s) = 0$ for $s \geq \bar{s}$ and $\forall t$; $\bar{s}$ can also be $+\infty$. Let us remark that the dependence of the technology parameter on t and s marks both innovation (t) and quality depreciation and/or learning by doing effects (s) for the capital goods being employed in the firm. If α only depends on s then there is not innovation in the traditional sense, but only changes in technology productivity of the capital goods employed in the firm. $\alpha(t,s)$ can also be interpreted as the world price of a good of vintage s at time t, see [Lucas, 1991, Stokey, 1988]. In what follows we will develop the analysis assuming that $\alpha'(t,\cdot) \leq 0$, $\forall t \geq 0$.

The set-up of the model allows us to disentangle two aspects related to the vintage of the capital goods: λ describes the connection between the time flow and the vintage of the capital goods, $\alpha(t,s)$ describes the dependence of the technology on exogenous innovation and vintage.

There are no spillover effects and complementarities among the capital goods: the productivity of a technology is not related to the stock of capital employed in different technologies. The assumption of constant returns to scale is similar to the one done in models with learning by doing or with goods displaced along quality ladders, see for example [Judd, 1985, Grossman and Helpman, 1991a, Segerstrom, 1991, Stokey, 1988, Lucas, 1991, Young, 1993a].

The entrepreneur maximizes firm profits over an infinite horizon. In our setting, the firm profit at each instant of time is the integral over the domain of the technology/vintage, $[0, \bar{s}]$, of the returns the firm realizes from each technology minus the adjustment costs that the firm has to bear in the capital accumulation process. In our model we assume that the firm has to bear two different kinds of investment costs: a cost to buy capital goods, the unit cost $q(t,s)$, and adjustment costs. Adjustment costs are a quadratic function of the investments with a coefficient $\beta(t,s)$. For new capital goods, $u(t,0)$, the firm has to bear an extra cost due to innovation, $\beta_0(t) > 0$; so by $\beta(0,t)$ we represent adjustment costs for investments in new capital goods and by $\beta_0(t)$ we represent innovation costs. Innovation costs can be explained thinking to the fact that new technologies are protected by a patent for a while after being invented and you have to pay a price to use them. In the following we assume that this extra cost should be paid only for new capital goods, $s = 0$.

The entrepreneur maximizes the following objective function:

$$J(k_0; u)$$

$$= \int_0^{+\infty} e^{-\rho t} \left[-\beta_0(t) u^2(t,0) + \int_0^{\overline{s}} \left[\alpha(t,s)k(t,s) - q(t,s)u(t,s) - \beta(t,s)u^2(t,s) \right] ds \right] dt \tag{2}$$

overall state trajectory–control pairs $\{k, u\}$ which are solutions in a suitable sense of equation (1).

The unit investment cost $q(t,s)$ is related to the vintage of the capital goods; the shape of $q(t,\cdot)$ is similar to the one of $\alpha(t,\cdot)$: $q'(t,\cdot) \leq 0 \; \forall t \geq 0$, vintage capital goods are less expensive than new capital goods. Adjustment costs are related to the vintage of the capital goods, we assume that it is easier to install vintage goods rather than new capital goods, $\beta'(t,\cdot) \leq 0 \; \forall t \geq 0$. The dependence of $q(t,s)$ on s can be avoided assuming that there are homogeneous capital goods in the market and that differentiation only occurs in the plant because of the technology to which each good is specific, in that case $q(t,s)$ is constant in s. If $\overline{s} = +\infty$ then $\alpha(t,\cdot)$, $\beta(t,\cdot)$, $q(t,\cdot)$ go to zero as $s \to +\infty$.

In the following we will first consider a technology with α, β, q not dependent on t, there is no innovation as time goes but only differentiation due to the vintage of the capital goods. Then, in Section 9 we will consider exogenous innovation: new technologies arrives continuously in the market, their productivity grows with t at a constant rate.

3 The state equation

We begin by recalling some basic mathematical definitions and results that we will use throughout the paper. We refer the reader to [Brezis, 1983]. Let $\overline{s} \in]0, +\infty]$. We will denote by $L^2(0,\overline{s})$, or, when no confusion will be possible, simply by L^2 the spaces of Lebesgue measurable functions $f : (0,\overline{s}) \to \mathbb{R}$ such that

$$\int_0^{\overline{s}} |f(s)|^2 ds < +\infty.$$

Moreover, we will denote by $H^n(0,\overline{s})$ (or simply H^n), $n = 1, 2, ..$, the Sobolev space of functions $f \in L^2$ such that the n-th distributional derivative of f still belongs to L^2. We denote by $H^1(0, +\infty; H^1)$ the space of all functions $f : (0, +\infty) \to H^1$ that are square integrable with their first distributional derivative.

Now we study the state equation in an infinite dimensional setting. Take a controlled dynamical system (for a review of the optimal control theory for this kind of systems see [Lions, 1972, Bensoussan et al., 1992, Li and Yong, 1995]) whose behavior is described by the following partial differential equation in the strip $[0,+\infty) \times [0,\overline{s}]$ ($[0,+\infty) \times [0,+\infty)$ if $\overline{s} = +\infty$):

$$\begin{cases} \dfrac{\partial k(t,s)}{\partial t} + \lambda \dfrac{\partial k(t,s)}{\partial s} + \mu k(t,s) = u(t,s); & t > 0,\ s \in [0,\overline{s}] \\ k(t,0) = u(t,0); & t > 0 \\ k(0,s) = k_0(s); & s \in [0,\overline{s}] \end{cases} \tag{3}$$

where $u : [0,+\infty) \times [0,\overline{s}] \to \mathbb{R}$ is the control function and $k : [0,+\infty) \times [0,\overline{s}] \to \mathbb{R}$ is the state function. We want to express this *Partial Differential Equation* (PDE) as an *Ordinary Differential Equation* (ODE) in the Hilbert space $L^2 = L^2(0,\overline{s})$, by using the language of semigroups. In the following we will often omit the variable s; $k(t), u(t)$ denote the element $k(t,\cdot), u(t,\cdot) \in L^2$. We will employ the variable s only when it will be needed to avoid misunderstandings.

We consider the following linear closed operator on L^2

$$\begin{cases} D(A) = \{f \in H^1 : f(0) = 0\} \\ Af(s) = -\lambda f'(s) - \mu f(s). \end{cases} \tag{4}$$

Let us remark that the problem is not well-posed if we substitute the boundary condition

$$k(t,0) = u(t,0); \qquad t > 0$$

with a boundary condition for $s = \overline{s}$ (when $\overline{s}$ is finite)

$$\frac{\partial k(t,\overline{s})}{\partial s} = 0 \ \text{ or } \ k(t,\overline{s}) = 0; \qquad t > 0 \tag{5}$$

which would establish that nothing goes out of the set of productive technologies and that the capital outside the set of productive technologies is zero, respectively. This is due to the presence of the first derivative with respect to s in (1) which generates, in some sense, a transport phenomenon in the solution.

To the PDE (3) a semigroup $T(t)$ is associated; $T(t)$ and its properties are described in the following Proposition, a sketch of the proof is in Appendix A.

Proposition 3.1 *The operator A is a linear closed dissipative operator on L^2 and generates a strongly continuous semigroup $T(t)$. When $\overline{s} < +\infty$, $T(t)$ is given by*

$$[T(t)f](s) = e^{-\mu t} \begin{cases} f(s - \lambda t) & s \in [\lambda t, \overline{s}] \\ 0 & s \in [0, \lambda t) \end{cases}$$

for $t \in [0, \frac{\overline{s}}{\lambda}]$, and by

$$[T(t)f](s) = 0; \quad s \in [0,\overline{s}]$$

for $t > \frac{\overline{s}}{\lambda}$. When $\overline{s} = +\infty$ we have

$$[T(t)f](s) = e^{-\mu t} \begin{cases} f(s - \lambda t) & s \in [\lambda t, +\infty) \\ 0 & s \in [0, \lambda t). \end{cases}$$

The resolvent set of A contains the half plane $\{\mathrm{Re}\ \gamma > -\mu\}$. *For* $\mathrm{Re}\ \gamma > -\mu$, $f \in H$, $\bar{s} \in (0, +\infty]$, *we have*

$$[R(\gamma; A)f](s) = \int_0^{+\infty} e^{-\gamma t}[T(t)f](s)dt = \frac{1}{\lambda}\int_0^s e^{-\frac{\gamma+\mu}{\lambda}(s-\sigma)} f(\sigma)d\sigma \quad s \in [0, \bar{s}].$$

Let A^* the adjoint operator of A. The following Proposition can be stated about A^*.

Proposition 3.2 *The operator* A^* *is given by*

$$\begin{cases} D(A^*) = \left\{f \in H^1 : f(\bar{s}) = 0\right\} \\ [A^*f](s) = \lambda[f'(s)] - \mu f(s) \end{cases}$$

when $\bar{s} < +\infty$. *When* $\bar{s} = +\infty$ *we have that* $\lim_{s\to+\infty} f(s) = 0$ *for* $f \in H^1(0, +\infty)$, *so that*

$$\begin{cases} D(A^*) = H^1 \\ [A^*f](s) = \lambda[f'(s)] - \mu f(s). \end{cases}$$

The operator A^* *is the generator of a strongly continuous semigroup on* L^2, $T^*(t)$; *when* $\bar{s} < +\infty$, $T^*(t)$ *is given by*

$$[T^*(t)f](s) = e^{-\mu t}\begin{cases} f(s + \lambda t) & s \in [0, \bar{s} - \lambda t] \\ 0 & s \in (\bar{s} - \lambda t, \bar{s}] \end{cases}$$

for $t \in [0, \frac{\bar{s}}{\lambda}]$, *and by*

$$[T^*(t)f](s) = 0; \quad s \in [0, \bar{s}]$$

for $t > \frac{\bar{s}}{\lambda}$. *When* $\bar{s} = +\infty$ *we have*

$$[T^*(t)f](s) = e^{-\mu t} f(s + \lambda t) \qquad s \geq 0.$$

For $\bar{s} \in (0, +\infty]$, *the resolvent of* A^*, $R(\gamma; A^*)$, *is given by the formula (for* $f \in H$*)*

$$[R(\gamma; A^*)f](s) = \int_0^{+\infty} e^{-\gamma t}[T^*(t)f](s)dt = \frac{1}{\lambda}\int_s^{\bar{s}} e^{-\frac{\gamma+\mu}{\lambda}(\sigma-s)} f(\sigma)d\sigma \quad s \in [0, \bar{s}].$$

Let us remark that the semigroups T and T^* (and consequently their resolvent operators) preserve the positivity in the sense that for every $t \geq 0$ we have

$$f(s) \geq 0 \quad \forall s \in [0, \bar{s}]$$

$$\Updownarrow$$

$$[T(t)f](s),\ [T^*(t)f](s),\ [R(\gamma; A)f](s),\ [R(\gamma; A^*)f](s) \geq 0 \quad \forall s \in [0, \bar{s}].$$

To handle the control problem with the boundary condition $k(t, 0) = u(t, 0)$, the control strategy should be well defined at $s = 0$ for every $t > 0$. To this end, the control strategy $u(t, s)$ has to be at least piecewise continuous with respect to the variable s. Moreover, to have a well defined solution of the state equation (see below) we will also require the control strategy $u(t,s)$ to be differentiable with square integrable derivative with respect to t. To take care of these requirements on the control strategy $u(t,s)$, we will assume that $u \in H^1(0,+\infty; H^1)$. We will see below how to generalize this setting to more general control strategies, see the end of Section 4.

We now define the solution of the state equation (3). Let first $u(t,0) \equiv 0$ for every $t \geq 0$. Given a control strategy $u \in H^1(0,+\infty; H^1)$ with $u(t,0) \equiv 0$ we define as usual (see e.g. [Pazy, 1983, §4.2]) the mild solution of (3) as the continuous function $k : [0,+\infty) \to L^2$

$$k(t) = T(t)k_0 + \int_0^t T(t-\tau)u(\tau)d\tau. \tag{6}$$

When $u(t,0) \not\equiv 0$, the solution of (3) can still be written in integral form by a standard procedure (see e.g. [Bensoussan et al., 1992, Lions and Magenes, 1968]). Denoting by $w(t,\cdot)$ the unique element of H^1 such that

$$-\lambda \frac{\partial w}{\partial s}(t,s) - \mu w(t,s) = 0 \qquad w(t,0) = u(t,0),$$

so that

$$w(t,s) = e^{-\frac{\mu}{\lambda}s} u(t,0) \stackrel{def}{=} w_0(s)u(t,0) \quad s \in [0,\bar{s}]$$

then the mild solution of (3) becomes

$$k(t) = T(t)k_0 - \left[A \int_0^t T(t-\tau)u(\tau,0)d\tau\right] w_0 + \int_0^t T(t-\tau)u(\tau)d\tau. \tag{7}$$

In the following we will denote by $k(t; k_0, u)$ the mild solution of (3) for given data k_0, u. We will omit the data, simply writing $k(t)$, when no confusion will be possible.

The term in (7)

$$-\left[A \int_0^t T(t-\tau)u(\tau,0)d\tau\right] w_0$$

represents the effect of the boundary condition $k(\tau,0) = u(\tau,0) \neq 0,\ 0 \leq \tau \leq t$, on the solution of the state equation (3) at time t.

In general, if the function $t \to u(t,0)$ is continuous then the term

$$\left[\int_0^t T(t-\tau)u(\tau,0)d\tau\right] w_0$$

does not belong to $D(A)$. This is guaranteed when the function $t \to u(t,0)$ has integrable derivative. In this case in fact we can write (see e.g. [Pazy, 1983, p.107])

$$\left[A \int_0^t T(t-\tau)u(\tau,0)d\tau\right] w_0 = T(t)w_0u(0,0) + \int_0^t T(t-\tau)\frac{\partial}{\partial \tau}u(\tau,0)w_0 d\tau - w_0u(t,0).$$

This ensures that the mild solution (7) of equation (3) is a continuous function $k : [0,+\infty) \to L^2$.

Given the framework described above, equation (3) can be written in the following differential form:

$$\begin{cases} k'(t) = A[k(t) - w(t)] + u(t) \\ k(0) = k_0 \end{cases} \tag{8}$$

which *does not make sense in the space* L^2. In fact the term $-Aw(t) = -Aw_0 u(t,0)$, which is the effect of the boundary condition $k(t,0) = u(t,0) \neq 0$ on the time derivative $k'(t)$ of the state k at time t, is not a function in the space L^2. More precisely $-Aw_0$ is a distribution and is equal to the Dirac delta function multiplied by λ, so we can write $-Aw_0 u(t,0) = \lambda\delta_0 u(t,0)$. The presence of the factor λ is due to the change of scale: the boundary condition is established at $s = 0$, but the state equation is written with respect to the variable t and so the component coming from the boundary condition on the state equation is affected by the change of scale $\delta s = \lambda \delta t$.

4 The reward functional and the value function

In the following we will drop the dependence of the technology on t, we will only consider technology change due to the employment of capital goods. Let us assume that $\beta_0 \in \mathbb{R}$, α, β and q depend only on s and are bounded elements of H^2. Let us assume that the firm technology satisfies the following Assumption

Assumption 4.1

(i) $\beta_0 > 0$

(ii) $\alpha(s), q(s) \geq 0$, *and* $\beta(s) \geq \epsilon > 0$ *for a given* $\epsilon > 0$, $\forall s \in [0, \overline{s}]$.

(iii) $\alpha(\overline{s}) = 0$

(iv) $\alpha'(s) \leq 0$, $q'(s) \leq 0$, $\beta'(s) \leq 0, \forall s \in [0, \overline{s}]$..

We consider the function

$$g : L^2 \to \mathbb{R}$$

$$g(k) = \int_0^{\overline{s}} \alpha(s) k(s) ds =< \alpha, k >_{L^2}$$

and

$$l : H^1 \to \mathbb{R}$$

$$\begin{aligned} l(u) &= \int_0^{\overline{s}} [-q(s)u(s) - \beta(s)u^2(s)]ds - \beta_0 u^2(0) \\ &= - < q, u >_{L^2} - < B_\beta u, u >_{L^2} - \beta_0 u^2(0) \end{aligned}$$

where $B_\beta : L^2 \to L^2$ is the continuous linear operator defined as $[B_\beta f](s) = \beta(s) f(s)$, $s \in [0, \overline{s}]$. As pointed out in the previous section we assume that the control strategy u belongs to the set

$$\mathcal{U} \stackrel{def}{=} H^1(0, +\infty; H^1).$$

The optimal control problem becomes:

(P): maximize the functional

$$J(k_0; u) = \int_0^{+\infty} e^{-\rho\tau} \left[g(k(\tau)) + l(u(\tau))\right] d\tau \tag{9}$$

overall control trajectories $u \in \mathcal{U}$ where k is the corresponding mild solution of the state equation (3).

Definition 4.2 *A control strategy $u^* \in \mathcal{U}$ will be called an optimal strategy if*

$$J(k_0; u^*) \geq J(k_0; u) \qquad \forall u \in \mathcal{U},$$

a state-control pair (k^*, u^*) will be called an optimal pair if u^* is an optimal control strategy and k^* is the corresponding state trajectory.

The value function of the problem is defined as

$$v(k_0) = \sup_{u \in \mathcal{U}} J(k_0; u).$$

By a classical argument of control theory (see e.g. [Bellman, 1977, Fleming and Rishel, 1975] and [Fleming and Soner, 1993, ch. 1]) we expect the value function to be the unique solution (in a suitable sense) of the following Hamilton-Jacobi equation:

$$\rho v(k) - < k, A^* Dv(k) >_{L^2} - H_0(Dv(k)) = g(k) \tag{10}$$

(see also Remark 4.8) where the Hamiltonian H_0 is given (for $p \in D(A^*)$) by

$$H_0(p) = \sup_{u \in H^1} F_0(u, p) \tag{11}$$

and the current value Hamiltonian F_0 is

$$F_0(u, p) \stackrel{def}{=} -u(0) < w_0, A^* p >_{L^2} + < u, p >_{L^2} + l(u). \tag{12}$$

We observe that in the classical context the current value Hamiltonian should be given by $F_1(u, p, k) = F_0(u, p) + g(k) - < k, A^* p >_{L^2}$. Here we have used the notation of [Fleming and Soner, 1993] by putting in the current value Hamiltonian only the terms depending on the control u.

Since g is linear and l is strictly concave, then the control problem enjoys nice properties. First we introduce the function

$$\overline{\alpha}(s) = [R(\rho; A^*)\alpha](s) = \frac{1}{\lambda} \int_s^{\overline{s}} e^{-\frac{\rho+\mu}{\lambda}(\sigma - s)} \alpha(\sigma) d\sigma, \tag{13}$$

$\bar{\alpha}(s)$ is the discounted return associated with a unit of capital of vintage s. As time goes, the return of capital goods of vintage $\sigma \geq s$ is associated with a unit of capital of vintage s. We remember that the connection between the vintage of a good and the flow of time is $\delta s = \lambda \delta t$; this means that a capital good of vintage s will be of vintage $\sigma > s$ after the time period $\frac{\sigma-s}{\lambda}$, so the discounted return associated with a unit of capital good of vintage s for being of vintage σ after the time period $\frac{\sigma-s}{\lambda}$ is $e^{-\rho\frac{\sigma-s}{\lambda}}\alpha(\sigma)$; meanwhile the unit of capital good is exponentially decreased and amounts to $e^{-\mu\frac{\sigma-s}{\lambda}}$. Therefore, the discounted return of a unit of capital of vintage s for being of vintage σ after the time period $\frac{\sigma-s}{\lambda}$ becomes

$$e^{-\frac{\rho+\mu}{\lambda}(\sigma-s)}\alpha(\sigma),$$

this is the integrand of (13) and explains the economic interpretation of $\bar{\alpha}(s)$. The following Proposition explains how the functional J can be rewritten in a simpler form involving $\overline{\alpha}$, the proof is in Appendix A.

Proposition 4.3 *The functional J is linear with respect to k_0 and has the following expression*

$$J(k_0, u) = <\overline{\alpha}, k_0>_{L^2} + \int_0^{+\infty} e^{-\rho t}F(u(t))dt \tag{14}$$

where

$$\begin{aligned} F(u) = & - <A^*\overline{\alpha}, w_0> u(0) - \beta_0 u^2(0) \\ & + <-q+\overline{\alpha}, u>_{L^2} + <B_\beta u, u>_{L^2} = F_0(u, \overline{\alpha}). \end{aligned} \tag{15}$$

Remark 4.4 (i) The above result follow simply by substituting the expression of the mild solution k of the state equation (3) into (9). Due to the linearity of the model the term $\int_0^{+\infty} e^{-\rho t} g(k(t)) = <\alpha, k(t)>_{L^2}$ can be split in two parts: the first one depends only on the initial state k_0, $<\overline{\alpha}, k_0>_{L^2}$, the second one depends on the control u, $-<A^*\overline{\alpha}, w_0>_{L^2} u(0) + <\overline{\alpha}, u>_{L^2}$. Then, also the functional J is naturally split in two parts: the first one only depends on the initial state k_0, $<\overline{\alpha}, k_0>_{L^2}$, while the second depends on the control and is given by the integral of $l(u)$ plus the term $-<A^*\overline{\alpha}, w_0>_{L^2} u(0) + <\overline{\alpha}, u>_{L^2}$ coming from the integration of the map $g(k(t))$.

(ii) The term

$$-<A^*\overline{\alpha}, w_0>_{L^2} u(t, 0)$$

represents the effect of the boundary condition $k(t, 0) = u(t, 0) \neq 0$ on the reward functional. Concerning the sign of this term we observe that $-<A^*\overline{\alpha}, w_0>_{L^2} \geq 0$ when $\alpha \geq 0$. This fact tells us that the sign of the term $-<\alpha, AR(\rho; A)w_0> u(t, 0)$ is the same as the one of the control $u(t, 0)$. In fact, since

$$w_0(s) = e^{-\frac{\mu}{\lambda}s}$$

we have, by straightforward calculations

$$-<A^*\overline{\alpha}, w_0>_{L^2} = \lambda\overline{\alpha}(0) \leq 0.$$

As seen in the discussion after the equation (8) the term $-Aw_0 u(t, 0)$ is the effect of the boundary condition $k(t, 0) = u(t, 0) \neq 0$ on the derivative of the state k and is equal to $\lambda\delta_0$ where δ_0 is the Dirac delta function. The effect of this term on the reward functional is given by

$$< \overline{\alpha}, -Aw_0 >_{L^2} u(t,0) = \lambda\overline{\alpha}(0)u(t,0).$$

Concerning the existence of the optimal control we have the following result.

Proposition 4.5 *There exists only one optimal control u^* for problem (P) if and only if the following condition is satisfied*

$$\frac{\lambda\overline{\alpha}(0)}{2\beta_0} = \frac{1}{2\beta(0)}\left[\overline{\alpha}(0) - q(0)\right]. \tag{16}$$

The optimal control u^* does not depend on the initial state k_0 and on the time t; u^* is

$$\begin{aligned} u^*(0) &= \frac{\lambda\overline{\alpha}(0)}{2\beta_0} \\ u^*(s) &= \frac{1}{2\beta(s)}\left[\overline{\alpha}(s) - q(s)\right]. \end{aligned} \tag{17}$$

u^* satisfies the following "maximum principle"

$$F_0\left(u^*, \overline{\alpha})\right) = \sup_{u\in H^1} F_0\left(u, \overline{\alpha})\right) = H_0(\overline{\alpha}) \tag{18}$$

and it is given by

$$u^*(s) = DH_0(\overline{\alpha}).$$

Proof. First we prove that, if the compatibility condition (16) is satisfied then u^* given by (17) is the unique optimal control for our problem. To this end we observe that, by Proposition 4.3 problem (P) reduces to maximize the functional

$$J_0(u) = \int_0^{+\infty} e^{-\rho t} F_0(u(t), \overline{\alpha})dt \tag{19}$$

subject to (1). The functional $J_0(u)$ does not depend on the initial datum k_0; moreover, by the definition of the Hamiltonian H_0 given in (11) it is easy to check that

$$J_0(u) \leq \int_0^{+\infty} e^{-\rho t} H_0(\overline{\alpha})dt = \frac{1}{\rho}H_0(\overline{\alpha}).$$

If we find an element $u^* \in H^1$ such that (18) is satisfied then the control strategy $u(t) = u^*$ for every $t \geq 0$ is optimal. Now we observe that the function $F_0(\cdot, \overline{\alpha})$ is continuous and strictly concave on H^1. Then it is also weakly upper semicontinuous (see e.g. [Brezis, 1983, ch. III]), but in general it is not coercive. However we can consider the map $G : \mathbb{R} \times L^2 \to \mathbb{R}$,

$$G(r,u) = \lambda\overline{\alpha}(0)r - \beta_0 r^2 + < u, \overline{\alpha} - q >_{L^2} - < B_\beta u(t), u(t) >_{L^2}.$$

By construction

$$\sup_{u\in H^1} F(u) \le \max_{\substack{r\in\mathbb{R}\\ u\in L^2}} G(r,u).$$

Moreover the map G is strictly concave, weakly upper semicontinuous and coercive and so it has a unique maximum point on $\mathbb{R}\times L^2$ given by

$$r = r^* = \frac{\lambda\overline{\alpha}(0)}{2\beta_0} \qquad u(s) = u^*(s) = \frac{\overline{\alpha}(s) - q(s)}{2\beta(s)},$$

the value of the maximum is

$$\max_{\substack{r\in\mathbb{R}\\ u\in L^2}} G(r,u) = H_0(\overline{\alpha}) = \frac{\lambda^2\overline{\alpha}^2(0)}{4\beta_0} + \int_0^{\overline{s}} \frac{[\overline{\alpha}(s) - q(s)]^2}{4\beta(s)} ds.$$

When the compatibility condition is satisfied, the control

$$u^*(s) = \frac{\overline{\alpha}(s) - q(s)}{2\beta(s)}$$

is such that

$$F(u^*) = G(u^*(0), u^*) = \max_{\substack{r\in\mathbb{R}\\ u\in L^2}} G(r,u) = \sup_{u\in H^1} F(u)$$

which gives the claim.

Now, to prove the converse, we argue by contradiction. Assume that the compatibility condition is not satisfied. Then by considering a sequence of controls $(u_n)_{n\in\mathbb{N}}$ such that

$$u_n(0) = \frac{\lambda\overline{\alpha}(0)}{2\beta_0}; \qquad u_n(s) = \frac{1}{2\beta(s)}\left[\overline{\alpha}(s) - q(s)\right], \quad s \in \left[\frac{1}{n}, \overline{s}\right]$$

(it is enough to connect in a smooth way the points 0 and $\frac{1}{n}$), we can see that we still have

$$\lim_{n\to+\infty} J_0(u_n) = \sup_{u\in\mathcal{U}} J_0(u) = \frac{1}{\rho} H_0(R(\rho, A^*)\alpha).$$

So, if there exists an optimal control strategy $u^*(t,s)$, then it still has to satisfy the "maximum principle" (18). But this implies that u^* is given by (17) and so, since (16) is not satisfied, it is not continuous with respect to s at $s = 0$. ■

When the compatibility condition (16) is not satisfied we still have from the proof of the previous Proposition that

$$\sup_{u\in H^1} F(u) = \max_{\substack{r\in\mathbb{R}\\ u\in L^2}} G(r,u) = H_0(\overline{\alpha}) = \frac{\lambda^2\overline{\alpha}^2(0)}{4\beta_0} + \int_0^{\overline{s}} \frac{[\overline{\alpha}(s) - q(s)]^2}{4\beta(s)} ds$$

and

$$\sup_{u\in\mathcal{U}} J_0(u) = \frac{1}{\rho} H_0(\overline{\alpha}).$$

The supremum is not a maximum as proved before since the class of control strategies is too small. If we consider a larger class of control strategies

$$\overline{\mathcal{U}} = \Big\{u : [0, +\infty) \times [0, \overline{s}] \to \mathbb{R}; u(t, \cdot) \text{ left continuous, right limit } \forall t \geq 0;$$
$$u(\cdot, s) \in H^1(0, +\infty)\ \forall s \in [0, \overline{s}]\Big\}$$

then we can still give sense to the mild solution of the state equation (3) and also

$$\sup_{u \in \overline{\mathcal{U}}} J_0(u) = \frac{1}{\rho} H_0(\overline{\alpha}).$$

This implies that the control strategy $u(t) = u^*$ defined in (17) $\forall t \geq 0$ is optimal in the class $\overline{\mathcal{U}}$.

It is also possible to consider control strategies in a more general class, allowing the strategy u to be piecewise continuous with respect to the variable t. In this case we can not give sense to the solution in the space L^2 since the boundary term

$$A \left[\int_0^t T(t - \tau) u(\tau, 0) d\tau \right] w_0$$

does not belongs in general to L^2. Any way we can give sense to this term in a suitable space of distributions. The study of the control problem does not change (except for technical difficulties) and the optimal control strategy is still the one defined above.

Thanks to Assumption 4.1 the optimal control u^* is a function of L^2 and is differentiable for $s > 0$. From the optimal control expression it follows that $u^* \geq 0$ if and only if $\overline{\alpha} - q \geq 0$, i.e. investments in capital goods of a specific vintage are positive if the associated discounted return is larger than the unit investment cost. The economic interpretation of the optimal control u^* is equivalent to the one obtained in the finite dimensional capital accumulation problem with adjustment costs, see for example [Lucas, 1967, Gould, 1968, Abel, 1990]. Gross investments in capital goods of vintage s are equal to the difference between the future discounted return, $\overline{\alpha}(s)$, and the unit investments cost, $q(s)$, divided by the adjustment cost, $\beta(s)$.

Ceteris paribus, a larger discount factor ρ or a larger decay rate μ lead to a lower level of investments, the same thing happens in absolute value if $\beta(s)$ or β_0 are increased. If $\alpha'(s) \leq 0$ then $\overline{\alpha}(s)$ is monotonic (decreasing) in s, this implies that if $\beta(s)$ and $q(s)$ are constant then $(u^*)'(s) \leq 0$.

Let us remark that

$$\lim_{\lambda \to 0^+} \overline{\alpha}(s) = \frac{\alpha(s)}{\rho + \mu} \qquad \lim_{\lambda \to +\infty} \overline{\alpha}(s) = 0 \qquad \lim_{\lambda \to 0^+} \lambda \overline{\alpha}(s) = 0 \qquad \lim_{\lambda \to +\infty} \lambda \overline{\alpha}(s) = \int_s^{\overline{s}} \alpha(\sigma) d\sigma$$

for every $s \in [0, \overline{s}]$. $\lambda \overline{\alpha}(s)$ is always monotonic (increasing) in λ, moreover $\overline{\alpha}(s)$ is monotonic in λ (decreasing) if $\alpha'(s) < 0$. Therefore, investments in new capital goods are always positive and increasing in λ while investments in vintage capital goods are decreasing in λ if $\alpha'(s) < 0$. So if capital goods are only exposed to quality depreciation then, as the rate of depreciation becomes larger, investments in new capital goods increase, while investments in vintage capital goods decrease.

Nothing can be said a-priori about $(u^*)'(s)$ which depends on the behavior of $\alpha(s)$, $q(s)$ and $\beta(s)$.

In our analysis we have considered the case of quality depreciation $\alpha'(s) \leq 0$. This assumption does not take account the so called learning effect or experience, i.e. the productivity of a new capital good increases after being installed because of externalities, input complementarities, learning by doing, etc.. $\alpha(s)$ first increasing in s and then decreasing can be easily introduced in our setting. We recall that

$$\overline{\alpha}'(s) = \frac{1}{\lambda} \int_s^{\overline{s}} e^{-\frac{\rho+\mu}{\lambda}(\sigma-s)} \alpha'(\sigma) d\sigma,$$

so $\alpha(s)$ first increasing and then decreasing does not necessarily imply $\overline{\alpha}'(s) > 0$ and $(u^*)'(s) > 0$ for s small. What is relevant for the shape of $u^*(s)$ is $\overline{\alpha}'(s)$ the weighted mean of $\alpha'(\sigma)$ with $s \leq \sigma \leq \overline{s}$.

Remark 4.6 From the above considerations it becomes clear that the optimal control $u^*(s)$ is in general discontinuous at $s = 0$, i.e. $u^*(0) \neq \lim_{s\to 0^+} u^*(s)$. This is due to the form of the functional J and is a consequence of the form of the function F that we have to maximize. In fact, the function F depends on the control u in two different ways:

- $- < A^*\overline{\alpha}, w_0 > u(0) - \beta_0 u^2(0)$ which depends only on $u(0)$
- $< -q + \overline{\alpha}, u >_{L^2} + < B_\beta u, u >_{L^2}$ which depends on the integral of u.

If the control can be discontinuous, then the maximization of the first component is independent of the maximization of the second component; in fact the integral of a function does not change if we change the value of this function at the single point $s = 0$. This allows us to write the optimal control as in (17) also when the compatibility condition (16) is not satisfied.

Remark 4.7 We observe that the presence of innovation cost, i.e. the term $-\beta_0 u^2(t, 0)$, in the function l is important because it gives the coercivity of the functional J with respect to $u(t, 0)$. Without this term it can be easily seen that the functional J is unbounded unless we consider a bounded set of control strategies.

Remark 4.8 An easy consequence of Propositions 4.3 and 4.5 is that the value function v is an affine function and is given by the

$$v(k_0) = < \overline{\alpha}, k_0 >_{L^2} + \frac{1}{\rho} H_0(R(\rho, A^*)\alpha), \tag{20}$$

in particular this fact implies that v is differentiable and it is a classical solution of the Hamilton-Jacobi equation (10). To state a uniqueness theorem we would need to introduce more specific techniques that are not in the scope of this paper. We refer the reader to [Barbu and Da Prato, 1982, Crandall and Lions, 1990, Cannarsa et al., 1993, Cannarsa and Tessitore, 1996, Iftode, 1989] for results in this direction.

5 Optimality conditions, Long run equilibrium and the Turnpike property

In this section we first study the optimality conditions for our problem and then we show the existence of a stationary long run equilibrium. We will work with control strategies belonging to the set $\overline{\mathcal{U}}$, in this class we have existence and uniqueness of the optimal control.

We start by observing that the Hamiltonian H_0 can be split into two parts due to the presence of the boundary control term. For $p \in D(A^*)$ the Hamiltonian function $H_0(p)$ can be written as

$$H_0(p) = \sup_{u \in H^1} F_0(u,p)$$

$$= \sup_{r\in\mathbb{R}} \left[-r < w_0, A^*p >_{L^2} -\beta_0 r^2\right] + \sup_{u\in L^2} \left[< u, p-q >_{L^2} - < B_\beta u, u >_{L^2}\right]$$

$$= \frac{< w_0, A^*p >_{L^2}^2}{4\beta_0} + \frac{1}{4} < B_{\frac{1}{\beta}}(p-q), p-q >_{L^2} = \frac{\lambda^2 p^2(0)}{4\beta_0} + \int_0^{\overline{s}} \frac{[p(s)-q(s)]^2}{4\beta(s)} ds.$$

Setting for $a \in \mathbb{R}$

$$H_{01}(a) = \sup_{r\in\mathbb{R}} \left[-ra - \beta_0 r^2\right] = \frac{a^2}{4\beta_0},$$

and for $p \in L^2$

$$H_{02}(p) = \sup_{u\in L^2} \left[< u, p-q >_{L^2} - < B_\beta u, u >_{L^2}\right] = \frac{1}{4} < B_{\frac{1}{\beta}}(p-q), p-q >_{L^2}$$

then taking $a =< w_0, A^*p >_{L^2}$ $(p \in D(A^*))$ we have

$$H_0(p) = H_{01}(< w_0, A^*p >_{L^2}) + H_{02}(p).$$

From (20) $Dv(k) = \overline{\alpha}$ and therefore for every $k \in L^2$ the optimal control becomes

$$u^*(0) = DH_{01}(< w_0, A^*\overline{\alpha} >_{L^2}) = DH_{01}(\lambda\overline{\alpha}(0)), \qquad u^*(s) = DH_{02}(\overline{\alpha})(s);\ s \in (0, \overline{s}].$$

The presence of the boundary term and the linearity of the problem induce to split the Hamiltonian and the control into two parts. We observe that the two functions H_{01} and H_{02} come from the boundary control and from the distributed control, respectively. They will appear explicitly in the Hamiltonian system (21).

We now pass to the statement of the Pontryagin Maximum Principle for our problem which in this case is an easy consequence of the results of the previous section (see e.g. [Fleming and Rishel, 1975] for the finite dimensional case, [Bensoussan et al., 1992, Cannarsa and Frankowska, 1992, Barron and Jensen, 1986] for the infinite dimensional case with no boundary control and [Cannarsa and Tessitore, 1994, Gozzi and Tessitore, 1994, Fattorini, 1968, Lasiecka and Triggiani, 1991] for the infinite dimensional case with boundary control).

Theorem 5.1 *Let (k^*, u^*) be an optimal pair for problem (P) starting from k_0. Then there exists a function $p^* \in L^\infty(0, +\infty; L^2)$, $p^*(t) \in D(A^*)$ for $t \in [0, +\infty[$ such that* *(i) p^* is a solution of the adjoint equation*

$$p'(t) = [\rho - A^*]p(t) - \alpha$$

with the trasversality condition

$$\lim_{t\to+\infty} e^{-\rho t}p(t) = 0$$

(ii) p^* satisfies the Maximum Principle

$$u^*(t,0) < w_0, A^*p(t) >_{L^2} - < u^*(t), p(t) >_{L^2} - l(u^*(t))$$
$$= \sup_{u\in H} \{u(t,0) < w_0, A^*p(t) >_{L^2} - < u, p >_{L^2} - l(u)\} = H_0(p(t))$$

(iii) p^* satisfies the so-called costate inclusion

$$p(t) = Dv(k^*(t))$$

(iv) for every $t \geq 0$

$$\rho v(k^*(t)) + < k^*(t), A^*p^*(t) > + H_0(p^*(t)) = -\alpha.$$

Sketch of Proof. The proof can be done by applying the method of [Tessitore, 1995]. However in this case everything is simple. In fact, since we already know that the optimal control u^* is unique and that v is affine then for every $k \in H$ we have

$$Dv(k) = R(\rho, A^*)\alpha = \overline{\alpha}$$

so we have

$$p^*(t) = R(\rho, A^*)\alpha = \overline{\alpha} \qquad \forall t \geq 0.$$

It is easy to verify that p^* satisfies all the claims. ∎

The necessity of the trasversality condition

$$\lim_{t\to+\infty} e^{-\rho t}p(t) = 0$$

is very easy to get in this case due to the linearity of the problem. Let us remark that no results are available in the literature about the trasversality condition for infinite horizon/infinite dimensional optimal control problems. This result can be considered as a first generalization to the infinite dimensional case of the big variety of results about the trasversality condition for finite dimensional control problems. Let us remark that since the state trajectory k is bounded, then the trasversality condition $\lim_{t\to+\infty} e^{-\rho t}p(t) = 0$ given in Theorem 5.1 implies the classical trasversality condition $\lim_{t\to+\infty} e^{-\rho t} < k(t), p(t) >= 0$. $\overline{\alpha}$ can now be interpreted as the costate variable since we have that

$$p^*(t) = R(\rho, A^*)\alpha = \overline{\alpha} \qquad \forall t \geq 0.$$

$p^*(t) = \overline{\alpha}$ has a straightforward economic interpretation: the discounted return $\overline{\alpha}$ is the marginal value associated by the optimal control u^* to the state along the optimal trajectory k^*.

Theorem 5.2 *Let (k^*, u^*) be an optimal pair for problem (P). Then there exists a function $p^* \in L^\infty(0, +\infty; L^2)$ such that (k^*, p^*) is a mild solution of the following system (which makes sense only in integral form)*

$$\begin{cases} k'(t) = Ak(t) - DH_{01}(< w_0, A^*p(t) >_{L^2})Aw_0 + DH_{02}(p(t)); \qquad k(0) = k_0 \\ p'(t) = [\rho - A^*]p(t) - \alpha \end{cases} \tag{21}$$

satisfying the trasversality condition

$$\lim_{t \to +\infty} e^{-\rho t} p(t) = 0. \tag{22}$$

The solution (k^*, p^*) to system (21)-(22) is unique and is given by

$$p^*(t) = \overline{\alpha}; \qquad t \geq 0$$

$$k^*(t) = T(t)k_0 - u^*(0)[T(t) - I]w_0 + R(0; A)[T(t) - I]u^*.$$

where u^* is given by (17). The system (21) has only one stationary equilibrium point (k_∞, p_∞),

$$k_\infty = R(0; A)u^* + u^*(0)w_0.$$

$$p_\infty = R(\rho; A^*)\alpha$$

k_∞ and p_∞ are positive functions in L^2.

Sketch of Proof. For the first part it is enough to verify that the costate defined in the proof of Theorem 5.1 satisfies (21)-(22). Uniqueness of the solution of (21)-(22) follows by observing that $\overline{\alpha}$ is the only solution p of the second equation of (21) that also satisfies (22).

The other claims easily follow by recalling that the semigroups T and T^* given in Propositions 3.1 and 3.2. are characterized by an exponential decay rate, see Appendix A. ∎

The only solution of (21) which satisfies the trasversality condition is characterized by a constant $p(t)$:

$$p(t) = R(\rho; A^*)\alpha = \overline{\alpha} \qquad t \geq 0,$$

this means that the line $(k, p) = (k, \overline{\alpha})$ in the phase space $L^2 \times L^2$ is the stable manifold of the stationary equilibrium point (k_∞, p_∞) of the system (21). $\forall\ k_0$, if $p_0 = \bar{\alpha}$ then the solution of the system (21) converges to (k_∞, p_∞) as $t \to +\infty$.

In analogy with the finite dimensional case we would expect that, starting from a point $p_0 \neq \overline{\alpha}$, the solution of (21) does not converge to the stationary equilibrium point (k_∞, p_∞). Instead what happens is different, if $p_0 \neq \overline{\alpha}$ then the second equation of (21) is not well posed:

(i) if $\overline{s} < +\infty$ then the second equation of (21) can not be solved in L^2. In fact the equation of the costate variable becomes

$$\begin{cases} \dfrac{\partial p(t,s)}{\partial t} + \lambda \dfrac{\partial p(t,s)}{\partial s} - (\rho+\mu)p(t,s) = -\alpha(s); & t>0,\ s \in [0,\overline{s}] \\ p(t,\overline{s}) = 0; \qquad t>0 \\ p(0,s) = p_0(s); \qquad s \in [0,\overline{s}] \end{cases} \tag{23}$$

this equation is analogous of the PDE of the state variable k, but the boundary condition $p(t,\overline{s}) = 0$ renders the Cauchy problem (23) unsolvable.

(ii) if $\overline{s} = +\infty$ then the second equation of (21) has no unique solution. In fact in this case the equation for the costate is the same as (23) but without the boundary condition $p(t,\overline{s}) = 0$ and every function of the type (in the simplified case $\alpha = 0$, the others are equivalent from this point of view)

$$p(t,s) = e^{(\rho+\mu)t} \begin{cases} p_0(s-\lambda t) & s \in [\lambda t, +\infty) \\ a & s \in [0, \lambda t) \end{cases}$$

where a is any constant, is a solution of (23). So, by an easy verification we can see that every possible solution blows up in the L^2-norm as $t \to +\infty$.

Let us consider now the finite horizon optimal control problem:

$$Maximize \quad J_{T_0}(k_0;u) = \int_0^{T_0} e^{-\rho t}\left[g(k(t)) + l(u(t))\right] dt \quad T_0 > 0 \tag{24}$$

subject to

$$\begin{cases} \dfrac{\partial k(t,s)}{\partial t} + \lambda \dfrac{\partial k(t,s)}{\partial s} + \mu k(t,s) = u(t,s) & t \in (0,T_0],\ s \in [0,\overline{s}] \\ k(t,0) = u(t,0) \qquad t \in (0,T_0] \\ k(0,s) = k_0(s) \qquad s \in [0,\overline{s}]. \end{cases} \tag{25}$$

The following Proposition can be stated.

Proposition 5.3 *Let $(k^*_{T_0}, p^*_{T_0})$ the state-costate couple representing the optimal solution for the problem (24)-(25) then, as $T_0 \to +\infty$ we have*

$$p^*_{T_0}(t) \to p_\infty;$$

uniformly on t belonging to bounded subsets of $[0,+\infty)$. Moreover

$$k^*_{T_0}(t) \to k_\infty;$$

uniformly on t belonging to compact subsets of $(0,+\infty)$.

Proof. It is enough to remark that in this case the costate is

$$p^*_{T_0}(t) = \int_t^{T_0} e^{-(T_0-\tau)\rho} T^*(t-\tau)\alpha d\tau$$

and that we have $p^*_{T_0}(t) \to p_\infty$ uniformly on bounded subsets of $[0,+\infty)$ for $T_0 \to +\infty$. The corresponding state trajectory

$$k^*_{T_0}(t) = T(t)k_0 - A\int_0^t T(t-\tau)DH(p^*_{T_0}(\tau))(0)w_0 d\tau + \int_0^t T(t-\tau)DH(p^*_{T_0}(\tau))d\tau$$

goes to k_∞ as $T_0 \to +\infty$ uniformly in t belonging on compact subsets of $(0,+\infty)$. ∎

It can be easily shown that the above Corollary implies the so-called *Turnpike property* for our optimal control problem, see [Carlson et al., 1991]. Given any $\varepsilon > 0$ there exists $\overline{T}_0 > 0$ such that for $T_0 \geq \overline{T}_0$ an $\eta > 0$ exists such that

$$\sup_{t\in[\eta,\frac{1}{\eta}]} \left[\|k^*_{T_0}(t) - k_\infty(t)\|_{L^2} + \|p^*_{T_0}(t) - p_\infty(t)\|_{L^2} \right] < \varepsilon.$$

The optimal solution for the finite horizon optimal control problem belongs to a neighborhood of (k_∞, p_∞) for a given period of time, the so-called turnpike.

6 The Long Run Equilibrium

The long run equilibrium is

$$p_\infty(s) = \overline{\alpha}(s) = R(\rho; A^*)\alpha(s) = \frac{1}{\lambda}\int_s^{\overline{s}} e^{-\frac{\rho+\mu}{\lambda}(\sigma-s)}\alpha(\sigma)d\sigma$$

$$k_\infty(s) = R(0;A)u^*(s) + u^*(0)w_0(s) = \frac{1}{\lambda}\int_0^s e^{-\frac{\mu}{\lambda}(s-\sigma)}u^*(\sigma)d\sigma + u^*(0)e^{-\frac{\mu}{\lambda}s}$$

where u^* is given by (17). We assume that the technology satisfies the following Assumption.

Assumption 6.1

$$\overline{\alpha}(0) - q(0) \geq 0.$$

Assumption 6.1 simply says that a new technology is profitable: the discounted return associated to a new capital good, $\overline{\alpha}(0)$, is larger than its unit investment cost, $q(0)$.

Let us remark that we have studied the problem without a positivity constraint on the state variable; in the following we restrict our attention to the analysis of a stationary equilibrium (k_∞, p_∞) characterized by a nonnegative state solution, $k_\infty(s) \geq 0$.

The optimal path associated with the capital accumulation problem gives two interesting pieces of information: the optimal stock of capital (k_∞) and the optimal path for investments. In the finite dimensional setting the sensitivity of the optimal stock of capital with respect to the parameters has been extensively analyzed, see [Treadway, 1971, Mortensen, 1973, Brock, 1986, Brock and Malliaris, 1989]. In particular, a well established result states that the optimal stock of capital is a decreasing function of the discount factor and of the interest rate. The result is confirmed in the infinite dimensional setting. For every $s \in (0, \overline{s})$ we have that $k_\infty(s)$ is decreasing in β_0, ρ, μ, $\beta(\sigma)$ and $q(\sigma)$ with $\sigma \in [0, s]$, and increasing in $\alpha(\sigma)$ with $\sigma \in [0, \overline{s}]$. In general, without further assumptions, it is difficult to analyze the behavior of k_∞ with respect to λ, see Section 8 for a numerical analysis.

Let us remark that $k_\infty(0) = u^*(0)$, therefore we have that the optimal stock of capital for new goods is increasing in λ and decreasing in μ, ρ and β_0. This reasoning can be replicated for s in a neighborhood of 0^+. A high rate of quantity depreciation and a high discount rate lead to a low stock of new capital goods, a high rate of quality depreciation leads to a high stock of new capital goods. If capital goods quickly go out of date then there is an incentive to invest in *young* capital goods. The length of the neighborhood of 0^+ for which this sensitivity analysis of $k_\infty(s)$ holds depends on other parameters of the model, see Section 8.

From easy calculations we obtain the following Proposition.

Proposition 6.2 *Let Assumption 4.1 and 6.1 be satisfied. Then the optimal control u^* is continuously differentiable on $(0, \overline{s})$ (possibly discontinuous at $s = 0$). The function k_∞ belongs to H^1 and its derivative k'_∞ is continuously differentiable out of $s = 0$. Moreover the function k_∞ is the unique solution of the equation*

$$\lambda k'(s) + \mu k(s) = u^*(s); \qquad k(0) = u^*(0) = \frac{\lambda \overline{\alpha}(0)}{2\beta_0}. \tag{26}$$

It follows that

$$k'_\infty(s) = -\frac{\mu}{\lambda} e^{-\frac{\mu}{\lambda}s} u^*(0) + \frac{1}{\lambda} u^*(s) - \frac{\mu}{\lambda} R(0; A) u^*(s)$$

and

$$k'_\infty(0^+) \stackrel{def}{=} \lim_{s \to 0^+} k'_\infty(s) = -\frac{\mu \overline{\alpha}(0)}{2\beta_0} + \frac{\overline{\alpha}(0) - q(0)}{2\lambda\beta(0)} = \frac{\overline{\alpha}(0)}{2} \left[\frac{1}{\lambda\beta(0)} - \frac{\mu}{\beta_0} \right] - \frac{q(0)}{2\lambda\beta(0)}. \tag{27}$$

Let $\overline{s} < +\infty$, since $\overline{\alpha}(\overline{s}) = 0$ we observe that

$$u^*(\overline{s}) = -\frac{q(\overline{s})}{2\beta(\overline{s})}$$

so that

$$u^*(\overline{s}) \leq 0 \quad \text{and} \quad u^*(\overline{s}) = 0 \Longleftrightarrow q(\overline{s}) = 0.$$

This means that it is not possible to have $(u^*)'(s) \geq 0 \forall s \in (0, \overline{s})$.

Let us remark that if $\alpha'(s) < 0$ and $q(s)$, $\beta(s)$ are constant non zero then $(u^*)'(s) < 0$ and an $s^\# \in [0, \overline{s}]$ exists such that $u(s^\#) = 0$, $u(s) > 0$ for $0 \leq s < s^\#$ and $u(s) < 0$ for $s^\# < s \leq \overline{s}$.

From (26) and the fact that $k_\infty \geq 0$, it follows that

$$k'_\infty(\bar{s}) = -\frac{\mu}{\lambda} k_\infty(\bar{s}) + \frac{1}{\lambda} u^*(\bar{s}) \leq 0. \tag{28}$$

If $\bar{s} = +\infty$, as $u^* \in L^2$ and $k_\infty \in H^1$ then we have $\lim_{s\to\infty} u^*(s) = \lim_{s\to\infty} k_\infty(s) = \lim_{s\to\infty} k'_\infty = 0$, the sign of $u^*(s)$ for s large depends on the rate of convergence to zero of $\alpha(s)$ and $q(s)$ and everything is possible, see Section 8 for the analysis of the exponential case.

The shape of $k_\infty(s)$ can have different characterizations. If $(u^*)'(s) \leq 0$ then the following Proposition can be stated.

Proposition 6.3 *If $(u^*)'(s) \leq 0$ for every $s \in (0, \bar{s})$ then we have*

- $k'_\infty(0^+) \leq 0 \Rightarrow k'_\infty(s) \leq 0 \quad \forall s \in [0, \bar{s}]$;
- $k'_\infty(0^+) > 0 \Rightarrow$ *$k_\infty(s)$ is single-peaked, there exists a point $s_0 \in (0, \bar{s})$ such that $k'_\infty(s_0) = 0$, $k'_\infty(s) > 0$ for $s < s_0$ and $k'_\infty(s) \leq 0$ for $s > s_0$.*

Proof.
Setting $z(s) \stackrel{def}{=} k'(s),\ s \in (0, \bar{s}]$, the following equation can be obtained from (26)

$$\lambda z'(s) + \mu z(s) = (u^*)'(s); \qquad z(0) = k'(0^+).$$

Let first $k'(0^+) \leq 0$. Let s_0 be a maximum point of z. If $s_0 \in (0, \bar{s})$ then we have

$$\mu z(s_0) = (u^*)'(s_0) \leq 0$$

If $s_0 = 0$ then $z(s_0) = k'_\infty(0^+) \leq 0$ and if $s_0 = \bar{s} < +\infty$ then $z'(s_0) \geq 0$ which implies

$$\mu z(s_0) \leq (u^*)'(s_0) \leq 0$$

(if $\bar{s} = +\infty$ we use that $\lim_{s\to+\infty} k'_\infty(s) = 0$ and then the same argument). It follows that $k'(s) = z(s) \leq 0$ for every $s \in (0, \bar{s})$. This gives the first part of the claim. For the second part we recall that $z(0) > 0$ and $z(\bar{s}) \leq 0$ by equation (28) (use thatlim$_{s\to+\infty} k'_\infty(s) = 0$ when $\bar{s} = +\infty$). This implies that there exists a first point $s_0 \in (0, \bar{s}]$ such that $z(s_0) = 0$. This point cannot be $\bar{s}$ since in this case we would have $k'_\infty \geq 0$ on $(0, \bar{s}]$ and, from (26) we would also have $k_\infty(\bar{s}) = u(\bar{s}) = 0$, impossible. We finally prove that $z(s) \leq 0$ for $s > s_0$. By contradiction there exists a maximum point $s_1 \in (0, \bar{s})$ such that $z(s_1) > 0$. But this is impossible by reasoning as in the first part of the proof. ∎

So if the technology of the model generates a flow of investments decreasing in s, $(u^*)'(s) \leq 0$, then the optimal stock of capital $k_\infty(s)$ can only be monotonically decreasing or *single-peaked* with a maximum in s_0, increasing for $s < s_0$ and decreasing for $s > s_0$. In this case we have the diffusion of the new technology.

From the analysis of (27) it turns out that if $\overline{\alpha}(0) - q(0) >> 0$ and β_0 is sufficiently high then $k'_\infty(0^+) > 0$ and $k_\infty(s)$ is *single peaked* with a maximum in s_0. So a new technology highly profitable and a high innovation cost give rise to the diffusion of the new technology; if a new technology is highly profitable and the cost to buy capital goods specific to that technology is very high (innovation cost), then it is better to wait, not to pay the innovation cost and to install the technology after a while.

Let us remark that $k'_\infty(0^+)$ is monotonic (decreasing) in λ, ρ, and μ if $k'_\infty(0^+) > 0$, moreover we have that $\lambda, \mu, \rho \nearrow +\infty$ implies $k'_\infty(0^+) < 0$, and $\lambda, \mu, \rho \searrow 0^+$ implies $k'_\infty(0^+) > 0$. To have a diffusion of a new technology, the discount rate and the rate of quality and quantity depreciation should be low. If λ and μ are high then there is no space for a diffusion of the new technology, the optimal stock of capital is strictly decreasing in s; as the capital goods depreciate quickly, it is not worthwhile to wait and to install a new capital good after a while without paying the innovation cost. The same thing happens for the discount factor, if the entrepreneur heavily discounts future returns then there is no reason to waste time.

There are two competing factors characterizing the shape of the optimal stock of capital $k_\infty(s)$. The profitability of the new capital good/technology, $\overline{\alpha}(0) - q(0)$, and its rate of depreciation λ, μ.

The existence of a diffusion of a new technology can be explained thinking to the fact that $k_\infty(s)$, due to the linearity of the problem, is made up of two components:

$$k_\infty(s) = k_1(s) + k_2(s)$$

where

- $$k_1(s) = u_0^*(0)e^{-\frac{\mu}{\lambda}s} = \frac{\lambda\overline{\alpha}(0)}{2\beta_0}e^{-\frac{\mu}{\lambda}s}$$

  comes from the boundary control (investments in new capital goods);

- $$k_2(s) = R(0;A)u^*(s) = \frac{1}{\lambda}\int_0^s e^{-\frac{\mu}{\lambda}(s-\sigma)}\frac{\overline{\alpha}(\sigma) - q(\sigma)}{2\beta(\sigma)}d\sigma$$

  comes from the distributed control (investments in vintage capital goods).

This gives an explanation to the presence of the diffusion of a new technology. The effect of the boundary control $u^*(0)$, innovation/investments in new capital goods, on the optimal stock of capital $k_\infty(s)$, $k_1(s)$, is strictly decreasing in s because of the exponential quantity depreciation rate μ. The effect of investments in vintage capital goods, $k_2(s)$, is increasing in s at 0^+, thanks to Assumption 6.1. The presence of the diffusion of a new technology, a *single peaked* $k_\infty(s)$ with a maximum in s_0, depends to the fact that $k_2(s)$, increasing in s, compensates $k_1(s)$. The extent of the diffusion of a new technology $[0, s_0]$ depends on the level of investments, $\frac{\overline{\alpha}(s)-q(s)}{2\beta(s)}$, and therefore on the productivity of the capital goods.

The presence of a diffusion of a new technology has been demonstrated, now let us analyze the extension of this phenomenon, i.e. the value of s_0. Let us consider in particular the sensitivity of the maximum s_0 of $k_\infty(s)$ with respect to the parameters of the technology. For example, let us consider s_0 as a function of β_0, $s_0(\beta_0)$, which is defined implicitly by the equation

$$k'_\infty(\beta_0, s_0) = 0,$$

therefore (by using the implicit function theorem)

$$\frac{\partial s_0}{\partial \beta_0} = -\frac{\frac{\partial k'_\infty(\beta_0, s_0(\beta_0))}{\partial \beta_0}}{\frac{\partial k'_\infty(\beta_0, s_0(\beta_0))}{\partial s}}.$$

Since s_0 is a maximum point of $k_\infty(s)$, then

$$\frac{\partial k'_\infty(\beta_0, s_0(\beta_0))}{\partial s} \leq 0$$

so, the sign of $\frac{\partial s_0}{\partial \beta_0}$ is determined by the sign of

$$\frac{\partial k'_\infty(\beta_0, s_0(\beta_0))}{\partial \beta_0}.$$

We remember that, for $s(0, \bar{s}]$

$$k'_\infty(s) = \frac{1}{\lambda}[u(s) - \mu k_\infty(s)].$$

Hence, for every given $s \in (0, \bar{s})$ we have that $\beta_0 \nearrow \Longrightarrow k'_\infty(s) \nearrow$, therefore we have that $\frac{\partial k'_\infty(\beta_0, s_0(\beta_0))}{\partial \beta_0} > 0$, so a larger innovation cost leads to a larger diffusion of the new technology. If the innovation cost connected to the installation of a new capital good is high then there is a wide diffusion effect: it is better not to pay the innovation cost, to wait and to install vintage capital goods later.

We can follow the same method for all the parameters. Unfortunately a complete sensitivity analysis is not available without further assumptions, we will give more details in Section 8. However, fixed the other parameters of the problem then there exist ρ^*, λ^*, μ^* such that $k'_\infty(0^+) > 0$ for $\rho < \rho^*$, $\lambda < \lambda^*$ or $\mu < \mu^*$ and $k'_\infty(0^+) < 0$ for $\rho > \rho^*$, $\lambda > \lambda^*$ or $\mu > \mu^*$. This means that s_0 is decreasing in ρ, λ and μ at $(\rho^*)^-$, $(\lambda^*)^-$ and $(\mu^*)^-$. In this neighborhood a lower depreciation rate, both quantitative and qualitative, leads to a larger diffusion of a new technology. Without further assumptions we do not know a priori how broad is the neighborhood, a more precise analysis is provided in Section 8. Let us remark that these results are confirmed when s_0 is small by the sensitivity analysis developed above about $k'(0^+)$. In that case we have also that the diffusion effect is incraesing with respect to the profitability of the new technology.

If the technology of the firm is such that $u^*(s)$ is increasing and then decreasing then the analysis described above changes a little bit. The presence of a diffusion of a new technology is reinforced, we may also have that $k_\infty(s)$ has a minimum and then a maximum. Let us remark that an optimal investment policy of this type is not guaranteed by a sort of learning effect, i.e. $\alpha'(s) > 0$ for s small, see Section 4.

7 Investment irreversibility

Consider now the control problem (P) with irreversibility of investments, i.e. $u(t, s) \geq 0$, for every $t, s \in [0, +\infty) \times [0, \bar{s}]$. The study of the problem can be done exactly in the same way as in the unconstrained case. We still have existence and uniqueness of an optimal control $\overline{u}$ in the class $\overline{\mathcal{U}}$. The optimal control is still independent of the initial state k_0 and of t. The new optimal control which is the following

$$\overline{u}(s) = \begin{cases} u^*(s) & \text{if } u^*(s) \geq 0 \\ 0 & \text{if } u^*(s) < 0. \end{cases}$$

where u^* is given by (17). The properties of the long run equilibrium can be studied as in Section 6. From (26) we observe that if $\overline{u}(s) = 0$ with $s \in (s_1, s_2)$ then in this interval we have

$$k_\infty(s) = e^{-\frac{\mu}{\lambda}(s-s_1)} k_\infty(s_1)$$

which says that the long run equilibrium is exponentially decreasing in s when no investment is done.

The assumption of investment irreversibility reinforces the decreasing shape of $k_\infty(s)$ for mature technologies.

8 Quality depreciation: the exponential case

Let us consider a technology characterized by quality depreciation. In particular let us consider the case of exponential quality depreciation: $\alpha(s) = A_0 e^{-\alpha s}$, $\beta(s) = B_0 e^{-\beta s}$, $q(s) = Q_0 e^{-qs}$ $\forall s \geq 0$, where $A_0, B_0, Q_0, \alpha, \beta, q$ are positive constants, and $\overline{s} = +\infty$.

The solution of the optimal investment problem is

$$\overline{\alpha}(s) = \frac{A_0 e^{-\alpha s}}{\rho + \mu + \lambda\alpha}$$

$$u^*(0) = \frac{A_0}{2\beta_0(\frac{\rho+\mu}{\lambda} + \alpha)}$$

$$u^*(s) = \frac{1}{2B_0}\left[\frac{A_0 e^{(\beta-\alpha)s}}{\rho+\mu+\lambda\alpha} - Q_0 e^{(\beta-q)s}\right]; \qquad s > 0$$

$$k_\infty(s) = e^{-\frac{\mu}{\lambda}s}\left[\frac{A_0}{2\beta_0(\frac{\rho+\mu}{\lambda}+\alpha)}\right.$$

$$\left. + \frac{A_0}{2\lambda B_0(\rho+\mu+\lambda\alpha)} \cdot \frac{e^{(\beta-\alpha+\frac{\mu}{\lambda})s} - 1}{\beta - \alpha + \frac{\mu}{\lambda}} - \frac{Q_0}{2\lambda B_0} \cdot \frac{e^{(\beta-q+\frac{\mu}{\lambda})s} - 1}{\beta - q + \frac{\mu}{\lambda}}\right]; \qquad s \geq 0.$$

Moreover

$$(u^*)'(s) = \frac{1}{2B_0}\left[\frac{(\beta-\alpha)A_0 e^{(\beta-\alpha)s}}{\rho+\mu+\lambda\alpha} - Q_0(\beta-q)e^{(\beta-q)s}\right]; \qquad s > 0$$

and

$$k'_\infty(0^+) = -\frac{\mu A_0}{2\beta_0(\rho+\mu+\lambda\alpha)} + \frac{1}{2\lambda B_0}\left[\frac{A_0}{\rho+\mu+\lambda\alpha} - Q_0\right].$$

Assumption 6.1 implies

$$\frac{A_0}{\rho+\mu+\lambda\alpha} > Q_0.$$

We restrict our analysis by imposing $k_\infty(s) \geq 0$; moreover as a consequence of Assumption 4.1 we require that $\lim_{s\to+\infty} u^*(s) = \lim_{s\to+\infty} k_\infty(s) = \lim_{s\to+\infty} k'_\infty(s) = 0$. These conditions imply that $\beta < \min\{\alpha, q\}$. Then we have the following cases: $\beta < \alpha < q$ and $\beta < q < \alpha$; installation costs decrease at the lowest rate.

If $\beta < \alpha < q$ then $u^*(s) \geq 0$ $\forall s \geq 0$. About $(u^*)'(s)$ we get that

- if
$$\frac{A_0}{Q_0(\rho+\mu+\lambda\alpha)} \geq \frac{q-\beta}{\alpha-\beta} > 1$$
then $(u^*)'(s) \leq 0$ $\forall s \geq 0$,

- if
$$\frac{q-\beta}{\alpha-\beta} > \frac{A_0}{Q_0(\rho+\mu+\lambda\alpha)} > 1$$
then $u^*(s)$ is *single-peaked*, $(u^*)'(s) \geq 0$ for $s \in [0, s^o]$ and $(u^*)'(s) \leq 0$ for $s \geq s^o$, where
$$s^o = \frac{1}{q-\alpha}\log\left(\frac{q-\beta}{\alpha-\beta}\cdot\frac{Q_0(\rho+\mu+\lambda\alpha)}{A_0}\right).$$

If $\beta < \alpha < q$ then depending on the sign of $k'_\infty(0^+)$ and on the parameters of the model, $k_\infty(s)$ may have three different shapes:

1. If $k'_\infty(0^+) > 0$, then $k_\infty(s)$ is *single-peaked*, attains its maximum in $s_0 > 0$, $k_\infty(s)$ is increasing for $s \in [0, s_0)$, and decreasing for $s > s_0$.

2. If
$$\frac{A_0}{Q_0(\rho+\mu+\lambda\alpha)} \geq \frac{q-\beta}{\alpha-\beta} > 1 \qquad k'_\infty(0)^+ \leq 0,$$
then $k'_\infty(s) \leq 0$ $\forall s \geq 0$, $k_\infty(s)$ is decreasing $\forall s \geq 0$.

3. If
$$\frac{q-\beta}{\alpha-\beta} > \frac{A_0}{Q_0(\rho+\mu+\lambda\alpha)} > 1 \qquad k'_\infty(0^+) \leq 0,$$
then one the following is verified:

   - $k_\infty(s)$ is decreasing $\forall s \geq 0$.
   - $k_\infty(s)$ is first decreasing with a minimum point in s_1, $s_1 \in [0, s^o)$, then increasing with a maximum point in s_0, $s_0 \geq s^o$, and at the end decreasing.

If $\beta < q < \alpha$ then $u^*(s) \geq 0$ for $s \in [0, s_2]$ and $u^*(s) \leq 0$ for $s \geq s_2$, where

$$s_2 = \frac{1}{\alpha - q} \log \frac{A_0}{Q_0(\rho + \mu + \lambda\alpha)}.$$

About $(u^*)'(s)$ we have $(u^*)'(s) \leq 0$ for $s \in [0, s_3]$ and $(u^*)'(s) \geq 0$ for $s \geq s_3$, where

$$s_3 = \frac{1}{\alpha - q} \log \left(\frac{A_0}{Q_0(\rho + \mu + \lambda\alpha)} \frac{\alpha - \beta}{q - \beta} \right) > s_2.$$

If $\beta < q < \alpha$ then depending on the sign of $k'_\infty(0^+)$ and on the parameters of the model, $k_\infty(s)$ may have two different shapes:

1. If $k'_\infty(0^+) > 0$, then $k_\infty(s)$ is *single-peaked* with a maximum in $s_0 < s_2$.
2. If $k'_\infty(0^+) \leq 0$, then $k'_\infty(s) \leq 0$ $\forall s \geq 0$, $k_\infty(s)$ is decreasing for $s \geq 0$.

The general discussion developed in the above sections about $k_\infty(0), k_\infty(s), u^*(s)$, $k'_\infty(0^+), [0, s_0]$, applies also here. Some further considerations are now possible exploiting the simplified expression of the solution of the problem. In particular we can analyze the sensitivity of the solution with respect to the parameters α, β, q.

Whatever the rates of depreciation $\beta, \alpha,\ q$ are, if a new technology is highly profitable with a high innovation cost, then $k'(0^+) > 0$ and $k_\infty(s)$ shows a diffusion effect, increasing with a maximum and then decreasing.

Let us remark that investments in new capital goods are a decreasing function of the productivity depreciation rate α and an increasing function of A_0, the technology status for new capital goods. If the new technology is highly profitable, and its productivity slowly decreases, then it is profitable to invest in new capital goods. Investments in vintage capital goods, $u^*(s)$, are a decreasing function of $Q_0, \rho, \mu, \lambda, \alpha$ and an increasing function of A_0, β, q. Let us remark that the sensitivity analysis of investments with respect to the parameters Q_0, A_0 describing parallel shifts is dual to the one obtained for the rates of depreciation α, q.

The slope of $k'_\infty(0^+)$ is not affected by β, q, but only by α. It is easy to show that if $k'_\infty(0^+) > 0$ then $k'_\infty(0^+)$ is decreasing in α and increasing in A_0, a high rate of technology depreciation leads to a less pronounced diffusion of a new technology. In the exponential framework we can explicitly identify the switching values for $\rho, \lambda, \mu, \alpha$, i.e. $\rho^*,\ \lambda^*,\ \mu^*,\ \alpha^*$. Given the other parameters, the numerical analysis shows that the neighborhoods of $(\rho^*)^-,\ (\mu^*)^-,\ (\alpha^*)^-$ for which s_0 is decreasing in $\rho,\ \mu$ and in α are large; the neighborhood of $(\lambda^*)^-$ for which s_0 is decreasing in λ is quite small.

From the discussion developed in the above sections it follows that $k_\infty(s)$ is a decreasing function of α and increasing of β, q. The behavior of $k_\infty(s)$ with respect to λ is controversial. As λ is increased, for *young* capital goods we have two competing factors, $u^*(0)$ goes up and $u^*(s)$ goes down, this entails that if there is a diffusion of the new technology then $k'_\infty(0^+)$ decreases but the global effect on $k_\infty(s)$ is not clear if we do not fix the values of the other parameters.

Thanks to Assumption 4.1, gross investments are positive for every vintage if the rate of depreciation of the capital goods' productivity is lower than the rate depreciation of installation costs. In this case, if the rate of depreciation of the unit investment cost is not too high, i.e.

$$q \leq \beta + \frac{A_0(\alpha - \beta)}{Q_0(\rho + \mu + \lambda\alpha)}, \tag{29}$$

then gross investments are strictly decreasing in s. If the rate of depreciation of the unit investment cost for capital goods is high enough, i.e. condition (29) is not satisfied, then gross investments as a function of the vintage s are *single-peaked* with a maximum in s^o. s^o is increasing in $Q_0, \rho, \mu, \lambda, \beta, q$ and decreasing in A_0; the sensitivity analysis with respect to α is not univocal. The sensitivity analysis of s^o is almost dual to the sensitivity analysis obtained for $u^*(s)$, the reason for this fact is that the parameters change has more effect on *young* capital goods rather than on *old* capital goods.

If the rate of depreciation of the productivity is higher than the rate of depreciation of installation costs then we have a switching point for investments in s_2: $u^*(s)$ is positive for $s < s_2$ and negative for $s > s_2$. As we expected, it is easy to check that s_2 is decreasing in $\rho, \mu, \lambda, Q_0, \alpha, \alpha - q$, and increasing in A_0.

In this setting we can also have a $k_\infty(s)$ with a minimum point and then a maximum if the rate of depreciation of installation costs is higher than the rate of depreciation of the productivity of the technology. This happens if a new technology is not highly profitable $k'_\infty(0^+) \leq 0$ and q is high enough, i.e. condition (29) is not satisfied. Anyhow, if a technology is highly profitable and the innovation cost is high then $k'_\infty(0^+) > 0$ and there is the diffusion of a new technology.

9 Exogenous Innovation

In the model analyzed above we assumed α, β, β_0 and q independent of the time variable t. Setting $\bar{s} = +\infty$, we now consider a time dependent technology. In particular we consider a technology with a constant growth rate with respect to time t. The assumption of a constant exogenous technology improvement is of course implausible, it is more reasonable to assume that knowledge improvement is the outcome of a *R&D* activity which is resources demanding. On models with exogenous technical change see [Solow, 1959, Judd, 1985, Lucas, 1991, Chari and Hopehhayn, 1991].

The model analyzed in this section is similar to the one proposed by Solow in [Solow, 1959] about investments, vintage, and technological progress. In [Solow, 1959] a non optimizing model of technological progress with vintage capital goods but without quality depreciation is analyzed: vintage capital goods are differentiated because of exogenous technological progress, not because they have been employed in the firm. In what follows we consider a model of optimal capital accumulation with a technology characterized both by exogenous innovation and quality depreciation. Technological progress directly affects new capital goods and vintage capital goods in a limited measure.

Let us assume that

$$\alpha(t,s) = e^{\overline{\alpha}t}\alpha_1(s), \qquad \beta(t,s) = e^{\overline{\beta}t}\beta_1(s), \qquad \beta_0(t) = e^{\overline{\beta}_0 t}\beta_0, \qquad q(t,s) = e^{\overline{q}t}q_1(s),$$

where α_1, β_1, q_1 are given functions in H^2 and $\alpha, \beta, q, \beta_0, \overline{\beta}_0$ are given constants. Not only the technology productivity increases, also investment costs grow at a constant rate. The new objective function is

$$J(k_0; u) = \int_0^{+\infty} e^{-\rho t} \Bigg[-e^{\overline{\beta}_0 t} \beta_0 u^2(t, 0) + \int_0^{+\infty} \left[e^{\alpha t} \alpha_1(s) k(t, s) - e^{qt} q_1(s) u(t, s) - e^{\beta t} \beta_1(s) u^2(t, s) \right] ds \Bigg] dt \tag{30}$$

with $\alpha, \beta, q, \overline{\beta}_0 < \rho$ so that the integrability with respect to t is guaranteed. By reasoning as in Section 4 we define the function $\overline{\alpha}_1(s)$ as follows

$$\overline{\alpha}_1(s) = R(\rho - \alpha; A^*)\alpha_1 = \frac{1}{\lambda} \int_s^{+\infty} e^{-\frac{\rho-\alpha+\mu}{\lambda}(\sigma - s)} \alpha_1(\sigma) d\sigma.$$

The interpretation of $\overline{\alpha}_1(s)$ is similar to the one of $\overline{\alpha}(s)$, the only difference is given by the exponential term $e^{\alpha t}$ which is due to the fact that the technology is exponentially time dependent. As in Section 4 we have

$$J(k_0; u) = < \overline{\alpha}_1, k_0 >_{L^2} + \int_0^{+\infty} e^{-\rho t} F(t, u(t)) dt,$$

where

$$\begin{aligned} F(t, u) = & -u(0) e^{\alpha t} < \alpha_1, AR(\rho - \alpha; A) w_0 >_{L^2} - u^2(0) e^{\overline{\beta}_0 t} \beta_0 \\ & + e^{\alpha t} < \overline{\alpha}_1, u >_{L^2} - e^{qt} < q_1, u >_{L^2} - e^{\beta t} < B_{\beta_1} u, u >_{L^2} . \end{aligned}$$

By reasoning as in Section 4 we obtain that there exists a unique optimal strategy in the class $\overline{\mathcal{U}}$. For every $t \geq 0$ the optimal strategy has to satisfy the following

$$u^*(t, 0) = \frac{1}{2\beta_0} e^{(\alpha - \overline{\beta}_0)t} < \alpha_1, -AR(\rho - \alpha; A) w_0 >_{L^2} = \frac{1}{2\beta_0} e^{(\alpha - \overline{\beta}_0)t} \int_0^{+\infty} \alpha_1(s) e^{-\frac{\rho - \alpha + \mu}{\lambda} s} ds$$

$$u^*(t, s) = \frac{1}{2\beta_1(s)} \left[e^{(\alpha - \beta)t} \overline{\alpha}_1(s) - e^{(q - \beta)t} q_1(s) \right]; \qquad s > 0.$$

The Hamilton-Jacobi equation for this problem is a time dependent one. For $p \in D(A^*)$, we set

$$H_1(t, p) = \sup_{u \in L^2} F_1(t, u, p)$$

where

$$\begin{aligned} F_1(t, u, p) = & -u(0) e^{\alpha t} < w_0, A^* p >_{L^2} + e^{\alpha t} < u, p >_{L^2} - u^2(0) e^{\overline{\beta}_0 t} \beta_0 \\ & - e^{qt} < q, u >_{L^2} - e^{\beta t} < B_{\beta_1} u, u >_{L^2}, \end{aligned}$$

the Hamilton-Jacobi equation becomes

$$v_t(t, k) = \rho v(t, k) - H_1(t, D_k v(t, k))$$

(our value function is given by $v(0,k)$). By reasoning as in Section 5, for $a \in \mathbb{R}$ and $p \in L^2$, we can define

$$H_{11}(t,a) = \sup_{r\in\mathbb{R}} \left[-re^{\alpha t}a - r^2 e^{\overline{\beta}_0 t}\beta_0\right]$$

$$H_{12}(t,p) = \sup_{u\in L^2} \left[e^{\alpha t} < u,p >_{L^2} - e^{qt} < u,q >_{L^2} - e^{\beta t} < B_{\beta_1}u,u >_{L^2},\right]$$

so that, for $p \in D(A^*)$ and $a =< w_0, A^*p >_{L^2}$ we have

$$H_1(t,p) = H_{11}(t, < w_0, A^*p >_{L^2}) + H_{12}(t,p)$$

and

$$u^*(t,0) = D_p H_{11}(t, < w_0, A^*\overline{\alpha}_1 >_{L^2})$$
$$u^*(t,s) = D_p H_{12}(t,p) \qquad s \in (0,+\infty).$$

The Hamiltonian system becomes

$$\begin{cases} k'(t) = Ak(t) - DH_{11}(t, < w_0, A^*p(t) >_{L^2})Aw_0 + DH_{12}(t,p(t)); \qquad k(0) = k_0 \\ p'(t) = [\rho - \alpha - A^*]p(t) - \alpha_1 \end{cases} \tag{31}$$

(which make sense only in integral form, see (21)) with the trasversality condition

$$\lim_{t\to+\infty} e^{-(\rho-\alpha)t}p(t) = 0$$

which implies the well-known condition

$$\lim_{t\to+\infty} e^{-\rho t} < k(t), p(t) >= 0.$$

To write down explicitly $k(t)$ we assume that $\alpha - \overline{\beta}_0,\ \alpha - \beta,\ q - \beta > -\mu$. Since the resolvent set of the operator A surely contains the half plane $(\gamma > -\mu)$; then for $\gamma < -\mu$ it can be shown that the optimal state can be written down in explicit form as follows

$$k(t) = T(t)k_0 + AR(\alpha - \overline{\beta}_0; A)[T(t) - e^{(\alpha-\overline{\beta}_0)t}]C_1 w_0$$
$$+R(\alpha - \beta; A)[e^{(\alpha-\beta)t} - T(t)]u_1^* - R(q - \beta; A)[e^{(q-\beta)t} - T(t)]u_2^*$$

where

$$C_1 \stackrel{def}{=} \frac{1}{2\beta_0} < \alpha_1, AR(\rho - \alpha; A)w_0 >_{L^2} = \frac{1}{2\beta_0}\int_0^{+\infty} \alpha_1(s)e^{-\frac{\rho-\alpha+\mu}{\lambda}s}ds,$$

$$u_1^*(s) \stackrel{def}{=} \frac{1}{2\beta_1(s)}R(\rho - \alpha; A^*)\alpha_1(s); \qquad s > 0$$

and

$$u_2^*(s) \stackrel{def}{=} \frac{1}{2\beta_1(s)}q_1(s); \qquad s > 0.$$

By regrouping the terms we have

$$k(t) = T(t)k_0 + \left[e^{(\alpha-\overline{\beta}_0)t} - T(t)\right]k_1 + \left[e^{(\alpha-\beta)t} - T(t)\right]k_2 - \left[e^{(q-\beta)t} - T(t)\right]k_3,$$

where $k_1 = -AR(\alpha - \overline{\beta}_0; A)C_1 w_0$, $k_2 = R(\alpha - \beta; A)u_1^*$, $k_3 = R(q - \beta; A)u_2^*$.

The optimal state can be divided in four components. The first one, $T(t)k_0$, comes from the initial data which, as expected (being linear the model, the initial conditions are expected to be meaningless in the long run), exponentially goes to zero as $t \to \infty$. The other three components grow or decrease at a constant rate with a correction term due to the presence of $T(t)$. This correction term exponentially goes to zero with a constant rate μ as $t \to +\infty$ and therefore it does not influence the long run capital accumulation path.

The first component k_1 comes from innovation/investments in new capital goods, k_1 is exponentially decreasing in s. The second and the third component, k_2, k_3, come from investments in vintage capital goods.

The positivity of the state $k(t)$ requires that if $q > \beta$ then $\max\{\alpha - \overline{\beta}_0, \alpha - \beta\} > q - \beta$. As we expected from the assumption of a technological progress with a constant exogenous rate we have perpetual growth at a constant rate in the limit if one of the following conditions is satisfied

- $q \leq \beta$, $\max\{\alpha - \overline{\beta}_0, \alpha - \beta\} > 0$
- $q > \beta$, $\max\{\alpha - \overline{\beta}_0, \alpha - \beta\} > q - \beta > 0$.

In the limit, the rate of growth is given by

$$\max\{\alpha - \overline{\beta}_0, \alpha - \beta\},$$

the stock of capital is accumulated in the limit at a constant rate which is given by the rate of technological progress deflated by the growth rate of installation costs (if $\beta < \overline{\beta}_0$) or by the growth rate of innovation costs (if $\beta > \overline{\beta}_0$). In the limit for $t \to \infty$ $k(t,s)$ is either $k_1(s)$ (if $\beta > \overline{\beta}_0$) or $k_2(s)$ (if $\beta < \overline{\beta}_0$).

Let us consider the quality depreciation exponential case:

$$\alpha_1(s) = A_0 e^{-\alpha z s}, \ \beta_1(s) = B_0 e^{-\beta s}, \ q_1(s) = Q_0 e^{-qs}.$$

As we stressed in Section 2, the parameter λ describes the connection between the flow of time and the vintage of capital goods. $\alpha(t,s)$ describes the technology of vintage capital goods and relates it to the technology at time t and therefore to exogenous innovation. In the exponential setting, for a generic couple (λ, z) we have that if the technology productivity at time t of a capital good of vintage s is $A_0 e^{\alpha(t-zs)}$ then after the time period δt the capital good becomes of vintage $s + \lambda\delta t$ and its productivity becomes $A_0 e^{\alpha(t+\delta t - z(s+\lambda\delta t))}$. The evolution of the technology is defined by the parameter $\psi \stackrel{def}{=} \lambda \cdot z$. If $\psi = 1$ then the technology of a capital good of vintage s at time t is exactly the technology first arrived in the market (for new capital goods) at time $t - \frac{s}{\lambda}$ and the productivity of a capital good is constant as time goes. If $\psi > 1$ then the technology corresponds to a technology first arrived in the market before time $t - \frac{s}{\lambda}$ and the productivity of a capital good is decreasing as time time goes; if $\psi < 1$ then the technology corresponds to a technology first arrived in the market after time $t - \frac{s}{\lambda}$ and the productivity of a capital good is increasing as time time goes.

ψ is an index marking the ageing of a capital good taking into account both quality depreciation and technological progress: if $\psi < 1$ then the productivity of a capital good is increasing as time goes, if $\psi > 1$ then the productivity of a capital good is decreasing as time goes. The behavior of investment costs and adjustment costs with respect to time is exclusively determined by λ: if $\lambda < 1$ they are increasing, if $\lambda > 1$ they are decreasing. Let us remark z regulates how the technological improvement for new capital goods is reflected on vintage capital goods: a large z implies a small effect.

The optimal solution in this setting becomes

$$u^*(t,0) = \frac{A_0 e^{(\alpha-\overline{\beta}_0)t}}{2\beta_0\left(\alpha z + \frac{\rho-\alpha+\mu}{\lambda}\right)}; \qquad t \geq 0,$$

$$u^*(t,s) = \frac{e^{-\beta(t-s)}}{2B_0}\left[\frac{A_0 e^{\alpha(t-zs)}}{\rho-\alpha+\mu+\lambda\alpha z} - Q_0 e^{q(t-s)}\right]; \qquad t \geq 0,\ s > 0,$$

$$u_1^*(s) = \frac{A_0}{2B_0}\cdot\frac{1}{\rho-\alpha+\mu+\lambda\alpha z}e^{-(\alpha z-\beta)s}; \qquad s > 0,$$

$$u_2^*(s) = \frac{Q_0}{2B_0}e^{-qs}; \qquad s > 0,$$

$$k_1(s) = -C_1 AR(\alpha-\overline{\beta}_0; A)w_0(s) = \frac{A_0}{2\beta_0}\cdot\frac{\lambda}{\rho-\alpha+\mu+\lambda\alpha z}e^{-\frac{\alpha-\overline{\beta}_0+\mu}{\lambda}s}; \qquad s > 0,$$

$$k_2(s) = R(\alpha-\beta; A)u_1^*(s) = \frac{1}{\lambda}\int_0^s e^{-\frac{\alpha-\beta+\mu}{\lambda}(s-\sigma)}\frac{A_0}{2B_0}\frac{e^{-(\alpha z-\beta)\sigma}}{\rho-\alpha+\mu+\lambda\alpha z}d\sigma$$

$$= \frac{A_0}{2\lambda B_0}\cdot\frac{1}{\frac{\alpha-\beta+\mu}{\lambda}-(\alpha z-\beta)}\cdot\frac{1}{\rho-\alpha+\mu+\lambda\alpha z}\left[e^{-(\alpha z-\beta)s} - e^{-\frac{\alpha-\beta+\mu}{\lambda}s}\right]; \qquad s > 0,$$

and finally

$$k_3(s) = R(q-\beta; A)u_2^*(s) = \frac{Q_0}{2\lambda B_0}\cdot\frac{e^{(\beta-q)s} - e^{-\frac{q-\beta+\mu}{\lambda}s}}{\beta-q+\frac{q-\beta+\mu}{\lambda}}; \qquad s > 0$$

Notice that all the functions $k_1,\ k_2,\ k_3$ belong to H^1 and so they go to 0 (with their first derivative) as $s \to +\infty$. To this end we require

$$\alpha z - \beta > 0, \qquad q - \beta > 0.$$

As we stressed above the limit growth rate of the stock of capital for $s \neq 0$ depends on the sign of $\beta - \overline{\beta}_0$. Things are different for $s = 0$; $\forall t \geq 0$ we have that investments in new capital goods grow at the constant rate $\alpha - \overline{\beta}_0$, i.e. the rate of technological progress deflated by the rate of growth of innovation costs. Moreover we have that $u^*(t,0)$ is increasing in λ, A_0 and decreasing in $\beta_0, \overline{\beta}_0, \rho, \mu, z$. The behavior with respect to α is more articulated: if $\psi > 1 + \frac{\alpha(\rho+\mu)}{1-\alpha^2}$ then $u^*(t,0)$ is a decreasing function of α, otherwise is increasing. The analysis confirms what we have observed for the exponential quality depreciation model.

The function $k_1(s)$ is exponentially decreasing with respect to s. It represents the effect on the state of the boundary control, investments in new capital goods. In the limit, for $t \to \infty$, the optimal stock of capital $k(t,s)$ is going to be decreasing in s if the rate of technological progress, $\overline{\beta}_0$, is lower than the rate of growth of installation costs β.

The sensitivity analysis of $k_1(s)$ with respect to the parameters of the model is the following: $k_1(s)$ is a decreasing function of $\beta_0, \overline{\beta}_0, \rho, \mu, z, \alpha$, and increasing of A_0, λ.

$k_2(s)$ is *single-peaked* with a maximum in s_0; for new capital goods, $s = 0$, we have $k_2(0) = 0$. The maximum point s_0 depends on the parameters of the model; leaving aside the case $\frac{\alpha-\beta+\mu}{\lambda} = \alpha z - \beta$ we have

1. if $\frac{\alpha-\beta+\mu}{\lambda} > \alpha z - \beta$ then

$$s_0 = \frac{1}{\frac{\alpha-\beta+\mu}{\lambda} - (\alpha z - \beta)} \log\left(\frac{\alpha-\beta+\mu}{\lambda(\alpha z - \beta)}\right)$$

2. if $\frac{\alpha-\beta+\mu}{\lambda} < \alpha z - \beta$ then

$$s_0 = \frac{1}{\alpha z - \beta - \frac{\alpha-\beta+\mu}{\lambda}} \log\left(\frac{\lambda(\alpha z - \beta)}{\alpha-\beta+\mu}\right).$$

$k_2(s)$ is an increasing function in A_0 and a decreasing function in B_0, ρ, μ, z. The behavior with respect α, β, λ is more articulated. $k_2'(0^+)$ is decreasing function in $\lambda, B_0, \mu, \rho, z$, and in α if $\psi > 1$ and an increasing function in α if $\psi < 1$ and in A_0. Let us remark that this confirms what we have observed in the quality depreciation setting. Therefore, in the limit, for $t \to \infty$, the optimal stock of capital $k(t,s)$ is going to be *single-peaked* if if the rate of technological progress, $\overline{\beta}_0$, is higher than the rate of growth of installation costs β.

Whatever the technology of the firm is we have *capital deepening* type results, the optimal stock of capital is a decreasing function of ρ, μ, z.

10 Conclusions

In this paper we have analyzed the firm capital accumulation problem in a vintage capital setting. The evolution of the stock of capital with respect to time and vintage is described by a Partial Differential Equation. The technology of the firm is characterized by constant returns to scale; the entrepreneur has to bear adjustment costs installing capital goods and an innovation cost installing new capital goods/new technologies.

The optimal stationary stock of capital as a function of the vintage is obtained when the technology is constant over time, when it is time varying we have an optimal capital accumulation path which in the limit can be at a constant rate.

The optimal stationary stock of capital can be a function strictly decreasing of the vintage or a *single-peaked* function, increasing with a maximum and then decreasing. The shape depends on the technology, if a new capital good is highly profitable and the innovation cost is high then it is likely to assist to a diffusion of a new technology. We have a diffusion of a new technology if it is profitable, the discount rate and the rate of depreciation (qualitative and quantitative) are low enough.

When the technology is time varying with exogenous constant technological improvement then we have perpetual growth at a constant rate in the limit; the rate of growth is given by the rate of technological improvement deflated by the minimum between the rate of growth of innovation costs and the rate of growth of adjustment costs. If the minimum between the two is the rate of innovation costs then in the limit we will have a stock of capital strictly decreasing in the vintage, otherwise it will be *single-peaked* with a maximum. Therefore if innovation costs grow more quickly than adjustment costs we will have the diffusion of the new technology in the limit.

The assumption of constant returns to scale and exogenous technological progress are two strong limits to the analysis developed in the paper. The analysis of capital accumulation for a firm characterized by a nonlinear technology with input complementarities, spillover effects, and $R\&D$ would call for a refinement of the optimal control infinite dimensional techniques, in particular with respect to the trasversality condition and the analysis of the long run equilibrium of the Hamiltonian system.

A Appendix

In this Appendix we present some technical results useful in the analysis and we give the proofs of some Propositions. We recall the following result which is a particular case of the Sobolev embedding theorem.

Theorem A.1 *The space $H^1(0,\overline{s})$ is canonically embedded in $C([0,\overline{s}])$, ($C([0,+\infty))$ if $\overline{s}=+\infty$), i.e. given $h \in H$ there exists $\tilde{h} \in C([0,\overline{s}])$ such that $h=\tilde{h}$ almost everywhere. Moreover for a suitable constant $C_{\overline{s}}$ we have*

$$\|h\|_\infty \leq C_{\overline{s}} \|h\|_{H^1}.$$

The semigroup $T(t)$ generated by the operator A defined in (4) satisfies Proposition 3.1 and the following estimates are given.

Proposition A.2

$$\|T(t)\|_{\mathcal{L}(H)} \leq e^{-\mu t} \qquad 0 \leq t < \frac{\overline{s}}{\lambda}$$

$$\|T(t)\|_{\mathcal{L}(H)} = 0 \qquad t \geq \frac{\overline{s}}{\lambda}$$

when $\overline{s} < +\infty$ and

$$\|T(t)\|_{\mathcal{L}(H)} \leq e^{-\mu t} \qquad t \geq 0$$

when $\overline{s} = +\infty$ and finally

$$\|R(\gamma; A)\|_{\mathcal{L}(H)} \leq \frac{1}{\gamma+\mu}.$$

for $\overline{s} \in (0,+\infty]$.

Remark A.3 The operator A is dissipative. In fact, for $k \in D(A)$ we have

$$< Ak, k >_H = -\mu |k|_H^2 - \frac{\lambda}{2}[k^2(\overline{s}) - k^2(0)] \leq -\mu |k|_H^2,$$

the equality holds in the case $\overline{s} = +\infty$ since we have $\lim_{s\to+\infty} f(s) = 0$ for $f \in H^1(0,+\infty)$.

Sketch of proof of Proposition 3.1 and A.2. The proof is a standard application of the theory of strongly continuous semigroups on a Hilbert space (see e.g. [Brezis, 1983]). One simply has to use definitions to check that

- A is closed
- $T(\cdot)$ is a C_0-semigroup and generates A
- Re $\gamma \leq -\mu$ implies $\gamma \in \rho(A)$

then the resolvent can be directly calculated by solving the differential equation ($s \in [0, \bar{s}]$)

$$\lambda g'(s) + (\gamma + \mu) g(s) = f(s); \qquad g(0) = 0$$

or via Laplace transform. The estimates on the norm of $T(t)$ and of the resolvent are then immediate. ∎

Remark A.4 By applying classical functional analysis theorems on the adjoint operators it follows that (see e.g. [Yosida, 1980] p. 224)

- $\rho(A^*) = \rho(A)$;
- for $\gamma \in \rho(A)$, $R(\gamma; A^*) = R(\gamma; A)^*$;
- $\|T^*(t)\| = \|T(t)\|$ and $\|R(\gamma; A^*)\| = \|R(\gamma; A)\|$.

Proof of Proposition 4.3.
It is enough to write down explicitly the functional J :

$$J(k_0; u) = \int_0^{+\infty} e^{-\rho t} \left[g(k(t)) + l(u(t)) \right] dt$$

$$= \int_0^{+\infty} e^{-\rho t} \left[< \alpha, k(t) >_{L^2} - < q, u(t) >_{L^2} - < B_\beta u(t), u(t) >_{L^2} - \beta_0 u^2(t, 0) \right] dt.$$

Now, by (7)

$$< \alpha, k(t) >_{L^2}$$

$$= < \alpha, T(t) k_0 >_{L^2} - < \alpha, A \int_0^t T(t - \tau) w(\tau) d\tau >_{L^2} + < \alpha, \int_0^t T(t - \tau) u(\tau) d\tau >_{L^2}$$

so that

$$\int_0^{+\infty} e^{-\rho t} < \alpha, k(t) >_{L^2} = \int_0^{+\infty} e^{-\rho t} < \alpha, T(t) k_0 >_{L^2} dt$$

$$+ \int_0^{+\infty} e^{-\rho t} \left[< \alpha, \int_0^t T(t - \tau) u(\tau) d\tau >_{L^2} dt - < \alpha, A \int_0^t T(t - \tau) w(\tau) d\tau >_{L^2} \right] dt.$$

Now for the first term we have

$$\int_0^{+\infty} e^{-\rho t} < \alpha, T(t) k_0 >_{L^2} dt = \left\langle \alpha, \int_0^{+\infty} e^{-\rho t} \left(T(t) k_0 \right) dt \right\rangle_{L^2}$$

$$= \langle \alpha, R(\rho; A) k_0 \rangle_{L^2}$$

while for the second

$$\int_0^{+\infty} e^{-\rho t} < \alpha, \int_0^t T(t - \tau) u(\tau) d\tau >_{L^2} dt = \int_0^{+\infty} \int_0^t e^{-\rho t} < T^*(t - \tau) \alpha, u(\tau) >_{L^2} d\tau dt$$

$$= \int_0^{+\infty} e^{-\rho \tau} \left\langle \left[\int_\tau^{+\infty} e^{-\rho(t - \tau)} T^*(t - \tau) \alpha dt \right], u(\tau) \right\rangle_{L^2} d\tau$$

$$= \int_0^{+\infty} e^{-\rho \tau} \langle R(\rho; A^*) \alpha, u(\tau) \rangle_{L^2} d\tau = \int_0^{+\infty} e^{-\rho t} \langle \overline{\alpha}, u(t) \rangle_{L^2} dt.$$

For the third we use the same reasoning by recalling that $\alpha \in D(A^*)$ (in fact it is possible to prove the claim also when $\alpha \notin D(A^*)$). ∎

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