Report n.98

Generalized Concavity and Generalized Monotonicity Concepts for Vector Valued Functions

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Pisa, February 1996
This research has been partially supported by the Italian Ministry of Public Education

Generalized Concavity and Generalized Monotonicity Concepts for Vector Valued Functions ([1])

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Abstract. The aim of the paper is to give first order characterization for special classes of vector valued directionally differentiable generalized concave functions in terms of suitable generalized monotonicity property of the directional derivative.

KeyWords. Vector Optimization, Generalized Concavity, Generalized Monotonicity.

1. Introduction

Recently, several kinds of vector valued generalized concave functions have been introduced and studied; these functions are often used in vector optimization problems since their properties let to state several necessary or sufficient optimality conditions, see for instance [3-5].

For this reason it is important to state as much properties as possible related to these classes of functions, so that further results can be obtained.

The main objective of the present paper is to deepen the analysis of several classes of generalized concave vector valued functions, initiated in [7, 8], by elaborating first order characterizations for them. These characterizations extend on the one hand the classical Arrow-Enthoven characterizations of real valued quasiconcave functions to several classes of generalized concave vector

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[1] The research was completed during the stay of the second author in Pisa, February 1996. Sections 2 and 4 have been developed by the first author, whereas the others sections due to the second author.

[2] Research partially supported by M.U.R.S.T.

[3] The second author gratefully acknowledges the research support he received as Visiting Professor of the Department of Statistics and Applied Mathematics, University of Pisa, in February 1996.

valued functions, while on the other hand, they extend the generalized monotonicity concepts elaborated so far only for real valued functions [1, 2, 9, 10, 12-16].

One of the key tools in our analysis is the Diewert's Mean Value Theorem which, for the sake of convenience, will be stated as follows [11].

Diewert's Mean Value Theorem

Let the directionally differentiable real valued function $\phi(x)$ be defined on the line segment $[y,z]$. Then there exists $\lambda_0 \in [0,1)$ such that:

$$\phi'(x_0, z-y) \leq \phi(z)-\phi(y),$$

where $x_0=y+\lambda_0(z-y) \in [y,z)$ and $\phi'(x_0, z-y) = \lim_{\lambda \to 0^+} \frac{\phi(x_0+\lambda(z-y))-\phi(x_0)}{\lambda}$.

Another important role will be played in our analysis by the following scalar function $\phi_p(x)=p^T f(x)$, $p \in \Re^m$, for which we have:

$$\phi'_p(x,d) = \lim_{\lambda \to 0^+} \frac{\phi_p(x+\lambda d)-\phi_p(x)}{\lambda} = p^T f'(x,d) .$$

Note that this directional derivative is positively homogeneous in d, which means that for all $\mu>0$ it results $\phi'_p(x,\mu d) = \mu\, \phi'_p(x,d)$.

The above properties of $\phi'_p(x,d)$ will be frequently used in the sequel without any further references.

Note finally that if function f is directionally differentiable then it is also radially continuous; this property will be useful in the rest of the paper as regard to the properties of quasiconcave functions.

2. Concavity and monotonicity

In [7, 8] several classes of vector valued concave functions have been introduced and studied; in this paragraph we will focus our attention to the following three classes.

Definition 2.1 Consider the vector valued function $f:S \to \Re^m$, where $S \subseteq \Re^n$ is a convex set, and let $C \subset \Re^m$ be a closed convex cone with not empty interior. Set $C^* \in \{C, C^0, C^{00}\}$, we will say that f is *C*-concave* [C*.cv] if and only if $\forall x,y \in S$, $x \neq y$, $\forall \lambda \in (0,1)$ it holds:

$$f(\lambda x+(1-\lambda)y)-\lambda f(x)-(1-\lambda)f(y) \in C^*.$$

In the following we will characterize these classes by means of the following properties.

Definition 2.2 Consider the vector valued function $f:S\rightarrow\Re^m$, where $S\subseteq\Re^n$ is a convex set, and let $C\subset\Re^m$ be a closed convex cone with not empty interior. We will say that:

i) f is *polarly C-concave* [p.C.cv] if and only if condition (2.1) holds:

the real valued function $\phi_p(x)=p^Tf(x)$ is concave $\forall p\in C^+$, $p\neq 0$; (2.1)

ii) f is *polarly C^0-concave* [p.C^0.cv] if and only if (2.2a) and (2.2b) hold:

the real valued function $\phi_p(x)=p^Tf(x)$ is concave $\forall p\in C^+$, $p\neq 0$, (2.2a)

$\forall x,y\in S$, $x\neq y$, $\forall\lambda\in(0,1)$ $f(\lambda x+(1-\lambda)y)-\lambda f(x)-(1-\lambda)f(y)\neq 0$; (2.2b)

iii) f is *polarly C^{00}-concave* [p.C^{00}.cv] if and only if condition (2.3) holds:

the real valued function $\phi_p(x)=p^Tf(x)$ is strictly concave $\forall p\in C^+$, $p\neq 0$. (2.3)

The following monotonicity concepts are strightforward generalizations of the monotonicity concepts used for real valued functions.

Definition 2.3 Consider the vector valued directionally differentiable function $f:S\rightarrow\Re^m$, where $S\subseteq\Re^n$ is a convex set, and let $C\subset\Re^m$ be a closed convex cone with not empty interior.

Set $C^*\in\{C, C^0, C^{00}\}$, then $f'(x,d)$ will be said *C^*-monotone* [C^*.mon] if and only if $\forall x,y\in S$, $x\neq y$, it holds:

$$f'(x,y-x)+f'(y,x-y)\in C^*.$$

Definition 2.4 Consider the vector valued directionally differentiable function $f:S\rightarrow\Re^m$, where $S\subseteq\Re^n$ is a convex set, and let $C\subset\Re^m$ be a closed convex cone with not empty interior.

Then $f'(x,d)$ will be said:

i) *polarly C-monotone* [p.C.mon] if and only if $\forall p\in C^+$, $p\neq 0$, $\forall x,y\in S$, $x\neq y$, it holds: $p^T[f'(x,y-x)+f'(y,x-y)]\geq 0$;

ii) *polarly C^0-monotone* [p.C^0.mon] if and only if $\forall p\in C^+$, $p\neq 0$, $\forall x,y\in S$, $x\neq y$, it holds: $p^T[f'(x,y-x)+f'(y,x-y)]\geq 0$ and $f'(x,y-x)+f'(y,x-y)\neq 0$;

iii) *polarly C^{00}-monotone* [p.C^{00}.mon] if and only if $\forall p\in C^+$, $p\neq 0$, $\forall x,y\in S$, $x\neq y$, it holds: $p^T[f'(x,y-x)+f'(y,x-y)]>0$.

The following theorems provide interrelations and first order characterizations of the concavity concepts introduced.

Theorem 2.1 Consider the vector valued function $f:S\to\Re^m$, where $S\subseteq\Re^n$ is a convex set, and let $C\subset\Re^m$ be a closed convex cone with not empty interior. Suppose also function f to be directionally differentiable when necessary.
Then the following statements are equivalent:

i) f is polarly C-concave,
ii) f is C-concave,
iii) $f'(y, x-y)-f(x)+f(y)\in C \ \forall x,y\in S, x\neq y$,
iv) $f'(x,d)$ is C-monotone,
v) $f'(x,d)$ is polarly C-monotone.

Proof i)⇒ii) Suppose ab absurdo that f is not C-concave, that is to say that $\exists x,y\in S, x\neq y, \exists\lambda\in(0,1)$ such that $f(\lambda x+(1-\lambda)y)-\lambda f(x)-(1-\lambda)f(y)\notin C$; since C is a convex cone, by means of a known separation theorem, $\exists p\in C^+$ such that $p^T[f(\lambda x+(1-\lambda)y)-\lambda f(x)-(1-\lambda)f(y)]<0$ so that $\phi_p(x)=p^Tf(x)$ is not concave and this is absurd.

ii)⇒iii) Since f is C-concave then $\forall x,y\in S, x\neq y, \forall\lambda\in(0,1)$ it holds:

$$\frac{f(y+\lambda(x-y))-f(y)}{\lambda}-f(x)+f(y)\in C;$$

the thesis then follows approaching $\lambda\to 0$ since C is a closed cone.

iii)⇒iv) Let $x,y\in S, x\neq y$; by the hypothesis we then have $f'(y, x-y)-f(x)+f(y)\in C$ and $f'(x, y-x)-f(y)+f(x)\in C$, adding these two conditions it results, being C convex, $f'(x, y-x)+f'(y, x-y)\in C$.

iv)⇒v) Let $p\in C^+$, $x,y\in S, x\neq y$; by the hypothesis it is $f'(x, y-x)+f'(y, x-y)\in C$ so that $p^T[f'(x, y-x)+f'(y, x-y)]\geq 0$.

v)⇒i) Suppose ab absurdo that $\exists p\in C^+, \exists x,y\in S, x\neq y, \exists\lambda\in(0,1)$ such that $p^T[f(\lambda x+(1-\lambda)y)-\lambda f(x)-(1-\lambda)f(y)]<0$. Let $z=\lambda x+(1-\lambda)y$, it follows that:

$$p^T[f(z)-f(x)]<(1-\lambda)p^T[f(y)-f(x)] \text{ and } p^T[f(z)-f(y)]<\lambda p^T[f(x)-f(y)].$$

Applying the Diewert's Mean Value Theorem to the line segments [x,z] and [y,z] then $\exists w\in[x,z)$ and $\exists v\in[y,z)$ such that:

$$p^Tf'(w, z-x)\leq p^T[f(z)-f(x)] \text{ and } p^Tf'(v, z-y)\leq p^T[f(z)-f(y)].$$

Note also that, since $f'(x,d)$ is positively homogeneous in d, being $z-x=(1-\lambda)(y-x)$ and $z-y=\lambda(x-y)$ it results:

$$f'(w, z-x)=(1-\lambda)f'(w, y-x) \text{ and } f'(v, z-y)=\lambda f'(v, x-y).$$

By means of these results it then follows that:

$$(1-\lambda)p^Tf'(w, y-x)<(1-\lambda)p^T[f(y)-f(x)] \text{ and } \lambda p^Tf'(v, x-y)<\lambda p^T[f(x)-f(y)],$$

being $\lambda>0$ and $(1-\lambda)>0$ it then follows:

$$p^T f'(w, y-x)<p^T[f(y)-f(x)] \text{ and } p^T f'(v, x-y)<p^T[f(x)-f(y)],$$

and now, by adding the two inequalities we obtain:

$$p^T f'(w, y-x)+p^T f'(v, x-y)<0.$$

Since $w-v=(1/\alpha)(x-y)$, with $\alpha>0$, we can deduce, by means of the positive homogeneity of $f'(x,d)$, that $p^T f'(w, v-w)+p^T f'(v, w-v)<0$ and this contradicts the polarly C-monotonicity of $f'(x,d)$. ♦

Theorem 2.2 Consider the vector valued function $f:S\to\Re^m$, where $S\subseteq\Re^n$ is a convex set, and let $C\subset\Re^m$ be a closed pointed convex cone with not empty interior. Suppose also function f to be directionally differentiable when necessary. Then the following statements are equivalent:

i) f is polarly C^0-concave,
ii) f is C^0-concave,
iii) $f'(y, x-y)-f(x)+f(y)\in C^0$ $\forall x,y\in S$, $x\neq y$,
iv) $f'(x,d)$ is C^0-monotone,
v) $f'(x,d)$ is polarly C^0-monotone.

Proof *i)⇒ii)* The thesis follows directly from Theorem 2.1 since a C^0-concave function is a C-concave function such that condition (2.2b) holds.

ii)⇒iii) Let $x,y\in S$, $x\neq y$, $\lambda\in(0,1)$ and set $z=\lambda x+(1-\lambda)y$; since f is C^0-concave then it is also C-concave so that $f'(y, z-y)-f(z)+f(y)\in C$, note also that since f is C^0-concave then $f(z)-\lambda f(x)-(1-\lambda)f(y)\in C^0$ so that, being C a pointed convex cone, it results $f'(y, z-y)-\lambda f(x)+\lambda f(y)\in C^0$; being $f'(x,d)$ positively homogeneous in d, then $f'(y, z-y)=f'(y, \lambda(x-y))=\lambda f'(y, x-y)$ so that the thesis holds.

iii)⇒iv) Let $x,y\in S$, $x\neq y$; by hypothesis we have $f'(y, x-y)-f(x)+f(y)\in C^0$ and $f'(x, y-x)-f(y)+f(x)\in C^0$, adding these two vectors it results, being C convex and pointed, $f'(x, y-x)+f'(y, x-y)\in C^0$.

iv)⇒v) Let $p\in C^+$, $x,y\in S$, $x\neq y$; by hypothesis it is $f'(x, y-x)+f'(y, x-y)\in C^0$ so that $p^T[f'(x, y-x)+f'(y, x-y)]\geq 0$ and $f'(x, y-x)+f'(y, x-y)\neq 0$.

v)⇒i) If $f'(x,d)$ is polarly C^0-monotone then it is also polarly C-monotone so that, by means of Theorem 2.1, f is polarly C-concave; suppose now ab absurdo that f is polarly C-concave but not polarly C^0-concave, so that $\exists x,y\in S$, $x\neq y$, $\exists\lambda\in(0,1)$ such that $f(\lambda x+(1-\lambda)y)-\lambda f(x)-(1-\lambda)f(y)=0$.

Let $z=\lambda x+(1-\lambda)y$. Then $\forall p\in C^+$, $p\neq 0$, it results:

$$p^T[f(z)-f(x)]=(1-\lambda)p^T[f(y)-f(x)] \text{ and } p^T[f(z)-f(y)]=\lambda p^T[f(x)-f(y)].$$

From v) it follows that $f'(x, y-x)+f'(y, x-y)\in C^0$ and since C is pointed then there exists $r\in C^+$, $r\neq 0$, such that $r^T[f'(x, y-x)+f'(y, x-y)]>0$.

Let us consider now the real valued function $\phi_r(x)=r^T f(x)$ which is concave (since f is polarly C-concave); we have furthermore that:

$$\phi_r(z)=r^T f(\lambda x+(1-\lambda)y)=r^T[\lambda f(x)+(1-\lambda)f(y)]=\lambda\phi_r(x)+(1-\lambda)\phi_r(y).$$

It can easily be proved that from this conditions it follows that $\phi_r(x)$ is linear on the line segment [x,y]. An immediate consequence of this is that:

$$\phi'_r(x,y-x)=r^T f'(x,y-x)=\phi_r(y)-\phi_r(x) \text{ and } \phi'_r(y,x-y)=r^T f'(y,x-y)=\phi_r(x)-\phi_r(y);$$

by adding these two equations we obtain $r^T f'(x,y-x)+r^T f'(y,x-y)=0$ which is a contradiction. ♦

Theorem 2.3 Consider the vector valued function $f:S\rightarrow\Re^m$, where $S\subseteq\Re^n$ is a convex set, and let $C\subset\Re^m$ be a closed convex cone with not empty interior. Suppose also function f to be directionally differentiable when necessary.
Then the following statements are equivalent:

i) f is polarly C^{00}-concave,
ii) f is C^{00}-concave,
iii) $f'(y,x-y)-f(x)+f(y)\in C^{00}\ \forall x,y\in S,\ x\neq y$,
iv) $f'(x,d)$ is C^{00}-monotone,
v) $f'(x,d)$ is polarly C^{00}-monotone.

Proof i)⇒ii) Suppose ab absurdo that f is not C^{00}-concave, that is to say that $\exists x,y\in S,\ x\neq y,\ \exists\lambda\in(0,1)$ such that $f(\lambda x+(1-\lambda)y)-\lambda f(x)-(1-\lambda)f(y)\notin C^{00}$; since C is convex, by means of a known separation theorem, $\exists p\in C^+,\ p\neq 0$, such that $p^T[f(\lambda x+(1-\lambda)y)-\lambda f(x)-(1-\lambda)f(y)]\leq 0$ so that $\phi_p(x)=p^T f(x)$ is not strictly concave and this is absurd.

ii)⇒iii) The proof is similar to the one given for Theorem 2.2.

iii)⇒iv) The proof is similar to the one given for Theorem 2.1.

iv)⇒v) Let $p\in C^+$, $x,y\in S$, $x\neq y$; by the hypothesis it is $f'(x,y-x)+f'(y,x-y)\in C^{00}$ so that $p^T[f'(x,y-x)+f'(y,x-y)]>0$.

v)⇒i) The proof is similar to the one given for Theorem 2.1 ♦

3. Quasiconcavity and quasimonotonicity

In the very recent works [7, 8] several classes of vector valued quasiconcave functions were introduced and investigated; from these classes we will consider only the following ones.

Definition 3.1 Consider the vector valued function $f:S \rightarrow \Re^m$, where $S \subseteq \Re^n$ is a convex set, and let $C \subset \Re^m$ be a closed convex cone with not empty interior. We will say that:

i) f is *(C,C)-quasiconcave* [(C,C).qcv] if and only if $\forall x,y \in S$, $x \neq y$, $\forall \lambda \in (0,1)$ it holds: $$f(x)-f(y) \in C \Rightarrow f(\lambda x+(1-\lambda)y)-f(y) \in C;$$

ii) f is *(C^{00},C^{00})-quasiconcave* [(C^{00},C^{00}).qcv] if and only if $\forall x,y \in S$, $x \neq y$, $\forall \lambda \in (0,1)$ it holds:
$$f(x)-f(y) \in C^{00} \Rightarrow f(\lambda x+(1-\lambda)y)-f(y) \in C^{00};$$

iii) f is *(C,C^{00})-quasiconcave* [(C,C^{00}).qcv] if and only if $\forall x,y \in S$, $x \neq y$, $\forall \lambda \in (0,1)$ it holds:
$$f(x)-f(y) \in C \Rightarrow f(\lambda x+(1-\lambda)y)-f(y) \in C^{00};$$

In the following we shall relate the above classes with the following ones.

Definition 3.2 Consider the vector valued function $f:S \rightarrow \Re^m$, where $S \subseteq \Re^n$ is a convex set, and let $C \subset \Re^m$ be a closed convex cone with not empty interior. We will say that:

i) f is *polarly C-quasiconcave* [p.C.qcv] if and only if $\phi_p(x)=p^T f(x)$ is quasiconcave $\forall p \in C^+$, $p \neq 0$, that is to say if and only if $\forall p \in C^+$, $p \neq 0$, $\forall x,y \in S$, $x \neq y$, $\forall \lambda \in (0,1)$ it holds:
$$p^T f(x)-p^T f(y) \geq 0 \Rightarrow p^T f(\lambda x+(1-\lambda)y)-p^T f(y) \geq 0;$$

ii) f is *polarly C^0-quasiconcave* [p.C^0.qcv] if and only if $\phi_p(x)=p^T f(x)$ is semistrictly quasiconcave $\forall p \in C^+$, $p \neq 0$, that is to say if and only if $\forall p \in C^+$, $p \neq 0$, $\forall x,y \in S$, $x \neq y$, $\forall \lambda \in (0,1)$ it holds:
$$p^T f(x)-p^T f(y)>0 \Rightarrow p^T f(\lambda x+(1-\lambda)y)-p^T f(y)>0;$$

iii) f is *polarly C^{00}-quasiconcave* [p.C^{00}.qcv] if and only if $\phi_p(x)=p^T f(x)$ is strictly quasiconcave $\forall p \in C^+$, $p \neq 0$, that is to say if and only if $\forall p \in C^+$, $p \neq 0$, $\forall x,y \in S$, $x \neq y$, $\forall \lambda \in (0,1)$ it holds:
$$p^T f(x)-p^T f(y) \geq 0 \Rightarrow p^T f(\lambda x+(1-\lambda)y)-p^T f(y)>0;$$

Note that, when f is radially continuous, if $\phi_p(x)$ is strictly quasiconcave then it is also semistrictly quasiconcave and if it is semistrictly quasiconcave then it is also quasiconcave; it then results that, under such an hypothesis, polarly C^{00}-quasiconcavity implies polarly C^0-quasiconcavity and polarly C^0-quasiconcavity implies polarly C-quasiconcavity.

Proposition 3.1 Consider the vector valued function $f:S\to\Re^m$, where $S\subseteq\Re^n$ is a convex set, and let $C\subset\Re^m$ be a closed convex cone with not empty interior.

i) If f is polarly C-quasiconcave then it is also (C,C)-quasiconcave;

ii) If f is polarly C^0-quasiconcave then it is also (C^{00},C^{00})-quasiconcave;

iii) If f is polarly C^{00}-quasiconcave then it is also (C,C^{00})-quasiconcave;

Proof i) Suppose ab absurdo that f is not (C,C)-quasiconcave, that is to say that $\exists x,y\in S$, $x\neq y$, $\exists\lambda\in(0,1)$ such that $f(x)-f(y)\in C$ and $f(\lambda x+(1-\lambda)y)-f(y)\notin C$; since C is a convex cone, by means of a known separation theorem, $\exists p\in C^+$ such that $p^T[f(\lambda x+(1-\lambda)y)-f(y)]<0$; since $f(x)-f(y)\in C$ it is also $p^Tf(x)-p^Tf(y)\geq 0$ so that, being f polarly C-quasiconcave, $p^Tf(\lambda x+(1-\lambda)y)-p^Tf(y)\geq 0$ which is a contradiction.

ii) Suppose ab absurdo that f is not (C^{00},C^{00})-quasiconcave, that is to say that $\exists x,y\in S$, $x\neq y$, $\exists\lambda\in(0,1)$ such that $f(x)-f(y)\in C^{00}$ and $f(\lambda x+(1-\lambda)y)-f(y)\notin C^{00}$; since C is a convex cone, by means of a known separation theorem, $\exists p\in C^+$ such that $p^T[f(\lambda x+(1-\lambda)y)-f(y)]\leq 0$; since $f(x)-f(y)\in C^{00}$ it is also $p^Tf(x)-p^Tf(y)>0$ so that, being f polarly C^0-quasiconcave, $p^Tf(\lambda x+(1-\lambda)y)-p^Tf(y)>0$ which is a contradiction.

iii) The proof is similar to the ones given for i) and ii). ♦

The following example shows that a vector valued quasiconcave function is not polarly quasiconcave in general.

Example 3.1 Let $f(x)=(-2x, x^2+2x)$, $x\in S=[-1,1]$, and let $C\subset\Re^2$ be the Paretian cone. Function f is (C,C).qcv and (C^{00},C^{00}).qcv in S, since $\nexists x,y\in[-1,1]$, $x\neq y$, such that $f(x)-f(y)\in C$, but it is not polarly C.qcv (neither polarly C^0.qcv nor polarly C^{00}.qcv) in S, since for $p^T=(1,1)$ the real valued function $\phi_p(x)=p^Tf(x)=x^2$ is strictly convex and not quasiconcave in [-1,1].

We would like to notice that the concepts of quasiconcavity given in definition 3.1 are weak in order to ensure the convexity of the upper level sets of the given function, see [8] Example 3.3, while the polar concepts of C-quasiconcavity ensure the above property.

In the following we will characterize polarly C*-quasiconcavity by means of the following properties of the directional derivatives.

Definition 3.3 Consider the vector valued directionally differentiable function $f:S\to\Re^m$, where $S\subseteq\Re^n$ is a convex set, and let $C\subset\Re^m$ be a closed convex cone with not empty interior. Then $f'(x,d)$ will be said:

i) *polarly C-quasimonotone* [p.C.qmon] if and only if (3.1) holds $\forall p\in C^+$, $p\neq 0$, $\forall x,y\in S$, $x\neq y$:

$$p^T f'(x,y-x)<0 \Rightarrow p^T f'(y,x-y)\geq 0; \quad (3.1)$$

ii) *polarly C^0-quasimonotone* [p.C^0.qmon] if and only if both conditions (3.2a) and (3.2b) hold $\forall p\in C^+$, $p\neq 0$, $\forall x,y\in S$, $x\neq y$:

$$p^T f'(x,y-x)<0 \Rightarrow p^T f'(y,x-y)\geq 0; \quad (3.2a)$$

$$p^T f'(x,y-x)<0 \Rightarrow \exists z\in [\tfrac{x+y}{2},y) \text{ such that } p^T f'(z,y-x)<0 \quad (3.2b)$$

iii) *polarly C^{00}-quasimonotone* [p.C^{00}.qmon] if and only if both conditions (3.3a) and (3.3b) hold $\forall p\in C^+$, $p\neq 0$, $\forall x,y\in S$, $x\neq y$:

$$p^T f'(x,y-x)<0 \Rightarrow p^T f'(y,x-y)\geq 0; \quad (3.3a)$$

$$\exists z\in (x,y) \text{ such that either } p^T f'(z,x-y)<0 \text{ or } p^T f'(z,y-x)<0 \quad (3.3b)$$

Theorem 3.1 Consider the vector valued directionally differentiable function $f:S\to\Re^m$, where $S\subseteq\Re^n$ is a convex set, and let $C\subset\Re^m$ be a closed convex cone with not empty interior.

i) If $f'(x,d)$ is polarly C^0-quasimonotone then it is polarly C-quasimonotone

ii) If $f'(x,d)$ is polarly C^{00}-quasimonotone then it is polarly C^0-quasimonotone

Proof Since i) follows directly from the definition, we just have to prove ii).
Let $f'(x,d)$ be polarly C^{00}-quasimonotone and let $p\in C^+$, $p\neq 0$, $x,y\in S$, $x\neq y$, be such that $p^T f'(x,y-x)<0$; by means of the positive homogeneity of $f'(x,d)$ we then have $p^T f'(x,z-x)<0$ $\forall z\in(x,y]$ so that, by (3.3a), it results $p^T f'(z,x-z)\geq 0$ $\forall z\in(x,y]$, this implies also that $p^T f'(z,x-y)\geq 0$ $\forall z\in(x,y]$.
By (3.3b) $\exists z\in(\frac{x+y}{2},y)$ such that either $p^T f'(z,y-\frac{x+y}{2}) = =\frac{1}{2}p^T f'(z,y-x)<0$ or $p^T f'(z,\frac{x+y}{2}-y)=\frac{1}{2}p^T f'(z,x-y)<0$. The thesis follows since this last possibility cannot occure being $p^T f'(z,x-y)\geq 0$ $\forall z\in(x,y]$. ♦

In the rest of the paper will be used the following results [6].

Theorem 3.2 Let $S\subseteq\Re^n$ be a convex set and let $\phi:S\to\Re$ be a real valued quasiconcave function. It then results that:

i) ϕ is not semistrictly quasiconcave if and only if $\exists x,y\in S$, $x\neq y$, such that $\phi(x)>\phi(y)$ and $\exists\bar\lambda\in(0,1)$ such that:

$$\phi(\lambda x+(1-\lambda)y)=\phi(y)\ \ \forall\lambda\in(0,\bar\lambda] \quad \text{and} \quad \phi(\lambda x+(1-\lambda)y)\geq\phi(y)\ \ \forall\lambda\in(\bar\lambda,1);$$

ii) ϕ is not strictly quasiconcave if and only if $\exists x,y\in S$, $x\neq y$, such that:

$$\phi(\lambda x+(1-\lambda)y)=\phi(x)=\phi(y)\quad \forall\lambda\in(0,1).$$

Theorem 3.3 Consider the vector valued directionally differentiable function $f:S\rightarrow\Re^m$, where $S\subseteq\Re^n$ is a convex set, and let $C\subset\Re^m$ be a closed convex cone with not empty interior. Then the following statements are equivalent:

i) f is polarly C-quasiconcave,

ii) $p^Tf(x)-p^Tf(y)\geq 0 \Rightarrow p^Tf'(y,x-y)\geq 0 \quad \forall p\in C^+, \forall x,y\in S, x\neq y$,

iii) $f'(x,d)$ is polarly C-quasimonotone.

Proof i)⇒ii) Since f is polarly C-quasiconcave then $\forall p\in C^+$, $\forall\lambda\in(0,1)$, $\forall x,y\in S$, $x\neq y$, such that $p^Tf(x)-p^Tf(y)\geq 0$ it holds $\frac{p^Tf(y+\lambda(x-y))-p^Tf(y)}{\lambda}\geq 0$, so that the thesis follows approaching $\lambda\rightarrow 0^+$.

ii)⇒iii) Let $p\in C^+$, $x,y\in S$, $x\neq y$, be such that $p^Tf'(x,y-x)<0$; by means of the hypothesis this inequality implies that $p^Tf(y)-p^Tf(x)<0$ so that, by means of the same condition, $p^Tf'(y,x-y)\geq 0$.

iii)⇒i) Suppose ab absurdo that $\exists p\in C^+$, $\exists x,y\in S$, $x\neq y$, $\exists\lambda\in(0,1)$ such that $p^Tf(x)\geq p^Tf(y)>p^Tf(\lambda x+(1-\lambda)y)$.

Let $z=\lambda x+(1-\lambda)y$; applying the Diewert's Mean Value Theorem to the line segments $[x,z]$ and $[y,z]$ then $\exists w\in[x,z)$ and $\exists v\in[y,z)$ such that:

$$p^Tf'(w,z-x)\leq p^T[f(z)-f(x)]<0 \quad\text{and}\quad p^Tf'(v,z-y)\leq p^T[f(z)-f(y)]<0.$$

Since $w-v=(1/\alpha)(z-y)$ and $v-w=(1/\beta)(z-x)$, with $\alpha,\beta>0$, we can deduce, by means of the positive homogeneity of $f'(x,d)$, that:

$$p^Tf'(w,v-w)<0 \quad\text{and}\quad p^Tf'(v,w-v)<0.$$

and this contradicts the polarly C-quasimonotonicity of $f'(x,d)$. ♦

Theorem 3.4 Consider the vector valued directionally differentiable function $f:S\rightarrow\Re^m$, where $S\subseteq\Re^n$ is a convex set, and let $C\subset\Re^m$ be a closed convex cone with not empty interior. Then the following statements are equivalent:

i) f is polarly C^0-quasiconcave,

ii) both the following conditions hold $\forall p\in C^+$, $\forall x,y\in S$, $x\neq y$:

$$p^Tf(x)-p^Tf(y)\geq 0 \Rightarrow p^Tf'(y,x-y)\geq 0, \tag{3.4a}$$

$$p^Tf'(x,y-x)<0 \Rightarrow \exists z\in[\tfrac{x+y}{2},y) \text{ such that } p^Tf'(z,y-x)<0, \tag{3.4b}$$

iii) $f'(x,d)$ is polarly C^0-quasimonotone,

Proof i)⇒ii) Since f is polarly C^0-quasiconcave then it is also polarly C-quasiconcave so that, by means of Theorem 3.3, condition (3.4a) holds, so that we just have to verify condition (3.4b).
Let $p\in C^+$, $p\neq 0$, $x,y\in S$, $x\neq y$, be such that $p^Tf'(x,y-x)<0$; from (3.4a) then $p^Tf(x)-p^Tf(y)>0$ and this inequality implies, by the polarly C^0-quasiconcavity of f, that $p^Tf(u)-p^Tf(y)>0$ $\forall u\in(x,y)$. By means of the Diewert's Mean Value Theorem applied to $[u,y]$ then $\exists z\in[u,y)$ such that $p^Tf'(z,y-u)\leq p^T[f(y)-f(u)]<0$. Set $u=\frac{x+y}{2}$, and the thesis follows.
ii)⇒iii) Condition (3.4a) implies that f is polarly C-quasiconcave so that, due to Theorem 3.3, (3.2a) holds; the other condition holds by hypothesis.
iii)⇒i) Suppose ab absurdo that $\exists p\in C^+$, $p\neq 0$, such that $\phi_p(x)=p^Tf(x)$ is not semistrictly quasiconcave. Since $f'(x,d)$ is polarly C^0-quasimonotone then it is also polarly C-quasimonotone and thus f is polarly C-quasiconcave, it then results that $\phi_p(x)$ is a quasiconcave function. By means of Theorem 3.2, being $\phi_p(x)=p^Tf(x)$ quasiconcave but not semistrictly quasiconcave, it then follows that $\exists x,y\in S$, $x\neq y$, such that $p^Tf(x)>p^Tf(y)$ and $\exists w\in(x,y)$ such that $p^Tf(v)=p^Tf(y)$ $\forall v\in[w,y)$ and $p^Tf(v)\geq p^Tf(y)$ $\forall v\in(x,w)$; note that, due to the radial continuity of $\phi_p(x)$, we can suppose, without loss of generality, that $w\in(x,y)$ is such that $p^Tf(v)>p^Tf(w)=p^Tf(y)$ $\forall v\in(x,w)$; note also that, since $p^Tf(v)=p^Tf(y)$ $\forall v\in[w,y)$, it is $p^Tf'(v,y-x)=0$ $\forall v\in(w,y)$. Let now $u\in(x,w)$ be such that $\frac{u+y}{2}\in(w,y)$; applying the Diewert's Mean Value Theorem to the line segment $[u,y]$ then $\exists z\in[u,y)$ such that $p^Tf'(z,y-u)\leq p^T[f(y)-f(u)]<0$; then, being $f'(x,d)$ positive homogeneous, $p^Tf'(z,y-z)<0$ so that by the hypothesis:

$$\exists k\in\left[\frac{z+y}{2},y\right)\subseteq\left[\frac{u+y}{2},y\right)\subset(w,y) \text{ such that } p^Tf'(k,y-z)<0;$$

By means of the positive homogeneity of $f'(x,d)$, we then have that $\exists k\in(w,y)$ such that $p^Tf'(k,y-x)<0$ and this is a contradiction. ♦

Theorem 3.5 Consider the vector valued directionally differentiable function $f:S\to\Re^m$, where $S\subseteq\Re^n$ is a convex set, and let $C\subset\Re^m$ be a closed convex cone with not empty interior. Then the following statements are equivalent:

i) f is polarly C^{00}-quasiconcave,
ii) both the following conditions hold $\forall p\in C^+$, $\forall x,y\in S$, $x\neq y$:

$$p^Tf(x)-p^Tf(y)\geq 0 \Rightarrow p^Tf'(y,x-y)\geq 0, \quad (3.5a)$$

$$\exists z\in(x,y) \text{ such that either } p^Tf'(z,x-y)<0 \text{ or } p^Tf'(z,y-x)<0 \quad (3.5b)$$

iii) $f'(x,d)$ is polarly C^{00}-quasimonotone,

Proof i)⇒ii) Since f is polarly C^{00}-quasiconcave then it is also polarly C-quasiconcave so that, by Theorem 3.3, (3.5a) holds. Let $p\in C^+$, $p\neq 0$, $x,y\in S$, $x\neq y$, arbitrarily chosen. Assume first that $p^Tf(y)\geq p^Tf(x)$. Let $u\in(x,y)$, then by the polarly C^{00}-quasiconcavity of f, we have $p^Tf(u)-p^Tf(x)>0$; applying the Diewert's Mean Value Theorem in $[u,x]$ we have that $\exists z\in[u,x)$ such that:

$$p^Tf'(z,x-u)\leq p^Tf(x)-p^Tf(u)<0,$$

consequently we have $p^Tf'(z,x-y)<0$.

Assume now that $p^Tf(x)>p^Tf(y)$. By repeating the arguments of the previous case, we derive that $\exists z\in(x,y)$ such that $p^Tf'(z,y-x)<0$.

ii)⇒iii) Condition (3.5a) implies that f is polarly C-quasiconcave so that, due to Theorem 3.3, (3.3a) holds; the other condition holds by hypothesis.

iii)⇒i) Suppose ab absurdo that $\exists p\in C^+$, $p\neq 0$, such that $\phi_p(x)=p^Tf(x)$ is not strictly quasiconcave; since $f'(x,d)$ is polarly C^{00}-quasimonotone then it is also polarly C-quasimonotone so that, by Theorem 3.3, $\phi_p(x)$ is a quasiconcave function. By means of Theorem 3.2, being $\phi_p(x)=p^Tf(x)$ quasiconcave but not strictly quasiconcave, it then follows that $\exists x,y\in S$, $x\neq y$, such that:

$$p^Tf(\lambda x+(1-\lambda)y)=p^Tf(x)=p^Tf(y) \quad \forall\lambda\in(0,1);$$

we then have that $p^Tf'(z,y-x)=p^Tf'(z,x-y)=0$ $\forall z\in(x,y)$ and this contradicts condition (3.3b). ♦

4. Pseudoconcavity and pseudomonotonicity

In [7, 8] also several classes of vector valued pseudoconcave functions have been introduced and investigated; from these classes we will consider only the following ones.

Definition 4.1 Consider the vector valued directionally differentiable function $f:S\to\Re^m$, where $S\subseteq\Re^n$ is a convex set, and let $C\subset\Re^m$ be a closed convex cone with not empty interior. We will say that:

i) f is *(C^{00},C^{00})-pseudoconcave* [(C^{00},C^{00}).pcv] if and only if $\forall x,y\in S$, $x\neq y$, it holds: $f(x)-f(y)\in C^{00} \Rightarrow f'(y,x-y)\in C^{00}$;

ii) f is *(C^{0},C^{00})-pseudoconcave* [(C^{0},C^{00}).pcv] if and only if $\forall x,y\in S$, $x\neq y$, it holds: $f(x)-f(y)\in C^{0} \Rightarrow f'(y,x-y)\in C^{00}$;

iii) f is *(C,C^{00})-pseudoconcave* [(C,C^{00}).pcv] if and only if $\forall x,y\in S$, $x\neq y$, it holds: $f(x)-f(y)\in C \Rightarrow f'(y,x-y)\in C^{00}$;

In the following we shall relate the above classes with the following ones.

Definition 4.2 Consider the vector valued directionally differentiable function $f:S\to\Re^m$, where $S\subseteq\Re^n$ is a convex set, and let $C\subset\Re^m$ be a closed convex cone with not empty interior. We will say that:

i) f is *polarly C-pseudoconcave* [p.C.pcv] if and only if $\phi_p(x)=p^Tf(x)$ is pseudoconcave $\forall p\in C^+$, $p\neq 0$, that is to say if and only if $\forall p\in C^+$, $p\neq 0$, $\forall x,y\in S$, $x\neq y$, it holds:

$$p^Tf(x)-p^Tf(y)>0 \Rightarrow p^Tf'(y,x-y)>0;$$

ii) f is *polarly C^0-pseudoconcave* [p.C^0.pcv] if and only if $\forall p\in C^+$, $p\neq 0$, $\forall x,y\in S$, $x\neq y$, it holds:

$$p^Tf(x)-p^Tf(y)\geq 0 \text{ with } f(x)\neq f(y) \Rightarrow p^Tf'(y,x-y)>0;$$

iii) f is *polarly C^{00}-pseudoconcave* [p.C^{00}.pcv] if and only if $\phi_p(x)=p^Tf(x)$ is strictly pseudoconcave $\forall p\in C^+$, $p\neq 0$, that is to say if and only if $\forall p\in C^+$, $p\neq 0$, $\forall x,y\in S$, $x\neq y$, it holds:

$$p^Tf(x)-p^Tf(y)\geq 0 \Rightarrow p^Tf'(y,x-y)>0;$$

Note that, by means of the definitions, it follows that a polarly C^{00}-pseudoconcave function is also polarly C^0-pseudoconcave and that a polarly C^0-pseudoconcave function is also polarly C-pseudoconcave.

Proposition 4.1 Consider the vector valued function $f:S\to\Re^m$, where $S\subseteq\Re^n$ is a convex set, and let $C\subset\Re^m$ be a closed convex cone with not empty interior.

i) If f is polarly C-pseudoconcave then it is also (C^{00},C^{00})-pseudoconcave;

ii) If f is polarly C^0-pseudoconcave then it is also (C^0,C^{00})-pseudoconcave;

iii) If f is polarly C^{00}-pseudoconcave then it is also (C,C^{00})-pseudoconcave;

Proof i) Suppose ab absurdo that f is not (C^{00},C^{00})-pseudoconcave, that is to say that $\exists x,y\in S$, $x\neq y$, such that $f(x)-f(y)\in C^{00}$ and $f'(y,x-y)\notin C^{00}$; since C is a convex cone, by means of a known separation theorem, $\exists p\in C^+$ such that $p^Tf'(y,x-y)\leq 0$; since $f(x)-f(y)\in C^{00}$ it is also $p^Tf(x)-p^Tf(y)>0$ so that, being f polarly C-pseudoconcave, $p^Tf'(y,x-y)>0$ which is a contradiction.

ii) Suppose ab absurdo that f is not (C^0,C^{00})-pseudoconcave, that is to say that $\exists x,y\in S$, $x\neq y$, such that $f(x)-f(y)\in C^0$ and $f'(y,x-y)\notin C^{00}$; since C is a convex cone, by means of a separation theorem, $\exists p\in C^+$ such that $p^Tf'(y,x-y)\leq 0$; since $f(x)-f(y)\in C^0$ it is also $p^Tf(x)-p^Tf(y)\geq 0$ with $f(x)\neq f(y)$ so that, being f polarly C^0-pseudoconcave, $p^Tf'(y,x-y)>0$ which is a contradiction.

iii) The proof is similar to the ones given for i) and ii). ♦

The following statement is direct consequence of a result of Diewert [11].

Proposition 4.2 Consider the vector valued function $f:S\rightarrow\Re^m$, where $S\subseteq\Re^n$ is a convex set, and let $C\subset\Re^m$ be a closed convex cone with not empty interior. Then if f is polarly C-pseudoconcave the f is also polarly C-quasiconcave.

Remark 4.1 Note that Example 3.1 shows that a vector valued pseudoconcave function is not in general polarly pseudoconcave.
Function f is (C,C^{00}).pcv (and also both (C^0,C^{00}).pcv and (C^{00},C^{00}).pcv) in S, since $\nexists x,y\in[-1,1]$, $x\neq y$, such that $f(x)-f(y)\in C$, but it is not polarly C.pcv (neither polarly C^0.pcv nor polarly C^{00}.pcv) in S, since for $p^T=(1,1)$ the real valued function $\phi_p(x)=p^Tf(x)=x^2$ is strictly convex and not pseudoconcave in [-1,1].

In the following we will characterize polarly C*-pseudoconcavity by means of the following properties of the directional derivatives.

Definition 4.3 Consider the vector valued directionally differentiable function $f:S\rightarrow\Re^m$, where $S\subseteq\Re^n$ is a convex set, and let $C\subset\Re^m$ be a closed convex cone with not empty interior. Then $f'(x,d)$ will be said:

i) *polarly C-pseudomonotone* [p.C.pmon] if and only if $\forall p\in C^+$, $p\neq 0$, $\forall x,y\in S$, $x\neq y$, it holds:
$$p^Tf'(x,y-x)\leq 0 \Rightarrow p^Tf'(y,x-y)\geq 0;$$
ii) *polarly C^0-pseudomonotone* [p.C^0.pmon] if and only if $\forall p\in C^+$, $p\neq 0$, $\forall x,y\in S$, $x\neq y$, it holds:
$$p^Tf'(x,y-x)\leq 0 \text{ with } f(x)\neq f(y) \Rightarrow p^Tf'(y,x-y)>0;$$
iii) *polarly C^{00}-pseudomonotone* [p.C^{00}.pmon] if and only if $\forall p\in C^+$, $p\neq 0$, $\forall x,y\in S$, $x\neq y$, it holds:
$$p^Tf'(x,y-x)\leq 0 \Rightarrow p^Tf'(y,x-y)>0;$$

Theorem 4.1 Consider the vector valued directionally differentiable function $f:S\rightarrow\Re^m$, where $S\subseteq\Re^n$ is a convex set, and let $C\subset\Re^m$ be a closed convex cone with not empty interior. Then the following statements are equivalent:

i) f is polarly C-pseudoconcave,
ii) $f'(x,d)$ is polarly C-pseudomonotone.

Proof i)⇒ii) Let $p\in C^+$, $x,y\in S$, $x\neq y$, be such that $p^Tf'(x,y-x)\leq 0$; by means of the hypothesis this inequality implies that $p^Tf(y)-p^Tf(x)\leq 0$; being f polarly C-pseudoconcave then it is also polarly C-quasiconcave so that $p^Tf(x)-p^Tf(y)\geq 0$ implies $p^Tf'(y,x-y)\geq 0$.

ii)⇒i) Suppose ab absurdo that f is not polarly C-pseudoconcave, so that $\exists p\in C^+$, $\exists x,y\in S$, $x\neq y$, such that $p^Tf(x)-p^Tf(y)>0$ and $p^Tf'(y,x-y)\leq 0$.

Due to the positive homogeneity of the directional derivative it follows that $p^Tf'(y,z-y)\leq 0$ for all $z\in[x,y]$; since f is polarly C-pseudomonotone therefore it follows that $p^Tf'(z,y-z)\geq 0$ for all $z\in[x,y]$, so that by means of the positive homogeneity of $f'(x,d)$ we have $p^Tf'(z,y-x)\geq 0$ for all $z\in[x,y]$. Let us now apply the Diewert's Mean Value Theorem to the line segment $[x,y]$; then there exists $w\in[x,y)$ such that $p^Tf'(w,y-x)\leq p^T[f(y)-f(x)]<0$ and this is absurd. ♦

Theorem 4.2 Consider the vector valued directionally differentiable function $f:S\rightarrow\Re^m$, where $S\subseteq\Re^n$ is a convex set, and let $C\subset\Re^m$ be a closed convex cone with not empty interior. Then the following statements are equivalent:

i) f is polarly C^0-pseudoconcave,

ii) $f'(x,d)$ is polarly C^0-pseudomonotone.

Proof i)⇒ii) Let $p\in C^+$, $x,y\in S$, $x\neq y$, be such that $p^Tf'(x,y-x)\leq 0$ and $f(x)\neq f(y)$; by hypothesis this inequality implies that $p^Tf(y)-p^Tf(x)<0$; by means of the same hypothesis we then have $p^Tf'(y,x-y)>0$.

ii)⇒i) Since $f'(x,d)$ is polarly C^0-pseudomonotone then it is also polarly C-pseudomonotone so that f is polarly C-pseudoconcave. Suppose now ab absurdo that f is not polarly C^0-pseudoconcave, so that $\exists p\in C^+$, $\exists x,y\in S$, $x\neq y$, such that $p^Tf(x)-p^Tf(y)=0$, $f(x)\neq f(y)$ and $p^Tf'(x,y-x)\leq 0$. By hypothesis, it implies that $p^Tf'(y,x-y)>0$. It then follows that x-y is an ascent direction of $p^Tf(x)$ at y, consequently it follows that $\exists w\in(x,y)$ such that $p^Tf(w)-p^Tf(y)>0$; this implies, since $p^Tf(x)=p^Tf(y)$, that $p^Tf(x)-p^Tf(w)<0$.

Let us now apply the Diewert's Mean Value Theorem to the line segment $[w,x]$; then there exists $v\in[w,x)$ such that $p^Tf'(v,x-w)\leq p^T[f(x)-f(w)]<0$; it then follows that $p^Tf'(v,x-y)<0$ with $v\in(x,y)$.

Since $p^Tf'(x,y-x)\leq 0$, due to the positive homogeneity of the directional derivative it follows that $p^Tf'(x,z-x)\leq 0$ for all $z\in[x,y)$; since f is also polarly C-pseudomonotone therefore it follows that $p^Tf'(z,x-z)\geq 0$ for all $z\in[x,y)$, so that, being $f'(x,d)$ positive homogeneous, we have $p^Tf'(z,x-y)\geq 0$ for all $z\in[x,y)$, so that in particular for the above v we have $p^Tf'(v,x-y)\geq 0$ and this is a contradiction. ♦

Theorem 4.3 Consider the vector valued directionally differentiable function $f:S\to\Re^m$, where $S\subseteq\Re^n$ is a convex set, and let $C\subset\Re^m$ be a closed convex cone with not empty interior. Then the following statements are equivalent:

i) f is polarly C^{00}-pseudoconcave,

ii) f '(x,d) is polarly C^{00}-pseudomonotone.

Proof *i)⇒ii)* The proof is similar to the one given in Theorem 4.2.

ii)⇒i) The proof is similar to the one given in Theorem 4.1. ♦

5. Concluding remarks

The authors are convinced that the polar concepts of generalized concavity are more restrictive than the C* generalized concavity concepts, but in many cases these concepts could provide nicer structures to the vector optimization problem; expecially we are aware that, in the image space approach to vector optimization problems with polarly C* generalized concave functions, the image set will have some nice particular properties. This idea will be developed in other papers.

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