Report n.106

A survey of bicriteria fractional problems

A.Cambini,L.Martein,I.M.Stancu-Minasian

Pisa, Giugno, 1996

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A. Cambini - L. Martein - I.M. Stancu-Minasian[1]

1. Introduction

In the last decades vectorial mathematical programming has been widely studied. It is interesting to note that altough the first papers in vectorial mathematical programming were done since about 1950 and the ones on fractional programming in 1960, the case of vectorial fractional mathematical programming appears systematically after 1980.

In a survey note appeared in 1981 [73] about the methods used for solving the linear fractional programming problems with several objective functions, Stancu-Minasian has given only 5 references. Now there are about 150 papers on this subject [74, 75]. From this it results that a considerable amount of work has been done since 1980 on the problem of vectorial fractional mathematical programming and, in particular, the bicriterion case has received a considerable attention.

Evidently, any method and result referring to the vectorial fractional programming can be applied to the case of two objective functions; however the reduce number of functions offers some advantages; it is sufficient to think to parametric representation in uni- and bi-dimensional spaces. Consequently, for the case of two objective functions many papers appeared which take the advantage of the particular structure of the problem.

In this paper we present a review of the main theoretical results obtained up to now in bicriterion fractional programming with few exceptions related to general results for bicriteria problems. With respect to sequential methods, we limit ourselves to present computational approaches for linear fractional problems.

The structure of the paper is organized as follows: in Section 2 some definitions are given in order to present a self-contained paper; in Section 3

[1] One of the authors (I.M. S.-M.) gratefully acknowledges the research support he received as Visitor Research of the Department of Statistics and Applied Mathematics, University of Pisa, in October 1994 when this paper was started.

some properties of the efficient set of bicriteria problems are given; in Section 4 we present approaches to solving the bicriterion fractional programming problem; in Section 5 we consider the particular case when one of the objective function is linear and the other one is a linear fractional function; in Section 6 the bicriteria fractional transportation problem is considered; in Section 7 we present some sequential methods for generating the set of all efficient solutions for the bicriteria linear fractional problem; in Section 8 the relation between bicriteria problems and bicriterion mathematical programs is presented; finally concluding remarks are made in Section 9,

2. Statement of the problem

Consider the following bicriteria problem

$$P: \quad \text{"maximize"} \; F(x) = (F_1(x), F_2(x)), \; x \in S.$$

where S is a subset of $\mathbf{R}^n$ and F_1, F_2 are real-valued functions defined on an open set X containing S.

We will refer to P as the bicriteria linear fractional programming problem when F_1, F_2 are linear fractional functions and S is a polyhedral set, that is:

$F_1(x) = \dfrac{c_1x + c_{01}}{d_1x + d_{01}}$, $F_2(x) = \dfrac{c_2x + c_{02}}{d_2x + d_{02}}$, $S = \{x : Ax \leq b, x \geq 0\}$, where $A = ((a_{ij}))$, $i = 1, ..., m$, $j = 1, ..., n$ is an m×n matrix, $b = (b_1, ..., b_m) \in \mathbf{R}^m$ is an m-dimensional vector, c_1, d_1, c_2, and $d_2 \in \mathbf{R}^n$ are n-dimensional vectors, $x = (x_1, ..., x_n) \in \mathbf{R}^n$ is an n-dimensional unknown vector, c_{01}, d_{01}, c_{02}, and d_{02} are scalar constants; it is assumed that $d_1x + d_{01} > 0$, $d_2x + d_{02} > 0$ for all $x \in S$

The meaning of "maximize" in problem P is to understand in the sense of different solution notions in vector optimization, such as in the sense of efficient, weakly efficient or properly efficient points, or, equivalently, in the sense of non-dominated, noninferior or Pareto optimal solutions.

In order to have a self-contained paper, we recall the following definitions.

Definition 2.1

A point $x^* \in S$ is said to be an efficient solution of P if there does not exist a point $x \in S$ such that $F_i(x) \geq F_i(x^*)$ $(i = 1,2)$ where at least one of these inequalities is strict.

Definition 2.2

A point $x^* \in S$ is said to be a weakly efficient solution of P if there does not exist a point $x \in S$ such that $F_i(x) > F_i(x^*)$ $(i = 1,2)$.

The previous definitions are equivalent to the following ones:

Definition 2.1' (2.2')

A point x^* is said to be an efficient (weakly efficient) solution for P if $x^* \in S$ and $F_i(x) > F_i(x^*)$ for some $x \in S$ and some $i = 1,2$ implies that there exists at least one $j \in J$, such that $F_j(x) < F_j(x^*)$ (resp. $F_j(x) \leq F_j(x^*)$).

Sometimes when the functions $F_i(\cdot)$ are nonlinear and differentiable, instead of problem P we consider linear approximation at x^0 of this problem, namely :

$$\text{"maximize"} \ (\nabla F_1(x^0) \cdot x, \nabla F_2(x^0) \cdot x) \ , \ x \in S \tag{2.1}$$

Applying Definition 2.1 to problem (2.1), we obtain

Definition 2.3

A point $x^* \in S$ is said to be a Kuhn-Tucker (K-T) properly efficient solution of P if there does not exist a point $x \in S$ such that $\nabla F_i(x^*) \cdot x \geq \nabla F_i(x^*) \cdot x^*$ $(i = 1,2)$ where at least one of these inequalities is strict.

Definition 2.4

An efficient solution x^0 of problem P is said to be properly efficient (in the Geoffrion's sense) if there is a scalar $M > 0$ such that for each i, $F_i(x) > F_i(x^0)$ and $x \in S$ imply $F_i(x) - F_i(x^0) \leq M(F_j(x^0) - F_j(x))$ for some j such that $F_j(x) < F_j(x^0)$.

As is known, there are in the literature several definitions of properly efficiency; through the paper, if any specification is given, any properly efficient solution it is considered in the Geoffrion's sense.

We will denote the set of all efficient points of P by E, the set of all weakly efficient points by E^W and the set of all properly efficient points by E^p.

In multiple objective programming, the knowledge of the set E may be useful in the process of decision making, but its complete generation may be computationally expensive, since E is, in general, not finite and its elements can occur at the extreme points, at the edges and also in the interior of S.

Sometimes it is more convenient to approach the problem in the so called criterion space $Z = F(S) = \{(z_1, z_2) \mid z_1 = F_1(x), z_2 = F_2(x), x \in S\}$, instead of the decision space S.

The images

$F(E) = \{(F_1(x), F_2(x)) \mid x \in E\}$ and $F(E^W) = \{(F_1(x), F_2(x)) \mid x \in E^W\}$ are called the efficient frontier and the weakly efficient frontier of P, respectively.

Before presenting the results obtained in bicriteria fractional programming, we will give some other preliminary definitions.

Definition 2.5

Let $\Lambda \subseteq \mathbf{R}^2$ be an arbitrary set and let $\varphi : \Lambda \to \mathbf{R}$ be a real-valued function; $\varphi(x,y)$ is said to be increasing in each argument if $\varphi(x_1,y) < \varphi(x_2,y)$ for each $(x,y) \in S$ such that $x_1 < x_2$ and $\varphi(x,y_1) < \varphi(x,y_2)$ for each $(x,y) \in S$ such that $y_1 < y_2$.

Let $X \subseteq \mathbf{R}^n$ be a nonempty convex set and let $f: X \to \mathbf{R}$ be a real-valued function.

Definition 2.6

The function f is concave on X if for $\forall\, x^1, x^2 \in X$ and $\forall\, t \in [0,1]$,

$$f[tx^1+(1-t)x^2] \geq tf(x^1)+(1-t)f(x^2) \tag{2.2}$$

The function f is strictly concave on X if $\forall\, x^1, x^2 \in X$, $(x^1 \neq x^2)$ and $\forall\, t \in (0,1)$ the inequality (2.2) is strict.

Definition 2.7

The function f is quasiconcave on X if for $\forall\, x^1, x^2 \in X$ and $\forall\, t \in [0,1]$,

$$\min(f(x^1), f(x^2)) \le f[tx^1 + (1-t)x^2] \tag{2.3}$$

The function f is said to be explicitly quasiconcave or semistrictly quasiconcave on X if for $x^1, x^2 \in X$, $(f(x^1) \ne f(x^2))$ and $\forall\, t \in (0,1)$ the inequality (2.3) is strict.

The function f is said to be strictly explicitly quasiconcave or strictly quasiconcave on X if $\forall\, x^1, x^2 \in X$, $(x^1 \ne x^2)$ and $\forall\, t \in (0,1)$ the inequality (2.3) is strict.

The function f is explicitly quasiconvex on X if (-f) is explicitly quasiconcave.

Definition 2.8

The function f is quasimonotonic if it is both quasiconcave and quasiconvex.

Definition 2.9

The function f is explicitly quasimonotonic if it is explicitly quasiconcave and explicitly quasiconvex.

Definition 2.10

The function f is pseudoconcave on X if X is open, f is differentiable and if $\forall\, x^1, x^2 \in X$, $(x^1 - x^2)\nabla f(x^2) \le 0 \Rightarrow f(x^1) \le f(x^2)$.

Definition 2.11

The function f is pseudoconvex on X if (-f) is pseudoconcave.

3. The structure and the properties of the efficient (weakly, properly efficient) set

In this section we will present some properties of the solutions of a bicriteria problem, with particular attention to a bicriteria linear fractional problem.

First of all, let us note that from definitions 2.1, 2.2 and 2.4, it results $E \subseteq E^w$, $E^p \subseteq E$ (but not conversely); furthermore for a linear fractional vector maximum problem, we have $E^p = E$. This result was proved by Choo [18] for the general case of L (L > 1) criterion functions, assuming that S is a compact set:

Theorem 3.1 (Choo [18])
Let x^0 be an efficient solution of the bicriteria linear fractional problem. Then x^0 is properly efficient.

In what follows we will give a characterization of efficiency for a bicriteria linear fractional problem by means of the Kuhn-Tucker conditions. With this aim, we denote the row vectors of the matrix A by $p_1, \ldots, p_m$ and we refer to the following definition:

Definition 3.1 (Choo and Atkins [19])
A subset T of S is said to be a face of S if there exists $J \subset \{1, \ldots, m\}$, such that T consists of points satisfying $p_i x = b_i$, $\forall i \in J$ and $p_i x < b_i$, $\forall i \notin J$.
T is called the face corresponding to J.

We have :

Theorem 3.2 (Choo and Atkins [19])
A point x* belonging to the face T corresponding to $\{1, \ldots, k\}$ is efficient if and only if there exist real numbers $a_i \geq 0$, $i = 1, \ldots, k$ and $w_1 \geq 1$, $w_2 \geq 1$ such that

$$a_1 p_1 + \ldots + a_k p_k = w_1 \nabla f_1(x^*) + w_2 \nabla f_2(x^*) \tag{3.1}$$

The equation (3.1) is equivalent to a system of linear constraints as it results from the following:

Lemma 3.1
A point x* in T is efficient if and only if there exist real numbers $a_1, \ldots, a_k$, h_1, h_2, q_1, q_2, such that

$$a_1 p_1 + \ldots + a_k p_k = q_1 c_1 - h_1 d_1 + q_2 c_2 - h_2 d_2$$
$$c_1 x + c_{01} = h_1$$
$$d_1 x + d_{01} = q_1$$
$$t_1 h_1 + t_2 h_2 - r_1 q_1 - r_2 q_2 = a_1 b_1 + \ldots + a_k b_k$$
$$p_i x^* = b_i \quad i = 1, \ldots, k$$
$$p_i x^* < b_i \quad i = k+1, \ldots, m$$
$$a_i \geq 0 \quad i = 1, \ldots, k, \quad q_i > 0 \quad i = 1, 2.$$

From Lemma 3.1, the following results follow:

Corollary 3.1

The set of all efficient points in any given face of S is convex.

Theorem 3.3

The set $E \cap T$ is a linearly constrained set and E is a finite union of linearly constrained sets.

As we have pointed out previously, efficient points may be situated on the faces, on the edges or in the interior of S; the interior points of S can be efficient solutions without necessarily having S = E as it happens in the linear case.

Hughes [43] derived general conditions for the bicriteria linear fractional problem to have interior efficient solutions. He stated that the set of interior efficient solutions (if any) is contained in a hyperplane separating the points where each objective function is optimized.

Let $R_1 = [F_1(x^*_2), \bar{f}_1]$ be an interval where $\bar{f}_k = \max\{F_k(x) : x \in S\}$, k = 1,2 and x^*_2 satisfies $F_2(x^*_2) = \bar{f}_2$.

Theorem 3.4 ([43], Lemma 1)

The bicriteria linear fractional problem has interior efficient solutions if and only if there exist scalars ϕ_c, θ_c, u_c such that

$$c_2 + \phi_c d_2 + \theta_c u_c d_1 = \theta_c c_1$$

$$c_{02} + \phi_c d_{02} + \theta_c u_c d_{01} = \theta_c c_{01}, \quad \theta_c < 0, u_c \in R_1.$$

Corollary 3.2 ([43])

The set of all interior efficient points (if any) of a bicriteria linear fractional problem lies on the hyperplane $c_1 x + c_{01} - u_c(d_1 x + d_{01}) = 0$ on which $F_1(x) = u_c$.

Corollary 3.3 ([43])

i) The hyperplane containing the interior efficient points (if any) of a bicriteria linear fractional problem passes through the intersection of the

hyperplanes $c_1x + c_{01} = 0$, $d_1x + d_{01} = 0$ and of the hyperplanes $c_2x + c_{02} = 0$, $d_2x + d_{02} = 0$
ii) $F_2(x) = - \phi_c$ for any interior efficient point.

With regards to closure property, Choo and Atkins ([19], Th.4.4) showed, for a bicriteria linear fractional programming having a compact feasible region, that E (and hence F(E)) is closed; such a property does not hold in general even if the feasible region S is compact.
Unlike the set E, the set E^w is always a closed set, as proved by Choo and Atkins ([20], Th.1) for $L \geq 2$ criteria, arbitrary continuous objective functions and for arbitrary constrained region S. With respect to problem P, we have:

Theorem 3.5 ([20])
The set E^w of all weakly efficient solutions of problem P is closed.

With regards to connectedness property, Choo and Atkins [19] showed, for a bicriteria linear fractional programming having a compact feasible region, that E and E^w are path-connected; more exactly, there exists a finite number of connected line segments in E and in E^w which describe the whole efficient (weakly efficient) frontiers F(E) and $F(E^w)$.
When the feasible region is unbounded, Cambini and Martein [10] established a necessary and sufficient condition for the connectedness of E together with the following Theorem:

Theorem 3.6 ([10])
The set E of all efficient points of a bicriteria linear fractional problem having an unbounded feasible region, is path-connected by a finite number of linear line when at least one of the objective function is linear.

In order to establish general results on the properties of the set E of all efficient solutions, Schaible [65] and Martein [56] considered the following parametric scalar problem associated to the bicriteria problem P:

$$P(\theta) : \max \{F_1(x) : x \in S, F_2(x) \geq \theta\}$$

Denote with $z(\theta)$ and $S(\theta)$ the optimal value and the set of optimal solutions of problem $P(\theta)$, respectively. Martein [56] gave a complete characterization of E as a suitable union of sets $S(\theta)$; furthermore Cambini and Martein [11, 12] proved that when F_1 does not have local maxima different from global, then θ belongs to an interval; such results generalize the ones given by Schaible [65] in the case of strictly quasiconcave functions.
Martein [56] established a necessary condition for the connectedness of E :

Theorem 3.7 ([56])
If E is connected, then $z(\theta)$ is a semistrictly quasiconcave function.

Schaible [65] generalized the results obtained in [17] and [19] :

Theorem 3.8 ([65])
If F_1 , F_2 are continuous and strictly quasiconcave functions and S is a compact convex set, then E is connected and closed.

Let us note that if one F_i is merely quasiconcave, E may be not connected.
As an application, Schaible considered the bicriteria non linear fractional program

$$\max_{x \in S} \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)} \right)$$

where f_i is concave on the convex set S, g_i is convex and positive in S, and f_i is nonnegative on S if g_i is not affine. According to [76] the functions $F_i(x)= \frac{f_i(x)}{g_i(x)}$ are strictly quasiconcave on S. Hence, the efficient set for the above problem is connected and closed.

At last, we give some results related to the connectedness of the efficient frontier.
Choo et al. [21] proved the following result:

Theorem 3.9 ([21])
The efficient frontier F(E) of problem P is path-wise connected if F_1 and F_2 are continuous and strictly quasiconcave and S is a compact convex set.

Recently, Marchi [53] generalized the previous result:

Theorem 3.10 ([53])
The efficient frontier F(E) of a bicriteria problem P having a compact feasible region is path-wise connected if the functions F_1 and F_2 are continuous and do not have local maxima different from global.

4. Methods for solving bicriterion fractional programming problems

In many problems involving conflicts among objectives, in a natural way a scalar optimization problem involving an utility function representing the decision maker's preference, is solved in order to find a suitable efficient solution for problem P.
More exactly the following bicriterion scalar problem is associated to the bicriteria problem P:

$$\max_{x \in S} \{z(x) = U[F_1(x), F_2(x)]\} \tag{4.1}$$

where U is an utility function defined on the set $F(S) = \{ (F_1(x), F_2(x)) : x \in S\}$
According to Geoffrion [32] at least one optimal solution to (4.1) is Pareto optimal.
This section is devoted to describe some methods for solving (4.1) when U is a "sum-type" operator, "product-type" operator and "minimum-type" operator .

Let us note that some other classes of bicriterion problems (i.e. problems involving two functions) can be transformed in problem (4.1) as it is shown in [77], [79], [80], [81], for the following bicriterion max-min fractional problem

$$\max_{x \in X} \min_{y \in Y} h(F(x), Q(x,y))$$

where $X \subset \mathbf{R}^n$ and $Y \subset \mathbf{R}^m$ are two compact convex polyhedral sets and $F: X \rightarrow \mathbf{R}$, $Q: X \times Y \rightarrow \mathbf{R}$ are fractional functions.

In a recent report [39] Hirche considers problem (4.1) when the objective functions $F_i(\cdot)$ $(i=1,2)$ are explicitly quasimonotonic; the class of explicitly quasimonotonic functions includes the class of linear fractional functions. Furthermore, it is assumed that the function U is strictly increasing in each argument.

Using an idea from [40], Hirche [39] choose the global maximizer of z(x) in a subset $S' \subset S$.

For $i = 1,2$ set $\bar{f}_i = \max \{F_i(x) : x \in S\}$ and set

$$\underline{f}_1 = \max \{F_1(x): x \in S, F_2(x) \geq \bar{f}_2\}, \quad (4.2)$$

$$\underline{f}_2 = \max \{F_2(x): x \in S, F_1(x) \geq \bar{f}_1\} \quad (4.3)$$

Define $S_j(w_j) = \{x \in S : F_j(x) \geq w_j\}$ for $w_j \in [\underline{f}_j, \bar{f}_j]$, $j = 1,2$ and $S' = S_1(\underline{f}_1) \cap S_2(\underline{f}_2)$.

Hirche [39] showed that : i) any global maximizer x^* of z(x) on S is in S'; ii) for each $w_j \in [\underline{f}_j, \bar{f}_j]$ the function $F_i(x)$ attains its maximum on $S_j(w_j)$ in the level surface $\{x \in S : F_j(x) = w_j\}$ and only there; iii) the function z(x) attains its local maxima (and consequently its global maximum) on S in edges or pieces of edges of S belonging to S'.

Warburton [83] showed that problem (4.1) can be solved by considering r one-dimensional subproblems defined by

(R_i) : $\max \{ u(F_1(x), F_2(x)) : x \in [x_{i-1}, x_i]\}$ for $1 \leq i \leq r$.

If $\hat{x}_i$ $(1 \leq i \leq r)$ solves (R_i), then $\hat{x}_k$ such that

$u(F(\hat{x}_k)) = \max\{u(F(\hat{x}_i)): 1 \leq i \leq r\}$ solves (4.1).

The problem (R_i) can be reduced to

$(\hat{R}_i)$: $\max \{ u(w, h_i(w)) : w_{i-1} \leq w \leq w_i \}$

where h_i is a linear fractional function and $w_i = F_1(x_i)$ i=1, ..., n.

According to Warburton [84], if $\hat{w}_i$ solves $(\hat{R}_i)$, then

$$\hat{x}_i = \hat{\alpha}_i x_{i-1} + (1 - \hat{\alpha}_i) x_i \quad (4.4)$$

where

$$\hat{\alpha}_i = \frac{\hat{w}_i (d_1 x_i + d_{01}) - (c_1 x_i + c_{01})}{\hat{w}_i d_1(x_i - x_{i-1}) + c_1(x_{i-1} - x_i)} \qquad (4.5)$$

is an optimal solution to (R_i).

Warburton's algorithm

Step 1 Apply the parametric procedure to obtain the values $\underline{f}_1 = w_0 \le w_1 \le \ldots \le w_r = \bar{f}_1$ and the points x_i for $0 \le i \le r$.

Step 2 For each i $(1 \le i \le r)$, let $\hat{w}_i$ be an optimal solution of $(\hat{R}_i)$.

Step 3 Let $\hat{u}_k = \max \{ u(\hat{w}_i , h_i(\hat{w}_i)) : 1 \le i \le r \}$, and let $\hat{x}_k$ be the corresponding solution of (R_k) given by (4.4) and (4.5). Then $\hat{x}_k$ solves (4.1).

Also, Warburton presented a simple finite procedure for solving (4.1) when U is a "sum-type" operator. Numerical experiments indicate that the Warburton's algorithm performs effectively for problems of moderate size. For larger problems the parametric phase may become rather expensive.

From now on, in this section we will consider problem (4.1) where F_1 and F_2 are linear fractional functions.

As is known, for a fractional programming problem with a single objective function a classic method of solving is that of the transforming variables. However, for the vectorial fractional programming in applying this method some difficulties occur. Due to this reason, in [78] the particular case where the denominators of F_1 and F_2 are identical is considered, that is the synthesis-function is under the form:

$$\max_{x \in S} U\left(\frac{c_1 x + c_{01}}{dx + d_0} , \frac{c_2 x + c_{02}}{dx + d_0} \right) \qquad (4.6)$$

This problem is called pseudo-fractional programming problem [78]; it is assumed that:

a) S is a nonempty and bounded set;

b) $dx + d_0 > 0 \quad \forall x \in S$

Using the variable transformation $y = tx$ $(t \ge 0)$ problem (4.6) becomes a pseudo-linear problem:

$$\max U(c_1 y + c_{01} t, c_2 y + c_{02} t) \tag{4.7}$$

subject to

$$Ay - bt \leq 0$$
$$dy + d_0 t = 1$$
$$y \geq 0, t \geq 0$$

The equivalence between problems (4.6) and (4.7) is given in the following Theorem:

Theorem 4.1 ([78], Tigan)

If the assumptions a) and b) hold and if (y^*, t^*) is an optimal solution for problem (4.7), then $\frac{y^*}{t^*}$ is an optimal solution for problem (4.6).

From Theorem 4.1 it results that $\frac{y^*}{t^*}$ is an efficient solution of the bicriteria linear fractional programming problem P.

Patkar et al. [61] consider an operator U of "product-type", so that the following problem is obtained:

$$\max_{x \in S} \frac{(c_1 x + c_{01})(c_2 x + c_{02})}{(d_1 x + d_{01})(d_2 x + d_{02})} \tag{4.8}$$

By a variable transformation, problem (4.8) reduces to a quadratic programming problem:

$$\max (c_1 y + c_{01} t)(c_2 y + c_{02} t) \tag{4.9}$$

subject to

$$Ay - bt \leq 0$$
$$(d_1 y + d_{01} t)(d_2 y + d_{02} t) \leq 1$$
$$y \geq 0, t \geq 0$$

If the set S is bounded, a result similar to the one given in Theorem 4.1 is true also for problems (4.8) and (4.9).

<u>Konno and Yajima's algorithm</u>

Konno and Yajima [47] proposed two algorithms (parametric simplex algorithm and branch and bound algorithm) for solving problem (4.1) in the case when the operator U is a "product-type". They transform the maximum problem into a minimum problem

$$\min_{x \in S} \frac{d_1x + d_{01}}{c_1x + c_{01}} \cdot \frac{d_2x + d_{02}}{c_2x + c_{02}} \tag{4.10}$$

under the following assumptions :

1) S is nonempty and bounded set;
2) $c_ix + c_{0i} > 0$, $d_ix + d_{0i} > 0$, $\forall\, i = 1,2$ and $\forall\, x \in S$.

After the variable transformation $y = tx$ $(t > 0)$, setting $w = (y, t)$, $\tilde{A} = (A, -b)$, $\tilde{c}_i = (c_i, c_{0i})$, $\tilde{d}_i = (d_i, d_{0i})$, $i = 1, 2$, problem (4.10) can be rewritten as follows:

$$\min \frac{\tilde{d}_1 w}{\tilde{c}_1 w} \cdot \frac{\tilde{d}_2 w}{\tilde{c}_2 w} \tag{4.11}$$

subject to

$\tilde{A}w = 0, \ w \geq 0$

According to Konno and Yajima [47], it can be assumed, without loss of generality, that $\tilde{c}_1 w \cdot \tilde{c}_2 w = 1$.

Then, (4.11) turns out to be the following:

$$\min \ \tilde{d}_1 w \cdot \tilde{d}_2 w \tag{4.12}$$

subject to

$\tilde{A}w = 0, \ w \geq 0$

$\tilde{c}_1 w \cdot \tilde{c}_2 w = 1$

Problem (4.12) can be solved by the following master problem with two parameters:

$$\min \xi \tilde{d}_1 w + \frac{1}{\xi} \tilde{d}_2 w \tag{4.13}$$

subject to

$\tilde{A}w = 0, \ w \geq 0$

$\tilde{c}_1 w = \eta, \quad \tilde{c}_2 w = \frac{1}{\eta}$

$\xi > 0, \ \eta > 0$

In the branch and bound algorithm presented by Konno and Yajima [47], the master problem associated to problem (4.10) is of the form

$$\min g(x, \xi) = \xi \frac{d_1x + d_{01}}{c_1x + c_{01}} + \frac{1}{\xi} \frac{d_2x + d_{02}}{c_2x + c_{02}} \tag{4.14}$$

subject to

$x \in S$ and $\xi > 0$.

The computational experiments performed by Konno and Yajima show that the total amount of computation time of the parametric algorithm is about eight times as much as that for solving an associated linear programming problem. Furthermore, when the size of the problem increases, the amount of total computation time of the branch and bound algorithm increases slower than the one of the parametric algorithm; this means that the branch and bound algorithm is preferred to parametric algorithm for large scale problems.

Hirche's algorithm

Hirche [39] proposed a one-parametric algorithm for solving problem (4.8). The algorithm, stated in a general form, can be used for solving the sum of two explicitly quasimonotonic functions.

Without loss of generality, in problem (4.11) it is assumed that $\tilde{c}_1 w = 1$. Hirche associates to this problem the following one-parametric problem :

$$\min \tilde{G}(w, \xi) = \xi \frac{\tilde{d}_2 w}{\tilde{d}_1 w} \qquad (4.15)$$

subject to

$$\tilde{A} w = 0, \quad \tilde{c}_1 w = 1, \quad \tilde{d}_1 w = \xi \quad , w \geq 0, \quad \xi_{min} \leq \xi \leq \xi_{max}$$

where $\xi_{min} = \min \{ \tilde{d}_1 w : \tilde{A} w = 0, \quad \tilde{c}_1 w = 1, w \geq 0 \}$,

$\xi_{max} = \max \{ \tilde{d}_1 w : \tilde{A} w = 0, \quad \tilde{c}_1 w = 1, w \geq 0 \}$.

By solving parametrically problem (4.15), a partition of the interval $[\xi_{min}, \xi_{max}]$ into finitely many subintervals $[\xi_k, \xi_{k+1}]$ is obtained. The minimum of $\tilde{G}(w, \xi)$ on $[\xi_k, \xi_{k+1}]$ is determined.

A solution of (4.11) (and also (4.10)) can be obtained by comparison of the minima on all subintervals $[\xi_k, \xi_{k+1}]$.

With minor modifications the algorithm can be adapted for maximizing the sum of two linear fractional functions. The resulting algorithm is related to Cambini, Martein and Schaible's algorithm [9] and to the Falk and Palocsay's algorithm [31].

Computational experiments performed by the author show that his algorithm is promising in comparison with the branch and bound algorithm and the two-parametric algorithms by Konno and Yajima. The relative expense was measured by the number of simplex iterations.

Cambini , Martein and Schaible's algorithm

In [9] , the problem of maximizing the sum of m concave-convex fractional functions on a convex set is shown to be equivalent to the one whose objective function f is the sum of m linear fractional functions defined on a suitable convex set; successively f is transformed into the sum of one linear function and (m-1) linear fractional functions. As a special case, the problem of maximizing the sum of two linear fractional functions subject to linear constraints is transformed in the one whose objective function is the sum of a linear and a linear fractional function. For such a problem two sequential methods are suggested for any feasible region (bounded or not); one of these methods is obtained by combining Cambini and Martein's algorithm [7] and Martein's algorithm [54], which will be described successively.

Bykadorov in [5] states that these two sequential methods are more preferable than other algorithms.

Falk and Palocsay's algorithm

Falk and Palocsay [31] elaborated a method for solving problem (4.1) in which U is a "sum-type" operator and S is a bounded set.

The algorithm determines $\bar{z} = \bar{z}_1 + \bar{z}_2 \in Z$, where $\bar{z}$ is the image of the optimal solution $\bar{x}$ of the problem

$$\max \{F(x) = \frac{c_1x + c_{01}}{d_1x + d_{01}} + \frac{c_2x + c_{02}}{d_2x + d_{02}} \ : x \in S\} \tag{4.16}$$

The objective function F is neither quasiconvex nor quasiconcave, and thus it can have multiple local minima and maxima. In general, a local maximum is not a global one. Bykadorov [5], [6] studied generalized concavity properties of sums of linear ratios and even of sums of ratios of polynomials. For location of optima for sum of linear fractional functions see also Craven [22, p. 137].

The idea of Falk and Palocsay's algorithm is to determine lower and upper bounds for $\bar{z}_1 + \bar{z}_2$ which are iteratively improved. More specifically, a triangular subset (z^0, l^0, v^0) in the criterion space containing $\bar{z}$ is founded and,

successively, the size of the triangle is reduced until either an optimal solution is obtained or the algorithm cannot improve the current bounds.
Let $z^0_i = \max \{F_i(x): x \in S\}$ $i = 1, 2$ and $x^{1,0}$, $x^{2,0}$ the optimal solutions. An upper bound for $\bar{z}_1 + \bar{z}_2$ is $z^0_1 + z^0_2$ and a lower bound is determined by one of the points $z(x^{1,0})$ and $z(x^{2,0})$ where $z(x^{1,0}) = (F_1(x^{1,0}), F_2(x^{1,0}))$; $z(x^{2,0}) = (F_1(x^{2,0}), F_2(x^{2,0}))$.
Find $f_l = \max \{F_1(x^{1,0}) + F_2(x^{1,0}), F_1(x^{2,0}) + F_2(x^{2,0})\}$ and determine $l^0 = (l^0_1, l^0_2) = (z^0_1, f_l - z^0_1)$ and $v^0 = (v^0_1, v^0_2) = (f_l - z^0_2, z^0_2)$

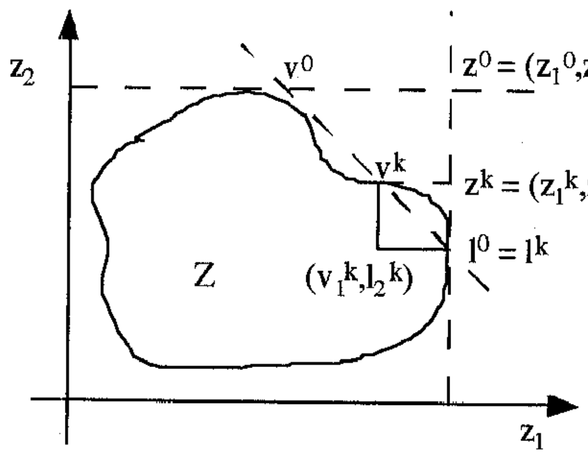

Fig.1

As it can be seen in Fig 1, $l^0 = z(x^{1,0})$ and hence $f_l = F_1(x^{1,0}) + F_2(x^{1,0})$. The point v^0 is the intersection between the line $z_2 = z^0_2$ and the line $z_1 + z_2 = f_l$ through l^0.
Then the points l^k , v^k , z^k, which provide better lower and upper bounds for $\bar{z}_1 + \bar{z}_2$, are constructed. For this purpose, a sequence of linear fractional programs are solved. For example, if we want to reduce the upper bound while the lower bound is not changed, we solve $\max\{F_i(x) : x \in S , F_1(x) \geq v^0_1\}$ and we obtain v^1 (in Fig. 1, v^1 is the point where the line joining the points v^0, l^0 , intersect the boundary of Z).
We obtain a new point $z^1 = (v^1_1, v^1_2)$ and so on.

It is possible that the algorithm stalls since neither lower nor upper bound change. In this case the point ($v^k{}_1$, $l^k{}_2$) is added to triangle (l^k , v^k , z^k) and with these points a square is constructed which is divided vertically into two equally size rectangles. The value of $\bar{z}_1 + \bar{z}_2$ can be increased in these rectangles or not. If $\bar{z}_1 + \bar{z}_2$ can be increased then a smaller triangle containing $\bar{z}$ can be found by solving a linear fractional program and the algorithm is restarted; otherwise the algorithm must be applied separately to each rectangle.

5. On the maximization of a linear and a linear fractional function

Since a lot of problems of pratical interest involve a linear and a linear fractional function, in this section we consider a particular case of the bicriteria linear fractional problem, namely :

$$\max_{x \in S^*} F(x) = \left(c_1x + c_{01}, \frac{c_2x + c_{02}}{d_2x + d_{02}} \right) \tag{5.1}$$

Such problem was considered by Dormány [23] which applied the simplex algorithm when the problem is reduced to a dual-parametric linear programming problem.

For problem (5.1) the following cases will be considered for the synthesis-function U:

$$\max_{x \in S^*} G_1(x) = c_1x + c_{01} + \frac{c_2x + c_{02}}{d_2x + d_{02}} \tag{5.2}$$

$$\max_{x \in S^*} G_2(x) = (c_1x + c_{01}) \frac{c_2x + c_{02}}{d_2x + d_{02}} \tag{5.3}$$

$$\max_{x \in S^*} \{G_3(x) = \min \{ c_1x + c_{01} ; \frac{c_2x + c_{02}}{d_2x + d_{02}} \}\} \tag{5.4}$$

where S^* is a nonempty polyhedron without degenerate vertices.

Problem (5.3), for instance, appears when the remuneration fund and the profitableness of an economic enterprise would be optimized.

Sometimes it is possible for a function to take a small value for an efficient solution, but it is not acceptable in practice. For such a reason a problem of the form (5.4) is considered. In turn, Belen'kiî [4] considers the minimization of the maximum function of a linear function and a linear fractional function and proposes a finite method for its solving; many optimization problems arising in the automatic control of diverse objects reduce to such a problem.

Also, the simultaneous optimization of absolute and relative terms [66] can be expressed as a problem of type (5.2); for instance, the simultaneous maximization of profit and return on investment may give rise to the following model

$$\underset{x \in S}{\text{Max}} \; \lambda f_1(x) + \frac{f_1(x)}{g_1(x)} \; , \quad \lambda \neq 0$$

whereas the maximization of a weighted sum of risk and expected return/risk may give rise to

$$\underset{x \in S}{\text{Max}} \; \mu f_1(x) + \frac{g_1(x)}{f_1(x)} \; , \quad \mu \neq 0.$$

In general, the problems of the form (5.1) are met within problems with a quantitative criterion (expressed by a linear function) and a qualitative criterion (expressed by a linear fractional function).

Guerra and Verdaguer [34] present a parametric algorithm in order to find efficient points of problem (5.1). The linear function is transformed in a parametric constraint. The solutions of the parametric problem offer efficient points for problem (5.1).

Now we present some methods for solving problems (5.2) and (5.3).

On solving problem (5.2)

In literature, several methods for solving problem (5.2) are given .

In [54] Martein proposed an algorithm, which works for any feasible region, based on the concept of optimal level solution defined as follows:

Definition 5.1

A feasible point $x^* \in S$ is said to be an optimal level solution if it is an optimal solution of the problem

$P(\xi) : \left\{ \frac{1}{\xi} \max[(\xi c_1 + c_2)x + c_{02}] \right\} = z(\xi), \; x \in S_\xi$

where $S_\xi = \left\{ x \in \mathbf{R}^n : Ax = b, \; d_2x + d_{02} = \xi \, , \; x \geq 0 \right\}$.

Let x_0 be an optimal level solution of the linear problem $P(\xi_0+\theta)$, where ξ_0 is a feasible level, that is such that $S_{\xi_0} \neq \emptyset$, with corresponding basis B, where $\xi_0 = d_2x_0 + d_{02}$; we partition the vectors c_1, c_2 as $c_1 = (c_{1B}, c_{1N})$, $c_2 = (c_{2B}, c_{2N})$ and the matrix A as $A = [B | N]$.

Set $\bar{c}_{1N} = c_{1N} - c_{1B}B^{-1}N$, $\bar{c}_{2N} = c_{2N} - c_{2B}B^{-1}N$.

The following parametric problem is considered:

$$P(\xi_0+\theta) : \left\{ \frac{1}{\xi_0+\theta} \max[(\xi_0+\theta)c_1 + c_2)x + c_{02}] \right\},\ x \in S_{\xi_0+\theta}$$

The idea of the algorithm is to find a local maximum point for problem (5.2) by generating a finite sequence of optimal level solutions; such a finite sequence is found by testing feasibility and optimality conditions with respect to the parameter θ.

More exactly, set:

$w = B^{-1}e^{m+1}$ where $e^{m+1} = (0,\dots, 0,1)$, $\overset{0}{\mu}_B = (\xi_0 c_{1B} + c_{2B})\, w$, $\overset{0}{\lambda}_B = c_{1B}w$,

$c_0 = c_1x_0$, $z_0 = z(\xi_0)$, $x(\theta) = x(\xi_0+\theta) = x_0 + \theta w$, $H_B(\theta) = \{\theta : x(\theta) \geq 0\}$,

$K_B(\theta) = \{\theta : (\xi_0+\theta)\, \bar{c}_{1N} + c_2)x + \bar{c}_{2N} \leq 0\}$, $U_B(\theta) = H_B(\theta) \cap K_B(\theta)$,

$$z(\theta) = \frac{\xi_0 z_0 + (\overset{0}{\mu}_B + c_0)\theta + \overset{0}{\lambda}_B\, \theta^2}{\xi_0+\theta}$$

and $\hat{\theta}$ the positive root (if one exists) of the derivative of $z(\theta)$.

The following Theorem gives suitable optimality conditions.

Theorem 5.1

i) If it results $\overset{0}{\mu}_B = z_0 - c_0$ and $\overset{0}{\lambda}_B \leq 0$, then x_0 is a local maximum point for problem (5.2) (as a particular case x_0 is a global maximum point if $\overset{0}{\lambda}_B = 0$);

ii) If $\hat{\theta} \in U_B(\theta)$ then $x(\hat{\theta})$ is a global maximum point for problem (5.2);

iii) If x_0 is a vertex of S and there exist two different basis B_1, B_2 such that $\overset{0}{\mu}_{B_1} < z_0 - c_0$ and $\overset{0}{\mu}_{B_2} > z_0 - c_0$ (or $\overset{0}{\mu}_{B_1} > z_0 - c_0$ and $\overset{0}{\mu}_{B_2} < z_0 - c_0$) then x_0 is a local maximum point for problem (5.2).

The algorithm suggested by Martein [54], is the following:

Step 0.
Solve the linear problem $\min_{x \in S} (d_2x + d_{02}) = \xi_0$ and problem $P(\xi_0)$; go to step 1.

Step 1. Calculate z_0, c_0, μ_B^0, λ_B^0. If $\mu_B^0 = z_0 - c_0$, then x_0 is a local maximum point for (5.2) and go to step 4.

If $\mu_B^0 \neq z_0 - c_0$, find $U_B(\theta)$. If $U_B(\theta) = [\theta^*, 0]$, then go to step 2, otherwise calculate $\hat{\theta}$. If $\hat{\theta} \in U_B(\theta)$, stop: $x(\hat{\theta})$ is a global maximum point for (5.2).

If $\hat{\theta} \notin U_B(\theta)$ and $U_B(\theta) = [\theta_1, +\infty[$, stop: problem (5.2) does not have optimal solutions.

If $\hat{\theta} \notin U_B(\theta)$ and $U_B(\theta) = [\theta_1, \theta^*]$, set $\theta = \theta^*$ and go to step 3.

Step 2. Find a new optimal feasible basis and go to step 1. If such a basis does not exist then x_0 is a local maximum point for (5.2) and go to step 4.

Step 3. If θ^* is an extremum point for $H_B(\theta)$ then go to step 1, otherwise go to step 2.

Step 4. Find a basis B such that $\bar{\theta} = \dfrac{z_0 - \mu_B^0 - c_0}{\lambda_B^0} > 0$ and go to step 5; if such a basis does not exist stop: x_0 is a global maximum point for (5.2).

Step 5. Solve $P(\xi_0 + \theta)$; if such a problem does not have solutions, stop: x_0 is a global maximum point for (5.2), otherwise go to step 1.

Since the set of local, non global, maxima is finite, the procedure of Martein finds a global maximum in finitely many steps or it shows that the objective function is not upper bounded.
Two variants of the Martein's algorithm are suggested in [9].
In [26] Ellero and Moretti Tomasin present some theoretical properties of the problem and a sensitivity algorithm for a class of local optimum points.
See also Ellero [25].

Ritter [63] gives an algorithm for a more general case of problem (5.2) i. e.

$$\max_{x \in S} G_1^*(x) = a'(c_1x + c_{01}) + a'' \frac{c_2x + c_{02}}{d_2x + d_{02}}$$

where a' and a" are real nonnegative numbers.

Krupitskiî [48] presents a parametric algorithm for the case in which the feasible set is not necessary bounded.

Chatterjee and Sen [16], with the aim of solving a bicriteria linear fractional problem with a linear objective function in which $c_{01} = 0$, consider problem (5.1), form a convex combination of the two objective functions and present an algorithm for solving the following problem:

$$\max_{x \in S^*} F(x) = \alpha c_1 x + (1-\alpha) \frac{c_2 x + c_{02}}{d_2 x + d_{02}}, \quad \alpha \in (0,1) \tag{5.5}$$

where $S^* = \{x \in R^n : Ax = b,\ x \geq 0\}$.

Chatterjee and Sen [16] approached directly this problem starting from a basic feasible solution x_B corresponding to the basis B and they established a condition under which the solution can be improved and also conditions for optimality criteria.

Denote $c_1x = z_1$, $c_2x + c_{02} = z_2$ and $d_2x + d_{02} = z_3$ and let $z_0^{(1)} = c_{1B}x_B$, $z_0^{(2)} = c_{2B}x_B + c_{02}$ and $z_0^{(3)} = d_{2B}x_B + d_{02}$

where c_{1B}, c_{2B} and d_{2B} denote the components of the vectors c_1, c_2 and d_2 associated with the basic variables. We assume to know the following vectors:
$y_j = B^{-1}a_j$; $z^{(1)}_j = c_{1B}\, y_j$; $z^{(2)}_j = c_{2B}\, y_j$; $z^{(3)}_j = d_{2B}\, y_j$.

Set $\theta = \min \{\frac{x_{Bi}}{y_{ij}} : y_{ij} > 0\}$ and let $z_0 = \alpha c_{1B}x_B + (1-\alpha) \frac{c_{2B}x_B + c_{02}}{d_{2B}x_B + d_{02}}$,
be the value of the objective function for the feasible solution $x_B = B^{-1}b$.

Theorem 5.2 [16]

Let x_B be a nondegenerate basic feasible solution for problem (5.5). If for every column a_j in A, either the condition

$z^{(1)}_j + \frac{1-\alpha}{\alpha} \frac{z^{(2)}_2}{z^{(3)}_0} \geq c_{1j} + \frac{1-\alpha}{\alpha} \frac{c_{2j}}{z^{(3)}_0}$ holds when $z^{(3)}_j - c_{3j} \geq 0$ or the condition

$$\theta (z^{(1)}_j - c_{1j}) - \frac{1-\alpha}{\alpha} \frac{2\theta z^{(2)}_0 (z^{(3)}_j - c_{3j}) - \theta z^{(3)}_0 (z^{(2)}_j - c_{2j}) - \theta^2 (z^{(2)}_j - c_{2j})(z^{(3)}_j - c_{3j})}{(z^{(3)}_0)^2 - (\theta(z^{(3)}_j - c_{3j}))^2} \geq 0$$

holds, when $z^{(3)}_j - c_{3j} \leq 0$, then z_0 is the maximum value of problem (5.5) and the nondegenerate basic feasible solution x_B is an optimal basic feasible solution.

Konno and Kuno [46] consider generalized linear fractional programming problems

$$P_1 : \quad \text{minimize } f_1(x) = g(x) + \frac{c_1x + c_{01}}{c_2x + c_{02}}$$

subject to

$$x \in S^{**} = \{x : Ax \geq b\}$$

and

$$P_2 : \quad \text{minimize } f_2(x) = g(x) - \frac{c_1x + c_{01}}{c_2x + c_{02}}$$

subject to

$$x \in S^{**} = \{x : Ax \geq b\}$$

where g is a convex function on S^{**}. When $g \neq 0$ the objective functions f_1 and f_2 are no longer quasiconvex nor quasiconcave, so that multiple local optima can occur.

Assuming

$$c_1x + c_{01} > 0 \text{ and } c_2x + c_{02} > 0 \quad \forall x \in S^{**} \tag{5.6}$$

Konno and Kuno [46] embedded P_1 and P_2 into an (n+1)-dimensional master problem and then applied a parametric associated approach.

The master problem associated with P_1 is

$$P_3 : \quad \min F_1(x, \xi) = g(x) + \xi \frac{(c_1x + c_{01})^2}{2} + \frac{1}{2\xi(c_2x + c_{02})^2}$$

subject to

$$(x, \xi) \in S' = \{(x, \xi) : Ax \geq b, \ \xi > 0\}$$

while the master problem associated with P_2 is

$$P_4 : \quad \min F_2(x, \xi) = g(x) - 2\xi \sqrt{c_1x + c_{01}} + \xi^2 (c_2x + c_{02})$$

subject to

$$(x, \xi) \in S'.$$

The objective functions F_1 and F_2 are convex for fixed values of $\xi > 0$ and if (x^*, ξ^*) is an optimal solution of P_3 (P_4), then x^* is an optimal solution of P_1 (P_2) under assumption (5.6).

In this way the algorithm developed by Konno and Kuno [46] for generalized linear multiplicative programming problem can be adopted for problems P_1 and P_2.

On solving problem (5.3)

Consider problem (5.3) where S^* is a nonempty compact polyhedron without degenerate vertices, and assume that $c_1x + c_{01} > 0$ and $d_2x + d_{02} > 0 \ \forall x \in S^*$. Hirche [41] showed that if the set of the optimal solutions of (5.3) is nonempty then there exists an optimal solution of (5.3) which belongs to an edge of the polytope S^*.

The problem (5.3) was also studied by Ellero and Moretti Tomasin [27] , [28]; they introduced the following definitions:

Definition 5.2

The real number ξ is said a feasible level for (5.3) if there exists $x^* \in S^*$ such that $c_1x^* + c_{01} = \xi$.

One can easily see that since S^* is a compact set, the set of the feasible levels (denoted by K) is an interval of $\mathbf{R}^+$.

Definition 5.3

The point $x^* \in S^*$ is an optimal level solution (briefly : o.l.s.) of (5.3) for the level $\xi = c_1x^* + c_{01} \in K$ iff x^* is an optimal solution of the following linear fractional programming problem

$$PLF(\xi) : \max_{x \in S_\xi} G_2(x) = \xi \frac{c_2x + c_{02}}{d_2x + d_{02}}$$

where $S_\xi = \{x \in S^* : c_1x + c_{01} = \xi\}$.

Let L_ξ be the set of optimal level solutions of (5.3) for the level ξ and let $L = \bigcup_{\xi \in K} L_\xi$ be the optimal level solutions set of (5.3).

It is easy to proof that if $x^* \in S^*$ is a global maximum point for G_2 on S^* then $x^* \in L_\xi \subseteq L$ (where $\xi = c_1x^* + c_{01}$) and if $L_\xi \neq \emptyset$ then there exists at least one point of L_ξ which belongs to an edge of S^*.

Ellero and Moretti Tomasin [28] proposed a simplex-like algorithm for solving the problem (5.3) when the set S^* is bounded.

The algorithm is based on the exploration of the set of optimal level solutions and it consists by the following steps.

Step 1 Determine an optimal solution x_0 of the problem

P_0 : min $\{ c_1x + c_{01} : x \in S\}$ and let $\xi = c_1x_0 + c_{01}$ (ξ is the current level).

Determine an optimal solution x_{0c} of the problem PLF(ξ) and let B the corresponding basis. Set $x_{0c} = x_0$ (x_{0c} is the current optimal solution).

Step 2. Let s be the edge of S* identified by the basis B. Compute the nonnegative numbers υ_A and υ_{ols}, representing the highest value with respect to feasibility and level optimality, respectively, of the points of s.
If $\upsilon_A > 0$ and $\upsilon_{ols} > 0$ then go to step 3; otherwise go to step 4.
Step 3. Set $\hat{\upsilon} = \min \{\upsilon_A, \upsilon_{ols}\}$. Compute the point υ_0 of global optimum of the restriction $v(\xi + \upsilon)$ of the objective function on the edge s with $\upsilon \in [0, \hat{\upsilon}]$.
If $\upsilon_0 \neq 0$ and $G_2(x(\upsilon_0)) > G_2(x_{0c})$ then $x_{0c} = x(\upsilon_0)$, $\xi = \xi + \hat{\upsilon}$ and go to step 2.
Step 4. Compute (if one exists) an optimal basic solution x_0 for PLF(ξ) chosen among the optimal basic solutions not yet explored and let B the corresponding basis. Go to step 2.
If there are no solutions of PLF(ξ) not yet explored then STOP : x_{0c} is a solution of problem (5.3).
Remark : According to Ellero and Moretti Tomasin [28] the previous algorithm can be adapted for solving problem (5.2). But unlike the algorithms of Cambini et al. [9] and Martein [54] this algorithm determine the maximum when the set S* is bounded.
In another paper [27] Ellero and Moretti Tomasin give a different definition of optimal level solutions and give an algorithm for solving the problem (5.3) even if S* is unbounded; in their approach the function $\frac{c_2x + c_{02}}{d_2x + d_{02}}$ is parametrized.

6. Bicriteria fractional transportation problem

As is known, some fractional transportation problems can be viewed as particular fractional problems, so that the previous approaches can be applied to these. However, there are very few specific results referred to this problem. In this section we present some of the results obtained in bicriteria fractional transportation problems.
Consider the feasible region S of the classical transportation problem defined by

$$\sum_{j=1}^{n} x_{ij} = a_i \ , \ i = 1, ..., m$$
$$\sum_{i=1}^{m} x_{ij} = b_j \ , \ j = 1, ..., n$$
$$x_{ij} \geq 0, \ i = 1, ..., m \ ; j = 1, ..., n$$

where x_{ij} represents the amount of the commodity to be shipped from source i to destination j, a_i is the available quantity at source i and b_j denotes the demand level at destination j. We assume that $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

Kanchan et al. [44] considered the problem of minimization on S of the function

$$f(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_{ij}}{\sum_{i=1}^{m} \sum_{j=1}^{n} q_{ij} x_{ij}} \qquad (6.1)$$

where it is assumed that $\sum_{i=1}^{m} \sum_{j=1}^{n} q_{ij} x_{ij} > 0$ for all feasible solutions $X = (x_{ij})$.

They suppose that c_{ij}, d_{ij} and q_{ij} are linearly dependent but c_{ij} and q_{ij} are linearly independent so that the objective function f(X) is quasimonotonic. Hence,

i) the optimal solution occurs at an extreme point of the feasible set;

ii) local optimum is global optimum.

Based on these properties Kanchan et al. [44] proposed a simplex-like algorithm for solving the problem of minimizing the function (6.1).

However, Hirche [38] found out an example in which the algorithm proposed in [44] does not converge, so that the objective function does not attain its minimum at an extreme point of the feasible set. He draw the conclusion that the minimizer of problems with explicitly quasiconvex functions would be determined by a simplex-like algorithm followed by a method of descent.

Misra and Das [57] generalized the results presented in [44] for three and multi-index transportation problem with the objective function of the form (6.1).

Let us consider now the case of two fractional objective functions

$$f_1(X) = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} l_{ij} x_{ij}}{\sum_{i=1}^{m} \sum_{j=1}^{n} n_{ij} x_{ij}} \quad \text{and} \quad f_2(X) = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} x_{ij}}{\sum_{i=1}^{m} \sum_{j=1}^{n} n_{ij} x_{ij}}$$

where $\sum_{i=1}^{m} \sum_{j=1}^{n} l_{ij} x_{ij} \geq 0$; $\sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} x_{ij} > 0$ and $\sum_{i=1}^{m} \sum_{j=1}^{n} n_{ij} x_{ij} > 0$.

We can construct a synthesis-function of the form

$$F(X) = f_1(X)\, f_2(X) = \frac{\left(\sum_{i=1}^{m} \sum_{j=1}^{n} l_{ij} x_{ij}\right)\left(\sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} x_{ij}\right)}{\left(\sum_{i=1}^{m} \sum_{j=1}^{n} n_{ij} x_{ij}\right)^2} \tag{6.2}$$

The function (6.2) is explicitly quasiconcave on S and also pseudoconcave. The local minimum occurs at a basic feasible solution but the local minimum is not global .

Sharma [69] considers the problem of minimization the function (6.2) on S and provides a simplex-like algorithm ensuring a local minimum.

Stancu-Minasian [71] generalized the results presented in [69] for three and multi-index transportation problem and Sharma [70] for three-index.

Furthermore, Stancu-Minasian [72] considers the problem of minimizing the function (6.2) over a feasible set S obtained by adding constraints of the form

$0 \leq x_{ij} \leq g_{ij}$, $i = 1, \ldots, m; j = 1, \ldots, n.$

Such kind of problems are then formulated as three-dimensional problems (without the condition of capacity of quantity which is transported) and solved by a method similar to the one described in [71] using a variable transformation method.

Chandra and Saxena [15] present a technique for shipment completion date-total shipping cost tradeoffs in the quadratic fractional transportation problem

$$\underset{X \in S}{\text{minimize}}\ F_1(X) = \frac{\left(\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \alpha\right)^2}{\left(\sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_{ij} + \beta\right)^2} \tag{6.3}$$

$$\underset{X \in S}{\text{minimize}}\ \{F_2(X) = \max\ \{t_{ij} : x_{ij} > 0\}\} \tag{6.4}$$

where t_{ij} is the time of transportation from the i^{th} source point to the j^{th} destination point and it is independent of the amount of commodity transported so long as $x_{ij} > 0$.

They propose an algorithm which generates all solutions which are Pareto-optimal with respect to cost and completion date. The algorithm is similar to the one elaborated for the linear case by Glickman and Berger [33].

Denote problem (6.3) by (P) (initially P_0) and let Z^* be the optimal value of (P). If $X^*_k = (x^*_{ij})_k$ is one of the K alternate optimal solutions of (P), we consider the set $S_k = \{(i,j) : x^*_{ij} \in X^*_k , x^*_{ij} > 0\}$ and $T^* = \min_{1 \le k \le K} \max_{(i,j) \in S_k} t_{ij}$.

Let us note that if problem (P) has not alternate optimal solutions, then $T^* = \max_{(i,j) \in S_k} \{t_{ij} : x^*_{ij} > 0\}$.

The algorithm finds, iteratively, all optimal schedules with earliest completion times less than T*, till no other feasible schedule is found on the permissible routes.

The steps of the algorithm are as follows:

Step 1 Determine the set of all optimal solutions $X^* = \{x^*_{ij}\}$ for problem P_0 using Aggarwal's method [1].

Step 2 Calculate Z^*_0 and T^*_0.

Step 3 Modify the cost matrices $[c^0_{ij}]$ and $[d^0_{ij}]$ to get the problem (P_1), in the following way:

$$c^1_{ij} = \begin{cases} M \text{ (arbitrarily large)} & \text{if } t \ge T^*_0 \\ c^0_{ij} & \text{if } t < T^*_0 \end{cases}$$

$$d^1_{ij} = d^0_{ij}, \text{ for all } (i, j)$$

Step 4 Optimize (P_1) using the optimal solution of (P_0) and a reoptimizing procedure similar to Glickman and Berger [33]

Step 5 If (P_1) has a feasible solution for the permissible routes, then new values (Z^*_1, T^*_1) are obtained such that $Z^*_1 > Z^*_0$ and $T^*_1 < T^*_0$.

The procedure is repeated: at each iteration we get (Z^*_2, T^*_2), (Z^*_3, T^*_3) ... till no other feasible solution is found on the permissible routes.

Saxena [64] applied previously the same algorithm (with minor modifications) to a problem in which the objective functions are (6.1) and (6.4).

Gupta and Puri [35], consider the problem

$$\max_{X \in S} z(X) = \frac{\left(\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \alpha\right)\left(\sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_{ij} + \beta\right)}{\sum_{i=1}^{m} \sum_{j=1}^{n} e_{ij} x_{ij} + \gamma} = \frac{f_1(X) f_2(X)}{f_3(X)} \qquad (6.5)$$

where $f_1 \geq 0, f_2 > 0, f_3 > 0$ for all $X = (x_{ij}) \in S$ and f_1, f_2 are of non-conflicting nature i.e. $f_1(X^1) > f_1(X^2)$ if and only if $f_2(X^1) > f_2(X^2)$.
The function z is a pseudoconvex function. The optimal solution of problem (6.5) appears at an extreme point of S, The problem (6.5) is shown to be related to "indefinite" quadratic programming (IQP) which deals with maximization of a convex function over S for which a local maximum is a global maximum (under certains conditions) and its optimal solution provides an upper bound on the optimal value of problem (6.5). The extreme point solutions of (IQP) are ranked to tighten the bounds on the optimal value of (6.5) and a convergent algorithm is developed to obtain the optimal solution.

Basu et al. [3] have developed an algorithm to find the optimum time-cost trade-off in a three-dimensional two-objective transportation problem, where one of the objectives functions is the sum of a linear and a quadratic fractional function and the other function is non linear. In their procedure, the possible time-cost trade-off pairs are determined first and then the optimum transportation plan is arrived at by using trade-off ratios.

As we have previously point out, it is clear that not much work has been done in bicriteria transportation fractional programming.

7. On generating the set of all efficient solutions for the bicriteria linear fractional problem

In this section we present some sequential methods suggested for solving the bicriteria linear fractional problem P, that is in finding the set of all efficient solutions of P. Since a convex combination of the two objectives functions does not have, in general, particular properties, some authors elaborated parametrical methods for solving problem P. In such methods one of the two objective functions is transformed in a parametric constraint. The set of all efficient points is generated by means of a suitable post-optimality analysis.

For $i = 1,2$ set $\bar{f}_i = \max\{F_i(x) : x \in S\}$, $\underline{f}_1 = \max\{F_1(x) : x \in S, F_2(x) \geq \bar{f}_2\}$,

and $\underline{f}_2 = \max\{F_2(x) : x \in S, F_1(x) \geq \bar{f}_1\}$.

We have $\underline{f}_1(x) \leq F_1(x) \leq \bar{f}_1$ for $x \in E$.

For any $w \in [\underline{f}_1, \bar{f}_1]$, consider the following parametric problem:

$P(w) : \max\{F_2(x) : x \in S, F_1(x) \geq w\}$

The following results were given by Warburton [83].

Lemma 7.1

If x^* is efficient, then there exists a scalar $w \in [\underline{f}_1, \bar{f}_1]$ such that x^* solves $P(w)$.

Lemma 7.2

A solution of P may be found among the solutions of P(w) over the interval $\underline{f}_1 \leq w \leq \bar{f}_1$.

Choo and Atkins [19] showed that any solution of P(w) for $\underline{f}_1 \leq w \leq \bar{f}_1$ is efficient and that P(w) can be solved as a parametric linear program.

More exactly, by the variable change $y = tx$, the program P(w) is transformed in the following row parametric linear program Q(w)

$Q(w) : \max(c_2 y + c_{02} t)$

subject to

$$Ay - bt \leq 0$$
$$d_2 y + d_{02} t = 1$$
$$c_1 y + c_{01} t \geq w(d_1 y + d_{01} t), \quad y \geq 0, t \geq 0.$$

Choo and Atkins [19] give a parametric algorithm to evaluate the efficient frontier. The algorithm uses only one-dimensional parametric linear programming problem.

Parametric procedure of Choo and Atkins

Step 1 Solve (4.2) and (4.3) in order to obtain $\underline{f}_1$, $\bar{f}_1$.

Step 2 Starting from $\underline{f}_1 = w_0$ and applying row parametric procedure, determine w_i, i=1, ..., r, such that $w_0 \leq w_1 \leq \ldots \leq w_r = \bar{f}_1$ and the corresponding points (y_i , t_i) , where w_i occurs at the i-th basis change in the parametric solution of $Q(w_{i-1})$.

Step 3 Let $x_i = \frac{y_i}{t_i}$, i = 0, ..., r, be the corresponding solution of $P(w_i)$. Then

$$f(E) = \bigcup_{i=1}^{r} F[x_{i-1}, x_i]$$

Parametric procedure of Cambini and Martein

Unlike Choo and Atkins, Cambini and Martein consider the bicriteria linear fractional problem P for any feasible region (bounded or not). By means of the Charnes - Cooper transformation applied to one of the two linear fractional objective function (for instance the first one), problem P reduces to an equivalent bicriteria problem where one of the objective function is linear. We will refer to such a problem as:

$$P^* : \sup \left(ax , \frac{cx + c_0}{dx + d_0}\right) , \; x \in R=\{x: Ax = b , x \geq 0\}.$$

Consider the following scalar parametric problem

$$P^*(\theta) : \sup \frac{cx + c_0}{dx + d_0} = z(\theta) , \; x \in R(\theta)$$

where $R(\theta) = \{x: Ax = b , ax = L - \theta, x \geq 0\}$.

First of all, let us note that for any fixed θ, the linear fractional problem $P^*(\theta)$ can be solved by means of a simplex-like procedure which works for any feasible region, suggested by Cambini and Martein in [7 , 8].

The following procedure is utilized for finding L.

Calculate $\sup_{x \in S} ax = M$; if M is finite then solve the following problem

$\sup \frac{cx + c_0}{dx + d_0}$, $x \in S \cap \{x: ax \geq M\}$. If such a supremum is not finite then $E=\emptyset$, otherwise set $L = \sup \frac{cx + c_0}{dx + d_0}$, $x \in S \cap \{x: ax \geq M\}$.

If $M = +\infty$, there exists a feasible halfline r, whose equation is of the kind $x=x_0 + t\,u$, $t \geq 0$, such that $\sup_{x \in r} ax = +\infty$. Consider $\sup_{x \in r} \frac{cx + c_0}{dx + d_0}$. If such a supremum is not finite then $E = \emptyset$, otherwise set $L = \sup_{x \in r} \frac{cx + c_0}{dx + d_0}$.

The relationship between the optimal solutions of $P^*(\theta)$ and the set E of all efficient points of P^* is given [10, 55] by:

$E = \bigcup_{\theta \in [0, \theta_{max}]} S(\theta)$ where $S(\theta)$ is the set of optimal solutions of $P^*(\theta)$ (it can be proved that $S(\theta)$ is nonempty for any $\theta \geq 0$) and $\theta_{max} = +\infty$ or is such that z is increasing in $[0, \theta_{max}]$ and constant in $[\theta_{max}, +\infty]$; so, E can be generated by performing a suitable post-optimality analysis on $P^*(\theta)$.

For a fixed value $\hat{\theta}$ of the parameter, let x^* be an optimal basic solution of $P^*(\hat{\theta})$ with corresponding basis B; we partition the vectors x^*, c, and d as $x^* = (x^*_B, x^*_N)$, $c = (c_B, c_N)$, $d = (d_B, d_N)$ and the matrix $\hat{A} = [(A;\ a)]$ as $\hat{A} = [B \mid N]$.

Set $\bar{c}_N = c_N - c_B B^{-1} N$, $\bar{d}_N = d_N - d_B B^{-1} N$, $\bar{c}_0 = cx^* + c_0$, $\bar{d}_0 = dx^* + d_0$, $\gamma = \bar{d}_0\, \bar{c}_N - \bar{c}_0\, \bar{d}_N$, $\gamma(\theta) = \gamma - \theta\, w$ with $w = \lambda_0\, \bar{d}_N - \mu_0\, \bar{c}_N$, where λ_0 and μ_0 are the last components of the vectors $c_B B^{-1}$ and $d_B B^{-1}$, respectively;
$x^*_B(\theta) = x^*_B - \theta\, h$ where h is the last column of B^{-1}.

The parametric analysis is performed by studying the optimality condition $\gamma(\theta) \leq 0$ and the feasibility condition $x^*_B(\theta) \geq 0$. With regard to the optimality condition, set $I_1 = \{i : w_i < 0\}$; if $I_1 = \emptyset$ then $\gamma(\theta) \leq 0$ for any $\theta \geq 0$, otherwise $\gamma(\theta) \leq 0 \quad \forall \theta \in [0, \theta_1]$ where $\theta_1 = \min_{i \in I_1} \frac{\gamma_i}{w_i} = \frac{\gamma_k}{w_k}$.

With regard to the feasibility condition, set $I_2 = \{i : h_i > 0\}$.

If $I_2 = \emptyset$ then $x^*_B(\theta) \geq 0$ for any $\theta \geq 0$, otherwise $x^*_B(\theta) \geq 0 \quad \forall \theta \in [0, \theta_2]$ where $\theta_2 = \min_{i \in I_2} \frac{x^*_{B_i}}{h_i} = \frac{x^*_{B_j}}{h_j}$.

As a consequence, for any $\theta \in [0, \bar{\theta}]$, where $\bar{\theta} = \min\{\theta_1, \theta_2\}$, $x^*_B(\theta)$ is the optimal solution of the problem $P^*(\theta)$; when $\theta > \bar{\theta}$ and $\bar{\theta} = \theta_2$, feasibility is restored by means of dual-simple like algorithm; when $\theta > \bar{\theta}$ and $\bar{\theta} = \theta_1$, optimality is restored by means of Cambini and Martein's algorithm [7].
The sequential method suggested for solving problem $P^*(\theta)$, $\theta \in [0, \theta_{max}]$ is the following:

Step 0. Solve $P^*(0)$ and let $x^*_B(0)$ be an optimal basic solution; set $i = 0$ and go to step 1.

Step 1. Consider $P^*(\theta^i + \theta)$, $\theta \geq 0$; calculate $\gamma(\theta)$, $x^*_B{}^i(\theta)$, $\bar{\theta}$ and set $\theta^{i+1} = \theta^i + \bar{\theta}$; $x^*_B{}^{i+1} = x^*_B{}^i(\bar{\theta})$ is an optimal basic solution for $P^*(\theta^{i+1})$. If $z(\theta^{i+1}) = z(\theta^i)$, then $\theta_{max} = \theta_1$ stop; otherwise go to step 2.

Step 2 If $\bar{\theta} = \theta_1 < +\infty$ then x_{N_k} enters the basis by means of a simplex-like pivot operation; set $i = i+1$ and return to step 1.

If $\bar{\theta} = \theta_2 < +\infty$, then x_{B_j} must leave the basis and a pivot operation is performed on a_{tj} such that $\frac{\gamma_t(\bar{\theta})}{a_{tj}} = \min_{a_{ij}<0,\, i \in I_2} \frac{\gamma_i(\bar{\theta})}{a_{ij}}$; set $i = i+1$ and return to step 1. Otherwise $x^*_B{}^{i+1}(\theta)$ is optimal for $P^*(\theta^i + \theta)$, $\theta_{max} = +\infty$; stop.

When $F_1(x)$ and $F_2(x)$ have the same denominator $dx+d_0$, problem P^* reduces to a bicriteria linear problem as outlined also by Dutta et al. in [24] which compare their method with the one proposed by Nykowski and Zolkiewski [58] which reduce the problem to solving the following three criteria linear programming:

$$\max_{x \in S} \ (c_1x + c_{01}, c_2x + c_{02}, -dx - d_0) \quad \text{for } F_i(x) > 0 \ \ \forall x \in S, \ i = 1,2,$$

and

$$\max_{x \in S} \ (c_1x + c_{01}, c_2x + c_{02}, dx + d_0) \quad \text{for } F_i(x) < 0 \ \ \forall x \in S, \ i = 1,2.$$

The method of Dutta et al. [24] is computationally less cumbersome than the one of Nykowski et al. [58] which is suitable for problem with different denominators.

Applying the variable transfomation y = tx to problem P , where $t \geq 0$ is such that

$$d_i y + d_{0i} t = \alpha_i , \quad \alpha_i \in R, \ i = 1,2 ,$$

problem P becomes

$$\max (c_1 y + c_{01} t , c_2 y + c_{02} t)$$

subject to

$$Ay - bt \leq 0$$
$$d_i y + d_{0i} t = \alpha_i , \ i = 1,2$$
$$y \geq 0, t \geq 0$$

Let Y be the feasible set of this problem. It is possible for Y to be empty for any chosen α_i. Kall's theorem [82] gives some conditions for Y to be nonempty for any chosen α_i.

8. Bicriteria problems and bicriterion mathematical programs

In section 4 it has been pointed out that an efficient solution of the bicriteria problem P can be found by solving the bicriterion program (4.1), where U is a suitable utility function.

The problem can be also reversed: the knowledge of the set E of all efficient points can be used in order to find an optimal solution of (4.1) for a given function U?

Geoffrion [32] gives an answer to this question when F_1 and F_2 are real-valued concave functions of x and U is a real-valued increasing function with respect to each argument. He suggested a method for solving problem (4.1) based on any known parametric programming algorithm for the parametric program:

$$P(t) : \quad \max_{x \in S} \{t F_1(x) + (1-t)F_2(x)\}, \quad t \in [0,1],$$

The procedure suggested by Geoffrion can be applied only for functions F_1 and F_2 which are concave because the more general class of functions is not closed with respect to addition and multiplication with positive scalars.

The results of Geoffrion are extended by Marchi [51] to classes of non necessarily concave functions. She showed that the optimal solutions of problems P_{h_1} and P_{h_2} are contained in the set E_1 of all efficient solutions of P_{B_1} and the set E_2 of all efficient solutions of P_{B_2}, where

$$P_{h_1}: \max_{x \in S} h_1(F_1(x), F_2(x)) \; ; \qquad P_{h_2}: \max_{x \in S} h_2(F_1(x), F_2(x))$$

h_1 is an increasing function in each argument, h_2 is a function increasing in the first argument and decreasing in the other, F_1 and F_2 are continuous functions and

$$P_{B_1}: (\max_{x \in S} F_1(x), \max_{x \in S} F_2(x)) \quad \text{and} \quad P_{B_2}: (\max_{x \in S} F_1(x), \min_{x \in S} F_2(x)).$$

Also, Marchi [51] considered classes of problems more general than P_{h_1} and P_{h_2}, i.e.

$$P^*_{h_1}: \max_{x \in S} h_1(F(F_1(x)), G(F_2(x))) \; ; \qquad P^*_{h_2}: \max_{x \in S} h_2(F(F_1(x)), G(F_2(x)))$$

where h_1 and h_2 have the above properties and F and G are increasing functions.

For other results related to bicriteria problems see also [52], [53].

Pasternak and Passy [59] considered the case where U is a strictly quasiconcave function and where F_1 and F_2 are linear functions, extended the Geoffrion's method for including boolean variables.

Prasad et al. [62], developed an algorithm for maximization and minimization of bicriterion quasiconcave function $g(c_1x, c_2x)$ subject to linear constraints. The algorithm for maximization is based on bisection approach whereas the one for minimization is an implicit enumeration method.

9 Conclusions

In this paper we have reviewed the main theoretical results and computational approach obtained up to now in bicriteria fractional programming (BFP). From the review it is clear that much work has been done in (BFP). It is found that certain methods offer appreciable computational advantage over the others in terms of computer time requirement. Thus, there is an urgent need for computer based numerical experimentation to compare various BFP methods.

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This research has been supported by the Italian National Research Council (C.N.R.).

Adresses of the authors:

A. Cambini, L. Martein - Department of Statistics and Applied Mathematics - University of Pisa - Via Ridolfi 10 , 56100 Pisa (Italy).

I.M. Stancu-Minasian - Centre of Mathematical Statistics - Calea 13 Septembrie, Nr 13 - RO-76100 Bucharest 5 (Romania).