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Generalized Concavity for Bicriteria Functions

Riccardo CAMBINI

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GENERALIZED CONCAVITY
FOR BICRITERIA FUNCTIONS

RICCARDO CAMBINI ([1])
Department of Statistics and Applied Mathematics,
University of Pisa, Via Ridolfi 10, 56124 Pisa, Italy

1. Introduction

One of the most widely studied vector optimization problem in social and economical sciences is the bicriteria problem.

For such a reason, some authors focused on their study on such a particular class of vector optimization problems, in order to find specific optimality conditions or specific properties of the objective function [4, 12, 15].

One of the aim of the paper is to compare, in the bicriteria case, some classes of vector valued generalized concave functions, recently introduced by some authors [3, 5, 6, 7, 8, 10, 11].

In particular, we will prove that, under continuity assumption, the classes of C-quasiconcave type functions, introduced by Luc, see [11], coincide with the classes of (C,C)-quasiconcave and (C,C^{00})-quasiconcave functions introduced by the author.

It is pointed out also that it is possible to give a first order characterization for (C,C)-quasiconcave functions (let us note that this is not possible [6] when the image of f is a subset of $\Re^n$, with n>2) and that it is possible to extend, in the bicriteria case, a classical result given by Martos [14].

Furthermore some new classes of generalized concave vector valued functions are defined by means of a polyhedral cone C and their inclusion relationships with other classes are studied.

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2. Definitions and preliminary results

Several classes of vector valued functions have been recently defined by some several authors in order to extend to the vector case the concept of scalar quasiconcave functions, see [2]; one of the aim of this paper is to compare the following classes of functions in the bicriteria case: consider a function $f:S\rightarrow\Re^m$, with $S\subseteq\Re^n$ convex, and a closed, convex cone $C\subset\Re^m$, with nonempty interior; from now on we will denote with C^0 the cone C without the origin, with C^{00} the interior of C and with C^+ the positive polar cone of C.

A function f is said to be:

C-quasiconcave [11] if $\forall x,y\in S$, $x\neq y$, $\forall z\in\Re^m$ it holds:

$$f(x)\in z+C,\ f(y)\in z+C \ \Rightarrow\ f(x+\lambda(y-x))\in z+C \quad \forall\lambda\in(0,1);$$

strictly C-quasiconcave if $\forall x,y\in S$, $x\neq y$, $\forall z\in\Re^m$ it holds:

$$f(x)\in z+C,\ f(y)\in z+C \ \Rightarrow\ f(x+\lambda(y-x))\in z+C^{00} \quad \forall\lambda\in(0,1);$$

$(C^*,C^\#)$-quasiconcave [3, 5, 6] if $\forall x,y\in S$, $x\neq y$, it holds:

$$f(y)\in f(x)+C^* \ \Rightarrow\ f(x+\lambda(y-x))\in f(x)+C^\# \quad \forall\lambda\in(0,1),$$

where $C^*,C^\#\in\{C,C^0,C^{00}\}$;

polarly C-quasiconcave [7-10] if $\forall p\in C^+$, $p\neq 0$:

the scalar function $f_p(x)=p^Tf(x)$ is quasiconcave;

polarly semistrictly C-quasiconcave if $\forall p\in C^+$, $p\neq 0$:

the scalar function $f_p(x)=p^Tf(x)$ is semistrictly quasiconcave;

polarly strictly C-quasiconcave if $\forall p\in C^+$, $p\neq 0$:

the scalar function $f_p(x)=p^Tf(x)$ is strictly quasiconcave.

It is easy to verify that the classes of polarly C-quasiconcave type functions are properly contained in the C-quasiconcave type ones defined by Luc; furthermore the classes of $(C^*,C^\#)$-quasiconcave functions contain properly the previous ones.

When C is a polyhedral cone, so that also its positive polar C^+ is a polyhedral cone[1], we can define the following new classes of vector quasiconcave functions.

Definition 2.1 Let $f:S\rightarrow\Re^m$, where $S\subseteq\Re^n$ is a convex set, and let $C\subset\Re^m$ be a polyhedral cone. Then a function f will be said to be:

i) *C^+-quasiconcave* if $d^Tf(x)$ is quasiconcave for every extreme vector d of C^+;
ii) *semi C^+-quasiconcave* if $d^Tf(x)$ is semi quasiconcave for every extreme vector d of C^+;

iii) *strictly C^+-quasiconcave* if $d^T f(x)$ is strictly quasiconcave for every extreme vector d of C^+;
iv) *semistrictly C^+-quasiconcave* if $d^T f(x)$ is semistrictly quasiconcave for every extreme vector d of C^+.

Let us note that these new classes of functions are more general than the polarly C-quasiconcave type ones and are more restrictive than the C-quasiconcave type ones.

Remark 2.1 When the polyhedral cone C is the Paretian cone $C=\Re^m_+$, then $C^+=C$ and the above defined C^+-quasiconcave type functions become the componentwise quasiconcave type functions; in other words these functions can be considered as a generalization, with respect to a polyhedral cone, of the componentwise quasiconcavity.

As we have already said, in this paper we will analize the previously reminded functions in the bicriteria case. With this aim, from now on we will consider a closed, convex, pointed, polyhedral cone $C\subset\Re^2$ with nonempty interior in the following form:

$$C=\{x\in\Re^2: x=\lambda_1 u_1+\lambda_2 u_2\ ,\ \lambda_i\geq 0,\ u_i\in\Re^2,\ i=1,2\}, \tag{2.1a}$$

so that its polar cone, verifying the same properties, can be considered in the form:

$$C^+=\{x\in\Re^2: x=\mu_1 d_1+\mu_2 d_2\ ,\ \mu_i\geq 0,\ d_i\in\Re^2,\ i=1,2\}. \tag{2.1b}$$

Note that C and C^+ have nonempty interiors if and only if vectors u_i and d_i respectively are linearly independent.

3. Relationships among the classes in the bicriteria case

Now we will study in the bicriteria case the relationships existing among the classes of functions introduced in the previous section.
As regards to (C,C)-quasiconcave and C-quasiconcave functions, the following Example 3.1 focus on that when the image of f is a subset of $\Re^3$ the class of C-quasiconcave functions is properly included in the one of (C,C)-quasiconcave functions; Example 3.2 points out that this inclusion is still proper in the bicriteria case when f is not continuous.

Example 3.1
Let $f:\Re^2_+\to\Re^3$, $f(x_1,x_2)=(x_1,x_2,-x_1x_2)$, and let $C=\Re^3_+$; this function is (C,C)-quasiconcave but not C-quasiconcave (neither strictly C-quasiconcave) in the sense of Luc. Set x=(1,0), y=(0,1), z=(0,0) and w=x+(1/2)(y-x)=(1/2, 1/2), then f(z)=(0,0,0), $f(x)=(1,0,0)\in f(z)+C$, $f(y)=(0,1,0)\in f(z)+C$ and $f(w)=(1/2,1/2,-1/4)\notin f(z)+C$, so that f is not C-quasiconcave. It can be verified that f is a (C,C^0)-quasiconcave function, since $f(y)\in f(x)+C$, $x\neq y$, implies that $x_1x_2=y_1y_2=0$, $y_1\geq x_1\geq 0$ and $y_2\geq x_2\geq 0$, and this happens if and only if $y_1>x_1\geq 0$ with $y_2=x_2=0$ or $y_1=x_1=0$ with $y_2>x_2\geq 0$ so that $f(x+\lambda(y-x))\in f(x)+C^0$ $\forall\lambda\in(0,1)$.

Example 3.2
Let us consider the Paretian cone $C=\Re^2_+$ and the function $f:S\to\Re^2$ with $S=\{(x,y): x\in[-1,1], y\leq 0\}$:

$$f(x,y)=\begin{cases} (x,-x) & \text{for } x\in[-1,1],\ x\neq 0, \text{ and } y=0 \\ (-2,2) & \text{for } x=0 \text{ and } y=0 \\ (-1/2,-1/2) & \text{for } x\in[-1,1] \text{ and } y<0 \end{cases};$$

f is (C,C)-quasiconcave in S even if, for $\mu=(-1/2,-1/2)$ ($\mu=f(x,y)$ $\forall y<0$, $x\in[-1,1]$) the following C-upper level set is not convex:

$$U(f,\mu)=\{(x,y): x\in[-1/2,1/2]\setminus\{0\}, y=0\}\cup\{(x,y): x\in[-1,1], y<0\}.$$

Since C-quasiconcavity is equivalent to the convexity of the C-upper level set [11], the function f is not C-quasiconcave.

The following theorem proves that in the bicriteria case, under continuity hypothesis, (C,C)-quasiconcavity is equivalent to C-quasiconcavity, as well as to C^+-quasiconcavity.

Theorem 3.1 Let $f:S\to\Re^2$, where $S\subseteq\Re^n$ is a convex set, be a continuous function and let $C\subset\Re^2$ and $C^+\subset\Re^2$ be as described in (2.1).
Then the following properties are equivalent:
i) f is a (C,C)-quasiconcave function;
ii) f is a C-quasiconcave function;
iii) f is a C^+-quasiconcave function.
Proof We firstly prove, as a preliminary result, that if $d_i^Tf(y)=d_i^Tf(x)$, $i\in\{1,2\}$, then either $f(y)\in f(x)+C$ or $f(x)\in f(y)+C$.
Let $j\in\{1,2\}$, $j\neq i$, and suppose ab absurdo that $f(y)\notin f(x)+C$ and $f(x)\notin f(y)+C$; then by means of a known separation theorem, $\exists p_1,p_2\in C^+$, $p_1,p_2\neq 0$, such that

$p_1{}^T f(y) < p_1{}^T f(x)$ and $p_2{}^T f(x) < p_2{}^T f(y)$; set $p_1 = \mu_i^1 d_i + \mu_j^1 d_j$ and $p_2 = \mu_i^2 d_i + \mu_j^2 d_j$; since $d_i{}^T f(y) = d_i{}^T f(x)$ we have:

$$\mu_j^1 d_j{}^T f(y) < \mu_j^1 d_j{}^T f(x) \text{ and } \mu_j^2 d_j{}^T f(x) < \mu_j^2 d_j{}^T f(y),$$

then μ_j^1 and μ_j^2 must be nonzero so that $d_j{}^T f(y) < d_j{}^T f(x)$ and $d_j{}^T f(x) < d_j{}^T f(y)$ which is a contradiction.

Using this preliminary result we are now able to prove the thesis.

By means of the results described in the previous section we just have to prove that condition i) implies condition iii). Suppose ab absurdo that $\exists i \in \{1,2\}$ such that $d_i{}^T f(x)$ is not quasiconcave, so that $\exists x,y \in S$, $x \neq y$, $\exists \lambda_1 \in (0,1)$ such that $d_i{}^T f(y) \geq d_i{}^T f(x)$ and $d_i{}^T f(x+\lambda_1(y-x)) < d_i{}^T f(x)$. By means of the continuity of f, there exists $\lambda_2 \in (\lambda_1, 1]$ such that $d_i{}^T f(x+\lambda_2(y-x)) = d_i{}^T f(x)$, so that either $f(x+\lambda_2(y-x)) \in f(x)+C$ or $f(x) \in f(x+\lambda_2(y-x))+C$.

The (C,C)-quasiconcavity of f implies:

$$f(x+\lambda(y-x)) \in f(x)+C \text{ or } f(x+\lambda(y-x)) \in f(x+\lambda_2(y-x))+C \quad \forall \lambda \in (0,\lambda_2),$$

so that $d_i{}^T f(x+\lambda(y-x)) \geq d_i{}^T f(x)$ or $d_i{}^T f(x+\lambda(y-x)) \geq d_i{}^T f(x+\lambda_2(y-x)) = d_i{}^T f(x)$ $\forall \lambda \in (0,\lambda_2)$ which is a contradiction since $\lambda_1 \in (0,\lambda_2)$. ❑

Remark 3.1 Let us note that the equivalence between ii) and iii) is also proved by Luc in [11].

Remark 3.2 Note that when C is the Paretian cone of $\Re^2$, the previous results state that the (C,C)-quasiconcavity and the C-quasiconcavity of the function can be characterized by means of its componentwise quasiconcavity.

The following theorem shows that in the bicriteria case, under continuity assumptions, (C,C^{00})-quasiconcavity is equivalent to strictly C-quasiconcavity, as well as strictly C^+-quasiconcavity.

Theorem 3.2 Let $f: S \to \Re^2$, where $S \subseteq \Re^n$ is a convex set, be a continuous function and let $C \subset \Re^2$ and $C^+ \subset \Re^2$ be as described in (2.1).

Then the following properties are equivalent:

i) f is a (C,C^{00})-quasiconcave function;

ii) f is a strictly C-quasiconcave function;

iii) f is a strictly C^+-quasiconcave function.

Proof By means of the results described in the previous section we just have to prove that condition i) implies condition iii); note also that, by means of Theorem 3.1, since a (C,C^{00})-quasiconcave function is also (C,C)-quasiconcave

then f is a C^+-quasiconcave function. Suppose ab absurdo that $\exists i\in\{1,2\}$ such that $d_i^T f(x)$ is quasiconcave but not strictly quasiconcave, so that $\exists x,y\in S$, $x\neq y$, $\exists\lambda_1\in(0,1)$ such that $d_i^T f(y)\geq d_i^T f(x)$ and $d_i^T f(x+\lambda_1(y-x))=d_i^T f(x)$.
If $d_i^T f(y)=d_i^T f(x)$ then, by means of the preliminary result proved in Theorem 3.1, we have that either $f(y)\in f(x)+C$ or $f(x)\in f(y)+C$ so that from the (C,C^{00})-quasiconcavity of f it follows:

$$f(x+\lambda(y-x))\in f(x)+C^{00} \text{ or } f(x+\lambda(y-x))\in f(y)+C^{00} \quad \forall\lambda\in(0,1),$$

so that $d_i^T f(x+\lambda(y-x))>d_i^T f(y)=d_i^T f(x)=d_i^T f(x+\lambda_1(y-x))$ $\forall\lambda\in(0,1)$ which is a contradiction since $\lambda_1\in(0,1)$.
Let now be $d_i^T(y)>d_i^T(x)$; since $d_i^T(x)=d_i^T(x+\lambda_1(y-x))$ then we have that either $f(x+\lambda_1(y-x))\in f(x)+C$ or $f(x)\in f(x+\lambda_1(y-x))+C$; in every case by means of the (C,C^{00})-quasiconcavity of f it follows:

$$d_i^T f(x+\lambda(y-x))>d_i^T f(x+\lambda_1(y-x))=d_i^T f(x) \quad \forall\lambda\in(0,\lambda_1);$$

since $d_i^T f(y)>d_i^T f(x)$, the continuity of f implies that $\exists\lambda_2\in(0,\lambda_1)$, $\exists\lambda_3\in(\lambda_1,1)$ such that $d_i^T f(y)>d_i^T f(x+\lambda_2(y-x))=d_i^T f(x+\lambda_3(y-x))>d_i^T f(x+\lambda_1(y-x))$; by means of the same arguments, applying the (C,C^{00})-quasiconcavity of f to the interval $[\lambda_2,\lambda_3]$, then we have that:

$$d_i^T f(x+\lambda(y-x))>d_i^T f(x+\lambda_2(y-x))=d_i^T f(x+\lambda_3(y-x))>d_i^T f(x+\lambda_1(y-x)) \quad \forall\lambda\in(\lambda_2,\lambda_3),$$

and this is a contradiction since $\lambda_1\in(\lambda_2,\lambda_3)$. ❑

As it is well known, in the scalar case quasiconcave functions have a first order characterization, on the contrary for (C,C)-quasiconcave functions this is not possible [6] when the image of f is a subset of $\Re^n$, with $n>2$.
When $n=2$, we are able to characterize in the differentiable case a (C,C)-quasiconcave function, as is pointed out in the following theorem.

Theorem 3.3 Let $f:S\to\Re^2$, where $S\subseteq\Re^n$ is a convex set, be a differentiable function and let $C\subset\Re^2$ and $C^+\subset\Re^2$ be as described in (2.1).
f is (C,C)-quasiconcave if and only if for every $x,y\in S$, $x\neq y$, it holds:

$$f(y)\in f(x)+C \;\Rightarrow\; J_f(x)(y-x)\in C.$$

Proof We firstly prove, as a preliminary result, that if f is continuous and $\exists x,y\in S$, $x\neq y$, $\exists i\in\{1,2\}$ such that $f(y)\neq f(x)$ and $d_i^T f(y)=d_i^T f(x)>d_i^T f(x+\lambda(y-x))$ $\forall\lambda\in(0,1)$ then f is not a (C^{00},C)-quasiconcave function and if it is also differentiable then it is not a weakly (C^{00},C)-quasiconcave function.
Set $j\in\{1,2\}$, $j\neq i$, we can easily prove that since $d_i^T f(y)=d_i^T f(x)$ and $f(y)\neq f(x)$ then $d_j^T f(y)\neq d_j^T f(x)$; we can also suppose, without loss of generality, that $d_j^T f(y)>d_j^T f(x)$.

By means of the continuity of f and the assumptions, $\exists\delta\in(0,1)$ such that $d_i^T f(x+\lambda(y-x))\geq d_i^T f(x+\delta(y-x))$ $\forall\lambda\in(0,1)$, and $\exists\lambda_1\in(0,\delta)$ such that $\forall\lambda\in[0,\lambda_1]$:

$$d_i^T f(y)=d_i^T f(x)>d_i^T f(x+\lambda_1(y-x))>d_i^T f(x+\delta(y-x)) \text{ and } d_j^T f(y)>d_j^T f(x+\lambda(y-x)).$$

Then it is $f(y)\in f(x+\lambda_1(y-x))+C^{00}$ otherwise, by means of a separation theorem, $\exists p\in C^+$, $p\neq 0$, such that $p^T[f(y)-f(x+\lambda_1(y-x))]\leq 0$ so that being $p=\mu_i d_i+\mu_j d_j$ we have $\mu_i d_i^T[f(y)-f(x+\lambda_1(y-x))]+\mu_j d_j^T[f(y)-f(x+\lambda_1(y-x))]\leq 0$ with $\mu_i\geq 0$ and $\mu_j\geq 0$ which is a contradiction.

Since $f(y)\in f(x+\lambda_1(y-x))+C^{00}$ and $f(x+\delta(y-x))\notin f(x+\lambda_1(y-x))+C$ with $\delta\in(\lambda_1,1)$ we have that function f is not (C^{00},C)-quasiconcave.

Suppose now f to be also differentiable; by means of the Lagrange theorem applied to the segment $[x, x+\lambda_1(y-x)]$ $\exists\xi\in(0,\lambda_1)$ such that:

$$d_i^T f(x+\lambda_1(y-x))=d_i^T f(x)+d_i^T J_f(x+\xi(y-x))((x+\lambda_1(y-x))-x),$$

so that, being $d_i^T f(x)>d_i^T f(x+\lambda_1(y-x))$ and $\lambda_1>0$, $d_i^T J_f(x+\xi(y-x))(y-x)<0$ which implies that $J_f(x+\xi(y-x))(y-x)\notin C$. Since it is $y-(x+\xi(y-x))=(1-\xi)(y-x)$ with $1-\xi>0$ we also have $J_f(x+\xi(y-x))(y-(x+\xi(y-x)))\notin C$, so that since $f(y)\in f(x+\xi(y-x))+C^{00}$ we have that function f is not weakly (C^{00},C)-quasiconcave.

Using this preliminary result we will now prove the thesis.

Since in general a (C,C)-quasiconcave function is also weakly (C,C)-quasiconcave we just have to prove the sufficiency. Suppose ab absurdo that f is not (C,C)-quasiconcave, that is to say, by means of Theorem 3.1, that f is not C^+-quasiconcave; then $\exists i\in\{1,2\}$ such that $d_i^T f(x)$ is not quasiconcave, so that $\exists x,y\in S$, $x\neq y$, $\exists\lambda_1\in(0,1)$ such that $d_i^T f(y)\geq d_i^T f(x)>d_i^T f(x+\lambda_1(y-x))$; set also $j\in\{1,2\}$, $j\neq i$.

By means of the continuity of f, $\exists\lambda_2,\lambda_3\in[0,1]$, $\lambda_2\neq\lambda_3$, such that $\lambda_1\in(\lambda_2,\lambda_3)$ and $d_i^T f(x)=d_i^T f(x+\lambda_2(y-x))=d_i^T f(x+\lambda_3(y-x))>d_i^T f(x+\lambda(y-x))$ $\forall\lambda\in(\lambda_2,\lambda_3)$.

Set $v=(x+\lambda_2(y-x))$ and $w=(x+\lambda_3(y-x))$; by means of the continuity of f and the hypothesis, $\exists\alpha_1\in(0,1)$ such that the function $g(\alpha)=d_i^T f(v+\alpha(w-v))$ is strictly decreasing in $[0,\alpha_1]$. By means of the Lagrange theorem applied to the segment $[v, v+\alpha_1(w-v)]$ $\exists\xi\in(0,\alpha_1)$ such that:

$$d_i^T f(v+\alpha_1(w-v))=d_i^T f(v)+d_i^T J_f(v+\xi(w-v))((v+\alpha_1(w-v))-v),$$

so that, being $d_i^T f(v)>d_i^T f(v+\alpha_1(w-v))$ and $\alpha_1>0$, $d_i^T J_f(v+\xi(w-v))(w-v)<0$ which implies that $J_f(v+\xi(w-v))(w-v)\notin C$. Since f is continuous and $g(\alpha)$ is strictly decreasing in $[0,\alpha_1]$, then $\exists\gamma\in(\alpha_1,1)$ such that:

$$d_i^T f(v+\xi(w-v))=d_i^T f(v+\gamma(w-v))>d_i^T f(v+\alpha(w-v)) \ \forall\alpha\in(\xi,\gamma).$$

If $f(v+\xi(w-v))\neq f(v+\gamma(w-v))$ then by means of the preliminary result we have that f is not weakly (C^{00},C)-quasiconcave so that it is not weakly (C,C)-

quasiconcave; if $f(v+\xi(w-v))=f(v+\gamma(w-v))$ then, since $0\in C$, $f(v+\gamma(w-v))\in f(v+\xi(w-v))+C$.

Since $J_f(v+\xi(w-v))(w-v)\notin C$ and $(\gamma-\xi)>0$ we have also that $J_f(v+\xi(w-v))[(v+\gamma(w-v))-(v+\xi(w-v))]\notin C$ and this implies that f is not weakly (C,C)-quasiconcave. ❑

Remark 3.3 Theorem 3.3 is equivalent to state that the class of (C,C)-quasiconcave functions coincides with the class of weakly (C,C)-quasiconcave functions (2) [6].

4. Increasness and decreasness in the bicriteria case

In [5], some relationships among (C,C)-quasiconcavity, C-increasness and C-decreasness (3) have been investigated for single variable vector valued functions.

The aim of this section is to prove that in the bicriteria case it is possible to generalize to vector valued single variable functions the well known characterization of scalar quasiconcave functions given by Martos [14].

With this aim we firstly prove that C-increasness and C-decreasness are equivalent to the increasness and decreasness of the scalar functions $d_1^Tf(x)$ and $d_2^Tf(x)$.

Theorem 4.1 Let $f:[a,b]\to\Re^2$, where $[a,b]\subseteq\Re$ is an interval, be a continuous function and let $C\subset\Re^2$ and $C^+\subset\Re^2$ be as described in (2.1).

Then the following properties are equivalent:

i) f is C-increasing iff both $d_1^Tf(x)$ and $d_2^Tf(x)$ are increasing scalar functions;

ii) f is C-decreasing iff both $d_1^Tf(x)$ and $d_2^Tf(x)$ are decreasing scalar functions.

Proof We will prove only case i), since to other is analogous.

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2 Consider the vector differentiable function $f:S\to\Re^m$, where $S\subseteq\Re^n$ is a convex set, and let $C\subset\Re^m$ be a closed cone with nonempty interior. Set $C^*\in\{C,C^0,C^{00}\}$; function f will be said to be:
weakly (C,C)-quasiconcave* if $\forall x,y\in S$, $x\neq y$, it holds: $f(y)\in f(x)+C^* \Rightarrow J_f(x)(y-x)\in C$.

3 Let $f:[a,b]\to\Re^m$, where $[a,b]\subseteq\Re$ is an interval, and let $C\subset\Re^m$ be a closed cone with nonempty interior. Function f will be said th be *C-increasing* if $f(y)\in f(x)+C$ $\forall x,y\in[a,b]$, $y>x$, while it will be said to be *C-decreasing* if $f(y)\in f(x)-C$ $\forall x,y\in[a,b]$, $y>x$.

For the necessity, suppose ab absurdo that $\exists i \in \{1,2\}$ such that $d_i^T f(x)$ is not increasing, so that $\exists x,y \in [a,b]$, $y>x$, such that $d_i^T f(y)<d_i^T f(x)$; then $d_i^T[f(y)-f(x)]<0$ so that $f(y)-f(x) \notin C$ which contradicts the C-increasness of f.

For the sufficiency, suppose ab absurdo that f is not C-increasing, so that $\exists x,y \in [a,b]$, $y>x$, such that $f(y) \notin f(x)+C$; then by means of a separation theorem $\exists p \in C^+$, $p \neq 0$, such that $p^T[f(y)-f(x)]<0$ so that, being $p=\mu_1 d_1+\mu_2 d_2$, we have $\mu_1 d_1^T[f(y)-f(x)]+\mu_2 d_2^T[f(y)-f(x)]<0$ with $\mu_1 \geq 0$ and $\mu_2 \geq 0$; then $\exists i \in \{1,2\}$ such that $d_i^T f(y)<d_i^T f(x)$ which contradicts the increasness of $d_i^T f(x)$. ❑

The following theorem generalizes the one given by Martos [14] in the scalar case.

Theorem 4.2 Let $f:[a,b] \rightarrow \Re^2$, where $[a,b] \subseteq \Re$ is an interval, be a continuous function and let $C \subset \Re^2$ and $C^+ \subset \Re^2$ be as described in (2.1). Then function f is (C,C)-quasiconcave if and only if both the two conditions (4.1) and (4.2) hold:

$$\begin{array}{c} \forall x,y \in S,\ x \neq y, \text{ such that } f(y)=f(x) \\ \text{if } f(x+\lambda(y-x)) \notin f(x)+C^0\ \forall \lambda \in (0,1) \text{ then } f(x+\lambda(y-x))=f(x)\ \forall \lambda \in (0,1) \end{array} \tag{4.1}$$

$\exists \alpha,\beta \in [a,b]$, $\alpha \leq \beta$, such that: (4.2)

a) f is C-increasing in $[a,\alpha]$,
b) $\nexists x,y \in [\alpha,\beta]$ such that $f(y) \in f(x)+C^{00}$,
c) f is C-decreasing in $[\beta,b]$.

Proof $\Rightarrow$) Condition (4.1) follows directly by the definition of (C,C)-quasiconcave functions; from Theorem 3.1 the scalar functions $d_1^T f(x)$ and $d_2^T f(x)$ are quasiconcave so that being $d_i^T f(x)$ continuous $\forall i \in \{1,2\}$ then for each $i \in \{1,2\}$ $\exists \alpha_i,\beta_i \in [a,b]$, $\exists M_i \in \Re$ such that (see Martos [14]):

$d_i^T f(x)$ is increasing in $[a,\alpha_i]$ with $d_i^T f(x)<M_i\ \forall x \in [a,\alpha_i]$,
$d_i^T f(x)$ is constant in $[\alpha_i,\beta_i]$ with $d_i^T f(x)=M_i\ \forall x \in [\alpha_i,\beta_i]$,
$d_i^T f(x)$ is decreasing in $[\beta_i,b]$ with $d_i^T f(x)<M_i\ \forall x \in [\beta_i,b]$.

Set $\alpha=\min\{\alpha_1,\alpha_2\}$ and $\beta=\max\{\beta_1,\beta_2\}$; by means of Theorem 4.1 f is C-increasing in $[a,\alpha]$ and C-decreasing in $[\beta,b]$; note finally that also condition (4.2b) holds, since in $[\alpha,\beta]$ at least one of $d_1^T f(x)$ and $d_2^T f(x)$ is nondecreasing and the other is nonincreasing so that it's not possible to have $f(y) \in f(x)+C^{00}$ with $x,y \in [\alpha,\beta]$.

$\Leftarrow$) We firstly prove, as a preliminary result, that if condition (4.1) holds and f is not (C,C)-quasiconcave then $\exists i \in \{1,2\}$, $\exists x,y \in S$, $x \neq y$, such that $f(y) \neq f(x)$ and $d_i^T f(y)=d_i^T f(x)>d_i^T f(x+\lambda(y-x))\ \forall \lambda \in (0,1)$.

If f is not (C,C)-quasiconcave then, from Theorem 3.1, $\exists i\in\{1,2\}$ such that $d_i^T f(x)$ is not quasiconcave, so that $\exists v,w\in S$, $v\neq w$, $\exists \lambda_1\in(0,1)$ such that $d_i^T f(w)\geq d_i^T f(v)>d_i^T f(v+\lambda_1(w-v))$; set also $j\in\{1,2\}$, $j\neq i$.
By means of the continuity of f, $\exists \lambda_2,\lambda_3\in[0,1]$, $\lambda_2\neq\lambda_3$, such that $\lambda_1\in(\lambda_2,\lambda_3)$ and $d_i^T f(v)=d_i^T f(v+\lambda_2(w-v))=d_i^T f(v+\lambda_3(w-v))>d_i^T f(v+\lambda(w-v))$ $\forall\lambda\in(\lambda_2,\lambda_3)$.
Set $x=(v+\lambda_2(w-v))$ and $y=(v+\lambda_3(w-v))$ and note that $f(x+\lambda(y-x))\notin f(x)+C$ $\forall\lambda\in(0,1)$. Then condition (4.1) implies that $f(y)\neq f(x)$, otherwise it is $f(x+\lambda(y-x))=f(x)$ $\forall\lambda\in(\lambda_2,\lambda_3)$ which is a contradiction, so that the preliminary result is proved. Using this preliminary result we will now prove the thesis.
Suppose ab absurdo that f is not (C,C)-quasiconcave, then from the preliminary result we have that $\exists i\in\{1,2\}$, $\exists x,y\in S$, $x\neq y$, such that $f(y)\neq f(x)$ and $d_i^T f(y)=d_i^T f(x)>d_i^T f(x+\lambda(y-x))$ $\forall\lambda\in(0,1)$, so that $f(x+\lambda(y-x))\notin f(x)+C$ and $f(x+\lambda(y-x))\notin f(y)+C$ $\forall\lambda\in(0,1)$. By means of these conditions, if $x\notin[\alpha,\beta]$ then the C-increasness of f in $[a,\alpha]$ and the C-decreasness of f in $[\beta,b]$ is contradicted, so that $x\in[\alpha,\beta]$; in the same way we have that also $y\in[\alpha,\beta]$. By means of the preliminary result proved in Theorem 3.3, we also have that f is not a (C^{00},C)-quasiconcave function in $[\alpha,\beta]$ which contradicts condition (4.2b). ❑

The following Example 4.1 points out that condition (4.1) in Theorem 4.2 cannot be relaxed.

Example 4.1 Let $f:\Re\to\Re^2$, with $f(x)=\sin(x)[1,-1]^T$, and let $C=\Re^2_+$, so that componentwise quasiconcavity can be studied instead of C^+-quasiconcavity.
f is (C^0,C^{00})-quasiconcave but not (C,C)-quasiconcave.

Note finally that some concepts of vector increasness and decreasness have been studied also in [13].

5. Bicriteria case and C^+-quasiconcavity

In [5, 6], several classes of $(C^*,C^\#)$-quasiconcave functions have been defined and studied; the aim of this section is to point out the inclusion relationships among these classes and the new defined classes of C^+-quasiconcave type functions. It is easy to verify that:

i) if f is C^+-quasiconcave then it is also (C,C)-quasiconcave,
ii) if f is semi C^+-quasiconcave then it is also (C^{00},C)-quasiconcave,
iii) if f is strictly C^+-quasiconcave then it is also (C,C^{00})-quasiconcave,

iv) if f is semistrictly C^+-quasiconcave then it is also (C^{00},C^{00})-quasiconcave. Note that Example 3.1 shows that these inclusion relationships are proper. In the bicriteria case and under continuity hypothesis we have that the inclusion relationships can be represented in the following diagram.

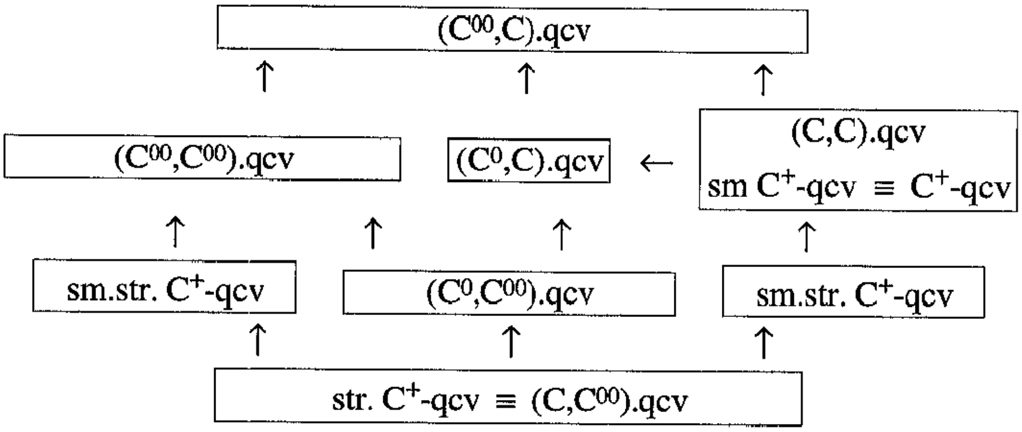

Diagram 1

Example 4.1 and the following Example 5.1 show that some of the represented inclusion relationships are proper.

Example 5.1 Let $f:\Re\rightarrow\Re^2$, with $f(x)=(0,|x|)$, and let $C=\Re^2_+$, so that componentwise quasiconcavity can be studied instead of C^+-quasiconcavity. f is (C^{00},C^{00})-quasiconcave (consequently f is (C^{00},C^{0})-quasiconcave and (C^{00},C)-quasiconcave, too) since $\nexists x,y\in\Re$ such that $f(y)\in f(x)+C^{00}$; on the other hand f is not (C^{0},C)-quasiconcave (so that it's neither (C^{0},C^{0})-quasiconcave nor (C^{0},C^{00})-quasiconcave) since $f(2)\in f(-1)+C^{0}$ but $f(0)\notin f(-1)+C$.

Remark 5.1 Note that it has been possible to characterize by means of the C^+-quasiconcavity and strictly C^+-quasiconcavity the (C,C)-quasiconcave and (C,C^{00})-quasiconcave functions, that is to say just those classes of functions having a fixed behaviour whenever $f(y)=f(x)$; Example 4.1 shows that nothing can be said when no properties of the functions are fixed for points x and y such that $f(y)=f(x)$, as we have for (C^{0},C^{00})-quasiconcave functions.

Assuming that condition (4.1) holds, we can furthermore deep on the relationships among the considered classes of generalized concave vector valued functions.

Theorem 5.1 Let $f:S\to\Re^2$, where $S\subseteq\Re^n$ is a convex set, be a continuous function and let $C\subset\Re^2$ and $C^+\subset\Re^2$ be as described in (2.1); suppose also that (4.1) holds. Then f is (C,C)-quasiconcave iff it is (C^{00},C)-quasiconcave.

Proof Since a (C,C)-quasiconcave function is also (C^{00},C)-quasiconcave we just have to prove the sufficiency. Suppose ab absurdo that f is not (C,C)-quasiconcave; then, by means of the preliminary result proved in Theorem 4.2, $\exists i\in\{1,2\}$, $\exists x,y\in S$, $x\neq y$, such that $f(y)\neq f(x)$ and $d_i^Tf(y)=d_i^Tf(x)>d_i^Tf(x+\lambda(y-x))$ $\forall\lambda\in(0,1)$.

By means of the preliminary result proved in Theorem 3.3, we then have that f is not a (C^{00},C)-quasiconcave function which contradicts the assumption. ❑

The following Diagram 2 summarizes the inclusion relationships among the considered classes of continuous bicriteria functions when condition (4.1) holds. Example 5.2 shows also that the inclusion relationships are proper.

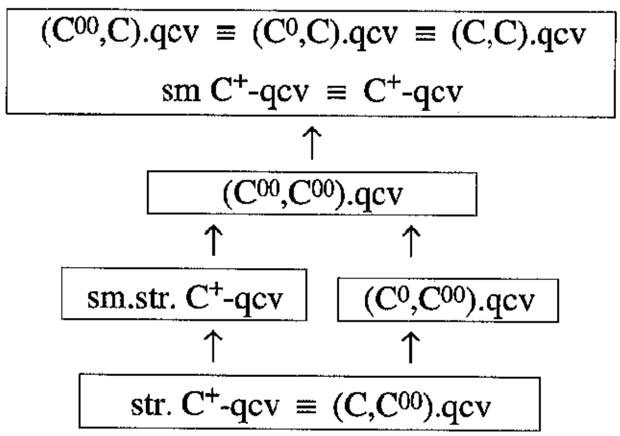

Diagram 2

Example 5.2 Let us consider the following continuous functions $f:S\to\Re^2$ such that $f(x)\neq f(y)$ $\forall x\neq y$ and let $C=\Re^2_+$.

i) $f(x)=(x,x^2+x|x|)$ is (C,C)-quasiconcave but it is not (C^{00},C^{00})-quasiconcave since for $x<0$ it is $f(x)=(x,0)$;

ii) $f(x)=(x^2-x|x|,x^2+x|x|)$ is (C^{00},C^{00})-quasiconcave but its components are not semistrictly quasiconcave and it is not (C^0,C^{00})-quasiconcave since for $x<0$ it is $f(x)=(2x^2,0)$;

iii) $f(x)=(x,0)$ is cw ss.quasiconcave (and also (C^{00},C^{00})-quasiconcave) but not (C^0,C^{00})-quasiconcave (nor cw strictly quasiconcave);

iv) $f(x)=(x^2+x|x|)[1,-1]^T$ is (C^0,C^{00})-quasiconcave (and also (C^{00},C^{00})-quasiconcave) but not cw ss.quasiconcave (nor cw strictly quasiconcave).

REFERENCES

[1] M.S. Bazaraa and C.M. Shetty, Foundations of optimization, Lecture Notes in Economics and Mathematical Systems 122, Springer-Verlag, 1976.

[2] A. Cambini and L. Martein, Multiobjective Programming and Generalized Concavity, submitted to the proceedings of the 5th Symphosium on Generalized Convexity, Luminy-Marsiglia (France), June 17-21, 1996.

[3] A. Cambini, L. Martein, and R. Cambini, Some optimality conditions in multiobjective programming, Proceedings of the XI International Conference on MCDM, Coimbra (Portugal), (1994)

[4] A. Cambini, L. Martein, and I.M. Stancu-Minasian, A survey of bicriteria fractional problems, Technical report n°106, Dipartimento di Statistica e Matematica Applicata all'Economia, Università di Pisa, 1996.

[5] R. Cambini, Composition theorems for generalized concave vector valued functions, to appear in JIOS.

[6] R. Cambini, Some new classes of generalized concave vector-valued functions, Optimization 36, n°1 (1996) 11-24.

[7] R. Cambini and S. Kómlosi, Generalized concavity and generalized monotonicity concepts for vector valued functions, Technical report n°98, Dipartimento di Statistica e Matematica Applicata all'Economia, Università di Pisa, 1996.

[8] R. Cambini and S. Kómlosi, Polar quasiconcavity and polar quasimonotonicity concepts for vector valued functions, submitted to JOTA.

[9] R. Cambini and S. Kómlosi, Generalized Concavity and Generalized Monotonicity in Vector Optimization, submitted to the proceedings of the 5th Symphosium on Generalized Convexity, Luminy-Marsiglia (France), June 17-21, 1996.

[10] R. Cambini and S. Kómlosi, Polar pseudoconcavity and polar pseudomonotonicity concepts for vector valued functions, submitted to the proceedings of the XIII International Conference on Mathematical Programming, Mátraháza (Hungary), March 24-28, 1996.

[11] D.T. Luc, Theory of vector optimization, Lecture Notes in Economics and Mathematical Systems 319, Springer-Verlag, Berlin, 1988.

[12] L. Martein, On the bicriteria maximization problem, in "Generalized Convexity and Fractional Programming with Economic Applications", edited by A. Cambini, E. Castagnoli, *et al.*, Springer-Verlag, Berlin (1990) 77-84.

[13] J.E. Martinez-Legaz, Quasiconvex duality theory by generalized conjugation methods, Optimization 19 (1988) 603-652.

[14] B. Martos, Nonlinear programming theory and methods, North-Holland, Amsterdam, 1975.

[15] S. Schaible, Bicriteria quasiconcave programs, Cahiers du C.E.R.O. 25 (1983) 93-101.