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On the Connectedness of the Efficient Frontier: Sets Without Local Maxima

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Abstract

In this paper the concept of Set Without Local Maxima is introduced. By means of this concept we shall investigate the connectedness of the efficient frontier for vector maximization problems defined by functions whose local maxima are global. Conditions under which the outcome of a vector function is a set without local maxima are established. Applications for bicriteria and three criteria problems are given.

0. Introduction

The connectedness of the efficient frontier for vector maximization problems is an important field of study because of its applications [3]. One of the most recent papers on this subject was given by Hu and Sun [2]. They introduced the concept of a strictly quasi concave set in order to prove the connectedness of the efficient outcome set. A crucial assumption in their work is the closure of this set. In order to consider a wider class of sets and relax the assumption of closure, we shall introduce, in this paper, the concept of set without local maxima and study some properties of this set. Relationships between a set without local maxima and a strictly quasi concave set are studied and some sufficient conditions for the connectedness of the efficient frontier are established in two dimensional and three dimensional spaces. The results obtained are more general than the ones given by Schaible [5] , Choo - Schaible - Chew [1] and Marchi [4] and the suggested approach allows us to find some of the results given by Warburton in [6], in a different way.

1. Sets Without Local Maxima and their properties

Let $T \subseteq \mathbb{R}^m$ be a not empty set. We shall consider the Projection function $P_i : T \to \mathbb{R}$ defined by $P_i(r) = r_i$ and we shall introduce the concept of Set Without Local Maxima.

Definition 1.1: $T \subseteq \mathbb{R}^m$ is said to be a Set Without Local Maxima (SWLM) iff P_i is a function whose local maxima[1] are global in T for any $i \in \{1,2,..,m\}$.

The upper level sets of functions P_i are:

$$L_{P_i}(\alpha_i) = \{ r \in T \mid r_i \geq \alpha_i \}$$

for every $\alpha_i \in G_i = \{ \alpha_i \in \mathbb{R} \mid L_{P_i}(\alpha_i) \neq \emptyset \}$, $i \in \{1,2,..,m\}$.

Definition 1.2: The point to set mapping $L_{P_i}(\alpha_i)$ is said to be lower semi continuous (lsc) at the point $\alpha_i \in G_i$ if $r \in L_{P_i}(\alpha_i)$, $\{\alpha^k_i\} \subset G_i$ $\{\alpha^k_i\} \to \alpha_i$ imply the existence of a natural number M and a sequence $\{ r^k \}$ such that $r^k \in L_{P_i}(\alpha^k_i)$ (with $r^k_i \geq \alpha^k_i$) $k = M, M+1,....$ and $\{ r^k \} \to r$.

Now we are able to prove the following theorem of characterization for sets without local maxima:

Theorem 1.1: $L_{P_i}(\alpha_i)$ is lsc $\forall \alpha_i \in G_i$ iff P_i is a function whose local maxima are global in T.

Proof: Directly, taking into account the results obtained by Zang in [7] regarding functions whose local maxima are global.

For the sake of simplicity, from now on we shall assume that T is a compact set.

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[1] **Definition:** A point $R^{i\text{-}loc} = (r_1^{loc}, r_2^{loc}, ..., r_m^{loc}) \in T$ is a local maximum for P_i in T but not global if there exists a $\delta > 0$ such that $P_i(r) \leq P_i(R^{i\text{-}loc})$, $\forall r \in B_\delta (R^{i\text{-}loc}) \cap T$, where $B_\delta(r)$ denotes an open ball with radius δ centered around r and there exists $\underline{r}$ such that $P_i(\underline{r}) > P_i(R^{i\text{-}loc})$.

Set $r_i^{i\text{-max}} = \max P_i(r),\ r \in T$ and $S_i = \{ r \in T \mid r_i = r_i^{i\text{-max}} \}$. Let us note, if T is a compact set then $S_i \neq \emptyset$ for any $i \in \{1,2,..,m\}$.

Theorem 1.2: T is a SWLM iff $r \in T$ and $r \notin S_i$ imply the existence of a natural number M and a sequence $\{ r^k \} \rightarrow r$ such that $r^k_i > r_i$ $k = M, M+1, \ldots$ for any $i = 1,.., m$.

Proof: Directly, taking into account Theorem 1.1 and the definition of local maximum for function P_i defined on T.

Theorem 1.3: Let $R^{i\text{-max}}$ be the unique element of S_i for any $i \in \{1,2,..,m\}$. If T is a SWLM then T is a connected set.

Proof: Suppose that T is disconnected; then there exist compact sets T_1 and T_2 of T such that $T = T_1 \cup T_2$, $T_1 \cap T_2 = \emptyset$. Consider the optimal solutions $\underline{r}^1$ and $\underline{r}^2$ of $\max P_i(r),\ r \in T_1$ and $\max P_i(r),\ r \in T_2$, respectively. Since $\underline{r}^1 \in T_1$ and $\underline{r}^2 \in T_2$, then $\underline{r}^1 \neq \underline{r}^2$. Taking into account that there exists a unique element belonging to S_i, $R^{i\text{-max}} = \underline{r}^1$ or $\underline{r}^2$. If $R^{i\text{-max}} = \underline{r}^1$ then $\underline{r}_i^1 > \underline{r}_i^2$. Since $\underline{r}_i^2 > r_i$ $\forall r \in T_2$ and $T_1 \cap T_2 = \emptyset$ then there exists a $\delta > 0$ such that $B_\delta(\underline{r}^2) \cap T_2 = B_\delta(\underline{r}^2)$ and $\underline{r}_i^2 > r_i$ $\forall r \in B_\delta(\underline{r}^2)$, where $B_\delta(r)$ denotes an open ball with radius δ centered around r. Hence, $\underline{r}^2$ is a local maximum for P_i on T and this contradicts the assumption.

2. Connectedness of E(T, $\mathbb{R}^m_+$) and WE(T, $\mathbb{R}^m_+$)

Let us consider the set of maximal elements of T denoted by E(T, $\mathbb{R}^m_+$), i.e.

$$E(T, \mathbb{R}^m_+) = \{ \underline{r} \in T \mid r \in T, \text{ such that } r \geq \underline{r}^2, \text{ does not exist} \},$$

and the set of weakly maximal elements of T denoted by WE(T, $\mathbb{R}^m_+$), i.e.

$$WE(T, \mathbb{R}^m_+) = \{ \underline{r} \in T \mid r \in T, \text{ such that } r > \underline{r}, \text{ does not exist} \}.$$

[2] For any $y^1, y^2 \subseteq \mathbb{R}^m$, $y^1 \geq y^2$ means $y^1 - y^2 \in \mathbb{R}^m \setminus \{0\}$, $y^1 > y^2$ means $y^1 - y^2 \in \operatorname{int} \mathbb{R}^m$.

In [2] the concept of Strictly Quasi Concave set is defined in the following way.

Let the set $Y \subset \mathbb{R}^m$ be non-empty.

Definition 2.1: Y is said to be Strictly Quasi Concave if for any $y^1, y^2 \in Y$ and $y^1 \neq y^2$, there exists some $z \in Y$ such that
i) $z \geq \min\{y^1, y^2\}$[3],
ii) $z_i > \min\{y^1_i, y^2_i\}, \forall i \in \{i \mid y^1_i \neq y^2_i\}$.

In the same way we are able to define the concept of Strongly Quasi Concave set:

Definition 2.2: Y is said to be Strongly Quasi Concave if for any $y^1, y^2 \in Y$, $y^1 \neq y^2$ there exists some $z \in Y$, $z \neq y^1 \neq y^2$ such that $z > \min\{y^1, y^2\}$.

As regards the connectedness of $E(T, \mathbb{R}^m_+)$, in [2] we have the following result.

Theorem 2.1: Let $T \subset \mathbb{R}^m$ be compact and $E(T, \mathbb{R}^m_+)$ be closed. If T is strictly quasi concave then $E(T, \mathbb{R}^m_+)$ is connected.

In order to relax the assumption of closure of set $E(T, \mathbb{R}^m_+)$ we will study for which sets this property is verified. First of all, we will establish under which conditions $E(T, \mathbb{R}^m_+) = WE(T, \mathbb{R}^m_+)$, since it is easy to prove the following theorem:

Theorem 2.2: If $X \subset \mathbb{R}^m$ is a closed set then $WE(X, \mathbb{R}^m_+)$ is a closed set.

Lemma 2.1: If S_i has a unique element, $R^{i\text{-max}}$, then $R^{i\text{-max}}$ is a maximal element of T, i = 1,2, ..m.

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[3] Let $y^j = (y^j_1, y^j_2, \ldots, y^j_m)$ j = 1,2 be two arbitrary points of $\mathbb{R}^m$, we denote by $\min\{y^1, y^2\}$ the infimum of y^1 and y^2, i.e. the greatest lower bound of y^1 and y^2 in the order generated by $\mathbb{R}^m_+$.

Proof: If $R^{i\text{-max}} \in T$ is the unique element belonging to S_i then $r \in T$, $r \neq R^{i\text{-max}}$ such that $r_i \geq R_i^{i\text{-max}}$ does not exist. Hence, $r \in T$ such that $r \geq R^{i\text{-max}}$ does not exist then $R^{i\text{-max}}$ is a maximal element of T.

Suppose T is a SWLM, the following theorems show under which conditions $E(T, \mathbb{R}^m_+) = WE(T, \mathbb{R}^2_+)$ and, consequently, we have the closure of $E(T, \mathbb{R}^m)$. Suppose S_i has a unique element $R^{i\text{-max}}$ $i = 1,..,m$ we have the following result:

Theorem 2.3: Let $T \subset \mathbb{R}^m$ be a SWLM and $w^0 \in T$ a weakly maximal element of T. If there do not exist other weakly maximal elements in the set $w^0 + \mathbb{R}_+^m$ except on the (n-1) dimensional face of $w^0 + \mathbb{R}_+^m$ then w^0 is a maximal element of T.

Proof: Taking into account Lemma 2.1, if $w^0 \in S_i$ then w^0 is a maximal element of T. Otherwise, if $w^0 \in T$ is a weakly maximal element of T then $r \in T$ such that $r > w^0$ does not exist and we have to prove that w^0 is a maximal element of T, i.e. $r^* \in T$ such that $r^* \geq w^0$ does not exist. We suppose that $r^* \in T$ such that $r^* \geq w^0$ exists. Taking into account the assumptions, we have that there exists an index j such that $w^0_j = r_j^*$ and $w^0_i < r_i^*$ for any $i \in \{1,2,..,m\}$ $i \neq j$. Hence, since $r > w^0$ does not exist, there exists a neighbourhood I_{r^*} of r^* such that $r_j^* \geq r_j$ for any $r \in I_{r^*} \cap T$, then r^* is a local maximum for function P_j on T. This is absurd since T is a set without local maxima.

Corollary 2.1: Let $T \subset \mathbb{R}^m$ be a SWLM and $w^0 \in T$ a weakly maximal element. If there do not exist other weakly maximal elements in the set $w^0 + \mathbb{R}_+^m$ except on the (n-1) dimensional face of $w^0 + \mathbb{R}_+^m$ then $E(T, \mathbb{R}^m_+) = WE(T, \mathbb{R}^2_+)$ is a closed set.

Proof: Directly from Theorems 2.2 and 2.3.

Regarding the Strongly Quasi Concave set, we have the following result:

Lemma 2.2: If $T \subset \mathbb{R}^m$ is a Strongly Quasi Concave set then:

i) T is a Strictly Quasi Concave set,

ii) S_i has a unique element $R^{i\text{-max}}$ for any $i \in \{1,2,..,m\}$.

Proof: i) Directly from definitions; ii)We shall suppose, ab absurd, that there exist y^1, $y^2 \in S_i$, i.e. $y^1_i = y^2_i$ and a $z \in T$ such that $y^1_i = y^2_i < z_i$ does not exist. If T is a Strongly Quasi Concave set then there exists $z \in T$, $z \neq y^1 \neq y^2$ such that $z > \min \{ y^1, y^2 \}$, hence, there exists a $z \in T$ such that $y^1_i = y^2_i < z_i$. This contradicts the assumption.

Let us note that result ii) of Lemma 2.2 is not true for the class of Strictly Quasi Concave sets.

Theorem 2.4 and 2.5 below will tell us the relationships between Strictly Quasi Concave sets and sets Without Local Maxima when T is a subset of $\mathbb{R}^3$ and $\mathbb{R}^2$.

Set $\text{Int}(Lp_i(\alpha_i)) = \{ r \in T \mid r_i > \alpha_i \}$ and $\text{Fr}(Lp_i(\alpha_i)) = \{ r \in T \mid r_i = \alpha_i \}$.

Remark 2.1: Let us note that if $r^* \in \text{Int}(Lp_i(\alpha_i))$ then for every sequence $\{ r^k \} \to r^*$ there exists M>0 such that $r^k_i > r^*_i$, $k = M, M+1, \ldots..$

Theorem 2.4: Let $T \subset \mathbb{R}^3$ be a SWLM and y^1, y^2 two maximal elements with one equal component. If there exists a $z > \min \{y^1, y^2\}$ then T is a Strictly Quasi Concave set.

Proof: We must prove that for any A, B $\in$ T and $A \neq B$, there exist some z $\in$ T, such that

i) $z \geq \min \{ A, B \}$,

ii) $z_i > \min \{ A_i, B_i \}$, $\forall i \in \{ i \mid A_i \neq B_i \}$.

If $A \geq B$ then $\min \{ A, B \} = A$ then $z = B$ verifies i) and ii).

If the condition $A \geq B$ is not verified then we can have the following cases:

a) $\min \{ A, B \} = [A_1, A_2, B_3]$ with $A_1 < B_1$, $A_2 < B_2$ and $A_3 > B_3$. Let us consider $Lp_1(A_1) = \{ r \in T \mid r_1 \geq A_1 \}$ and $Lp_2(A_2) = \{ r \in T \mid r_2 \geq A_2 \}$, we have $B \in \text{Int}(Lp_1(A_1))$ and $B \in \text{Int}(Lp_2(A_2))$, this means, for Remark 1.1, that there exits M>0 such that in every sequence $\{ r^k \} \to B$ $r^k_1 > A_1$ and $r^k_2 > A_2$ $\forall k = M, M+1, \ldots..$ Taking into account Theorem 1.1, $Lp_3(B_3)$ is lower semi continuous then for every $r \in Lp_3(B_3)$ and $\{ r^k_3 \} \to B_3$ there exists a M_3 and a sequence $\{ r^k \} \to r$ such that $r^k \in Lp_3(r^k_3)$, $\forall k = M_3, M_3+1, \ldots$ Let us consider the sequence $\{r^k_3\} = \{1/k\, A_3 + (1-1/k)\, B_3\}$. Taking into account B$\in$

L_{P3} (B_3) and { $r^k{}_3$ }→ B_3 then there exists a M_3 and a sequence { r^k } → B such that $B_3 < r^k{}_3 < A_3$, ∀ k =M_3,M_3+1..... This means that z = r^k ∀ k > min (M_3,M) verifies i) and ii).

b) min { A , B } = [A_1 , $A_2 = B_2$, B_3] with $A_1 < B_1$ and $A_3 > B_3$ then A ∈ Int (L_{P3} (B_3)), B ∈ Int (L_{P1} (A_1)) and A, B∈ Fr (L_{P2} (A_2)). Let us consider the optimal solution D∈T of the following problem:

$$D_2 = \max_{r_1 \geq A_1, r_3 \geq B_3} r_2 , r \in T.$$ We have that $D_2 \geq A_2 = B_2$.

If $D_2 > A_2 = B_2$ ($D_1 > A_1$ and $D_3 > B_3$) then z = D verifies i) and ii), otherwise A and B are maximal elements of T with one equal component. Taking into account the assumption, a z ∈ T exists such that z > min {A, B}.

In the particular case when T is a subset of $\mathbb{R}^2$ we have:

Theorem 2.5: If T⊂ $\mathbb{R}^2$ is a SWLM then T is a Strictly Quasi Concave set.

Proof: It is sufficient to observe that if T⊂ $\mathbb{R}^2$ is a SWLM the assumption of Theorem 1.4 is verified. In fact, we suppose that two maximal elements y^1, y^2∈T with one equal component exist and that a z> min {y^1 , y^2 } does not exist, then $y_1{}^1 = y_1{}^2$ and $y_2{}^1 < y_2{}^2$ and a z > min { y^1, y^2 } = [$y_1{}^1 = y_1{}^2$, $y_2{}^1$] does not exist. This means that there exist a neighbourhood I_{y2} of y^2 such that $y_1{}^2 \geq r_1$ for any r∈ I_{y2}, then y^2 is a local maximum point for function P_1 on T. This is absurd since T is a set without local maxima.

Remark 2.2: Let us note that the viceversa of Theorem 2.5 is not true since it is possible to give examples of Strongly Quasi Concave set which are not connected while a set without local maxima is connected (see Theorem 1.3).

As a consequence of the previous results, we have the connectedness of E(T, $\mathbb{R}^m$).

Corollary 2.2: Let T⊂ $\mathbb{R}^3$ be a SWLM, w^0 ∈ T a weakly maximal element of T and y^1, y^2 two maximal elements with one equal component. If there do not exist other weakly maximal elements in the set $w^0 + \mathbb{R}_+^m$ except on the (n-1) dimensional face of $w^0 + \mathbb{R}_+^m$ and a $z > \min \{y^1, y^2\}$ exists then $E(T, \mathbb{R}^3_+)$ is connected.

Proof: Directly from Theorem 2.4 and Corollary 2.1.

It is easy to find examples to show that the conditions in Corollary 2.2 are sufficient but not necessary.

In the particular case, $T \subset \mathbb{R}^2$, we have the following result.

Corollary 2.3: If $T \subset \mathbb{R}^2$ is a SWLM then $E(T, \mathbb{R}^2_+)$ is a closed and connected set.

Proof: Directly from Theorem 2.5 and Corollary 2.1. Taking into account, if w^0 is a weakly maximal element and another weakly maximal element r^* exists on the one dimensional face of $w^0 + \mathbb{R}_+^2$ then r^* is a local maximum element.

Let us note, with the same approach used for the previous results, that we obtain the closure of set $E(T, \mathbb{R}^m)$ when T is Strongly Quasi Concave and, therefore, the connectedness of $E(T, \mathbb{R}^m)$.

Lemma 2.3: If $T \subset \mathbb{R}^m$ is a Strongly Quasi Concave set then $E(T, \mathbb{R}^m_+) = WE(T, \mathbb{R}^2_+)$ is a closed set.

Proof : If $T \subset \mathbb{R}^m$ is a Strongly Quasi Concave set we have to prove that every weakly maximal element is a maximal element of T. For Lemmas 1.1 and 2.1, if $w^0 \in S_i$ then w^0 is a maximal element of T. If $w^0 \notin S_i$ $i \in \{1,2,..,m\}$ we shall suppose that w^0 is a weakly maximal element but not maximal element. Then $r > w^0$ does not exist but a $r^* \in Fr\,(w^0 + \mathbb{R}_+^m)$ exists, hence, for some components $r^*_i > w_i^0$, and for the others $r^*_i = w_i^0$, $\min \{ r^*, w^0 \} = w^0$. Since T is a Strongly Quasi Concave set then $z > \min \{ r^*, w^0 \} = w^0$ exists, this is absurd since w^0 is a weakly maximal element of T.

Corollary 2.4: If $T \subset \mathbb{R}^m$ is a Strongly Quasi Concave set then $E(T, \mathbb{R}^m)$ is connected.

Proof: Similar to the proof of Theorem 3.1 given in [2] and taking into account Theorem 2.2 and Lemma 2.3.

3. Vector Functions Without Local Maxima

Consider the vector maximization problem:

$$P_M: \ \max \ F(x) \, , x \in X$$

where $X \subset \mathbb{R}^n$ is a compact set and $F: X \rightarrow \mathbb{R}^m$ is continuous function, $F(x) = (f_1(x), f_2(x), .., f_m(x))$ and $F(X) \subset \mathbb{R}^m$.

Let $f_i(\underline{x}) = \max f_i(x), \ x \in X$ and $\underline{S_i} = \{ x \in X \mid f_i(\underline{x}) = f_i(x) \}$ for any $i \in \{1,2,..,m\}$.

In this section we shall state for which class of vector functions F, the outcome F(X) is a set without local maxima and we shall establish that this class is wider than the class of strictly quasi concave functions, considered by Hu and Sun in [2].

First of all, the following theorem states that the outcome of a vector function with functions whose local maxima are global is a set without local maxima.

Theorem 3.1: If $f_1(x), f_2(x), .., f_m(x)$ are functions without local maxima on X then F(X) is a set without local maxima.

Proof: If f_i does not have local maxima on X then for any $\underline{x} \in X$ and $\underline{x} \notin \underline{S_i}$, there is a sequence $\{x^k\} \rightarrow \underline{x}$ with $x^k \in X$ such that $f_i(x^k) > f_i(\underline{x})$ and, for the continuity of f_i, $\{f_i(x^k)\} \rightarrow f_i(\underline{x})$. Now, if we consider $F(\underline{x})$ and $\{ F(x^k) \}$ we have that $\{ F(x^k) \} \rightarrow F(\underline{x})$ with $f_i(x^k) > f_i(\underline{x})$, hence, $F(\underline{x})$ is not a local maximum for function P_i on F(X). For Theorem 1.3, F(X) is a se without local maxima.

Taking into account previous results, in the bicriteria case, we obtain the same results given by Marchi in [4]

Theorem 3.2: Let $F: X \rightarrow \mathbb{R}^2$, $F(x) = (f_1(x), f_2(x))$. If $f_1(x)$, $f_2(x)$ are functions without local maxima on X then $E(F(X), \mathbb{R}^2)$ is closed and connected.

Remark 3.1: Let us note that a strictly quasiconcave function does not have local maxima which are not global, hence, we have for this class of functions that F(X) is a set without local maxima.

In the following lemma, we will note that F(X) may be a set without local maxima also when a function f_i has local maxima on X.

Lemma 3.1: Suppose f_i has a local maximum on X, $x^{loc} \in X$. If $\underline{x} \in X$ such that $F(\underline{x}) = F(x^{loc})$ exists and $\underline{x}$ is not a local maximum for f_i on X then $F(x^{loc})$ is not a local maximum for function P_i on F(X).

Proof: If $\underline{x}$ is not a local optimal solution for f_i in X then there is a sequence $\{x^k\} \rightarrow \underline{x}$ with $x^k \in X$ such that $f_i(x^k) > f_i(\underline{x})$ and, for the continuity of f_i, $\{f_i(x^k)\} \rightarrow f_i(\underline{x})$. Now, if we consider $F(\underline{x})$ and $\{F(x^k)\}$ we have that $\{ F(x^k) \} \rightarrow F(\underline{x}) = F(x^{loc})$ with $f_i(x^k) > f_i(\underline{x})$, then F(X) is not a local maximum for P_i on F(X).

Example 3.1: Consider the following bicriteria problem:

$$P_B: \max F(x) = (f_1(x), f_2(x)), \; x \in X = [-1,4],$$

where $f_1(x) = x^2(x-3)$ and $f_2(x) = -x(x-3)^2$. It is easy to verify that x=0 is a local maximum while x=3 is not a local maximum for f_1 on X. Since $F(0) = F(3) = (0,0)$ then (0,0) is not a local maximum for P_1 on F(X).

Regarding a strongly quasi concave vector function, we obtain the same results given by Warburton in [6] with respect to the efficient frontier.

Theorem 3.3: Let X be a convex set. If $F: X \rightarrow \mathbb{R}^m$ is a continuous and Strongly Quasi Concave function then:

i) F(X) is a Strongly Quasi Concave set,

ii) $E(T, \mathbb{R}^m_+) = WE(T, \mathbb{R}^2_+)$ is a closed and connected set.

Proof : i) Let $x^1, x^2 \in X$ and $F(x^1) \neq F(x^2)$; since F is a continuous and Strongly Quasi Concave function, we have :

$$f_i(\theta x^1+(1-\theta)x^2) > \min[f_i(x^1), f_i(x^2)]$$

for any $i \in \{1,2,..,m\}$, $\theta \in (0,1)$. By the convexity of X, $\theta x^1+(1-\theta)x^2 \in X$, $i \in \{1,2,..,m\}$; therefore, there exists $z = F(\theta x^1+(1-\theta)x^2)$ such that $z > \min[F(x^1), F(x^2)]$;

ii) Directly, taking into account Corollary 2.4.

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