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Second Order Optimality Conditions in the Image Space

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SECOND ORDER OPTIMALITY CONDITIONS IN THE IMAGE SPACE

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The aim of this paper is to extend, for vector maximization problems, some recent results obtained in the image space regarding to scalar problems. Some second order necessary optimality conditions in the image space will be stated by means of two different kinds of regularity conditions. Several second order sufficient optimality conditions in the image space will be also established.

1. Introduction

In these last years, the image space has been frequently used [5-9, 11, 12] as a general framework within which some topics related to optimization have been studied [1, 2, 14, 16, 17]; recently, for a scalar problem, Cambini and Martein [10] have suggested a new approach for studying necessary second order optimality conditions based on a characterization of a suitable tangent cone in the image space [7, 8, 10].

The aim of this paper is to extend for a vector maximization problem the results given in [10] and, furthermore, to establish sufficient second order optimality conditions. The obtained results seems to give a new look inside second order optimality conditions.

Starting from a necessary optimality condition, stated without any use of constraints qualification and involving relationships in the image space between the Paretian cone $\Re_+^{s+m}$ and a set $(K_L+\overline{K}_Q)$ which is the sum of a linear manifold and a cone, the problem of the existence of an hyperplane which separates $(K_L+\overline{K}_Q)$ and $\Re_+^{s+m}$ arises. Such a problem suggests to define two different kinds of conditions, called regularity conditions (since they involve both the objective functions and the constraints), under which separation theorems hold. In this framework, some regularity conditions are stated.

By means of the same approach, several sufficient second order optimality conditions are established; specifying the given results to the scalar case we obtain sufficient optimality conditions more general than the ones commonly stated in the literature [3, 4, 15].

2. Preliminary results

Consider the following vector problem ([1]):

$$P: \begin{cases} \max f(x) \\ x \in S = \{x \in X : g(x) \geqq 0\} \end{cases},$$

where $X \subseteq \Re^n$ is an open set, $f: X \to \Re^s$ and $g: X \to \Re^m$ are continuous functions and $x_0 \in S$; we assume, from now on and without loss of generality, that x_0 is binding to all the constraints, so that $g(x_0)=0$. Point $x_0 \in S$ is said to be a *local efficient point* if there exists a suitable neighbourhood $I \subseteq \Re^n$ of x_0 such that:

$$\nexists y \in I \cap S \text{ such that } f(y) \geq f(x_0).$$

In order to state second order sufficient optimality conditions, from now on we will refer to $\Re^n$ as the *decision space* and to $\Re^{s+m}$ as the *image space*; we will also use the following cones and function:

i) $F: X \to \Re^{s+m}$ such that $F(x)=(f(x),g(x))$,
ii) $H = (\Re^s_+ \setminus \{0\}) \times \Re^m_+$.

These sets let us to characterize the local efficiency of $x_0 \in S$ in the image space.

Property 2.1 Consider problem P.
Then the three following conditions are equivalent:

i) $x_0 \in S$ is a local efficient point;
ii) $\exists I \subseteq \Re^n$, neighbourhood of x_0, such that:
$$\nexists y \in I \cap X,\ y \neq x_0, \text{ such that } F(y) \in F(x_0) + H;$$
iii) $K_I \cap H = \emptyset$, where $K_I = F(I \cap X \setminus \{x_0\}) - F(x_0)$.

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[1] Through the paper we will use the following notations: let $u,v \in \Re^s$; $u \geqq v$ denote that $u_i \geq v_i$ $\forall i \in \{1,\dots,s\}$, $u \geq v$ denote that $u \neq v$ and $u_i \geq v_i$ $\forall i \in \{1,\dots,s\}$, $u>v$ denote that $u_i > v_i$ $\forall i \in \{1,\dots,s\}$.

By means of these notations, we can define the following sets:

$$\Re^s_+ = \{r \in \Re^s : r \geqq 0\}, \quad \Re^s_+ \setminus \{0\} = \{r \in \Re^s : r \geq 0\}, \quad \Re^s_{++} = \{r \in \Re^s : r > 0\}.$$

In order to study the disjunction between K_I and H, several suitable sets have been introduced [5-9, 11-14]; in this paper we will use the following tangent cone defined in [7, 8, 10].

Definition 2.1 Let $X \subseteq \Re^n$ be an open set, $x_0 \in X$ and $F: X \to \Re^{s+m}$ be a vector function. We define the following tangent cone $T_1 \subseteq \Re^{s+m}$ in the image space:

$$T_1 = \left\{ t \in \Re^{s+m} : \exists \{x_k\} \subset X \setminus \{x_0\},\ x_k \to x_0,\ t = \lambda d,\ \lambda \geq 0,\ d = \lim_{k \to +\infty} \frac{F(x_k) - F(x_0)}{\|F(x_k) - F(x_0)\|} \right\}.$$

In order to state necessary or sufficient second order optimality conditions, from now on the vector valued function F will be considered twice differentiable. With this aim we will use the second order characterization of the tangent cone T_1 given in [10]; it results that:

$$T_1 = (K_L + \bar{K}_Q) \cup A_2, \tag{2.1}$$

where the sets K_L, K_Q and A_2 are defined as follows.

Definition 2.2 Let $X \subseteq \Re^n$ be an open set with $x_0 \in X$, let $F: X \to \Re^{s+m}$ be a twice differentiable vector function, let $J_F(x_0)$ be the Jacobian matrix of F at x_0 and let $H_F(x_0) = [\, H_{f_1}(x_0), \ldots, H_{f_s}(x_0), H_{g_1}(x_0), \ldots, H_{g_m}(x_0)\,]^T$ be the vector of the Hessian matrices of the functions $f_1, \ldots, f_s, g_1, \ldots, g_m$. We can then define the following sets:

$K_L = \{ t \in \Re^{s+m} : t = J_F(x_0)v,\ v \in \Re^n \}$ [2],

$K_Q = \{ t \in \Re^{s+m} : t = v^T H_F(x_0) v,\ J_F(x_0)v = 0,\ v \in \Re^n,\ v \neq 0 \}$,

$\bar{K}_Q = K_Q \cup \{0\} = \{ t \in \Re^{s+m} : t = v^T H_F(x_0) v,\ J_F(x_0)v = 0,\ v \in \Re^n \}$,

$$A_2 = \left\{ t \in T_1 \setminus \{0\} : \exists \{x_k\} \subset X \setminus \{x_0\},\ x_k \to x_0,\ t = \lambda d,\ \lambda > 0,\ d = \lim_{k \to +\infty} \frac{F(x_k) - F(x_0)}{\|F(x_k) - F(x_0)\|},\ \lim_{k \to +\infty} \frac{F(x_k) - F(x_0)}{\|x_k - x_0\|^2} = 0 \right\}.$$

In order to state optimality conditions for problem P, it is necessary to study the structure of the cone A_2. The following Lemma 2.1 has been stated in [10].

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[2] Note that K_L is a linear manifold and that $0 \in K_L \neq \emptyset$; note also that $0 \in \bar{K}_Q$ while not necessarily $0 \in K_Q$.

Lemma 2.1 Let $X \subseteq \Re^n$ be an open set with $x_0 \in X$ and let $F: X \rightarrow \Re^{s+m}$ be a twice differentiable vector function, consider also a sequence $\{x_k\} \subset X \setminus \{x_0\}$, $x_k \rightarrow x_0$, such that $v = \lim_{k \to +\infty} \frac{x_k - x_0}{\|x_k - x_0\|}$ and the associated sequence $\{h_k\} \subset \Re^{s+m}$ such that $h_k = \frac{F(x_k) - F(x_0)}{\|x_k - x_0\|^2}$. If $\{h_k\}$ is a bounded sequence then each of its limit points belongs to the cone $(K_L + K_Q)$.

The following Theorem 2.1 determine a property of A_2 which will be used in sections 4 and 5 in order to state sufficient optimality conditions.

Theorem 2.1 Let $X \subseteq \Re^n$ be an open set, $x_0 \in X$ and $F: X \rightarrow \Re^{s+m}$ be a twice differentiable vector function. Then it results:

$$A_2 = \emptyset \text{ if and only if } 0 \notin (K_L + K_Q).$$

Proof We will prove that $A_2 \neq \emptyset$ if and only if $0 \in (K_L + K_Q)$.
The necessity follows directly from Lemma 2.1 and the definition of A_2.
For the sufficiency, suppose that $0 \in (K_L + K_Q)$ and let $l = J_F(x_0)d \in K_L$ and $q = v^T H_F(x_0) v \in K_Q$, with $\|v\| = 1$, such that $l + q = 0$. Let us define the sequence $\{x_k\} \subset X \setminus \{x_0\}$, $x_k \rightarrow x_0$, such that $x_k = x_0 + \frac{1}{k}\left[\frac{d}{k} + \sqrt{2}\, v\right]$; we then have that:

$$\lim_{k \to +\infty} \frac{x_k - x_0}{\|x_k - x_0\|} = \lim_{k \to +\infty} \frac{(d/k) + \sqrt{2}\, v}{\|(d/k) + \sqrt{2}\, v\|} = v .$$

From the second order Taylor expansion for F in x_0 we have:

$$\frac{F(x_k) - F(x_0)}{\|x_k - x_0\|^2} = \frac{J_F(x_0)(x_k - x_0)}{\|x_k - x_0\|^2} + \frac{1}{2}\left(\frac{x_k - x_0}{\|x_k - x_0\|}\right)^T H_F(x_0)\left(\frac{x_k - x_0}{\|x_k - x_0\|}\right) + \sigma(x_k, x_0),$$

where $\lim_{k \to +\infty} \sigma(x_k, x_0) = 0$. Since $J_F(x_0)(x_k - x_0) = \frac{1}{k^2} J_F(x_0)d + \frac{\sqrt{2}}{k} J_F(x_0)v = \frac{1}{k^2} J_F(x_0)d$ and $k^2\|x_k - x_0\|^2 = \|(d/k) + \sqrt{2}\, v\|^2$, we then have that $\lim_{k \to +\infty} \frac{F(x_k) - F(x_0)}{\|x_k - x_0\|^2} = \frac{l+q}{2} = 0$, so that, substituting $\{x_k\}$ with a suitable subsequence if necessary, it results $t = \lim_{k \to +\infty} \frac{F(x_k) - F(x_0)}{\|F(x_k) - F(x_0)\|} \in A_2 \neq \emptyset$. ♦

Corollary 2.1 Let $X \subseteq \Re^n$ be an open set, $x_0 \in X$ and $F: X \rightarrow \Re^{s+m}$ be a twice differentiable vector function. If $K_Q \cap K_L = \emptyset$ then $T_1 = (K_L + \bar{K}_Q)$.

Proof The thesis follows from Theorem 2.1 since if $K_Q \cap K_L = \emptyset$ then, being K_L a linear manifold, $0 \notin (K_L + K_Q)$. ♦

The following theorem gives a result which will be used in section 4 in order to state sufficient second order optimality conditons.

Theorem 2.2 Let $X\subseteq\Re^n$ be an open set with $x_0\in X$, let $F:X\to\Re^{s+m}$ be a twice differentiable function and let $\alpha\in\Re^{s+m}$, $\alpha\neq 0$, be a vector such that $\alpha^T J_F(x_0)=0$. Consider also a sequence $\{x_k\}\subset X\setminus\{x_0\}$, $x_k\to x_0$, such that $v=\lim_{k\to+\infty}\frac{x_k-x_0}{\|x_k-x_0\|}$.
Then we have: $\lim_{k\to+\infty}\frac{\alpha^T[F(x_k)-F(x_0)]}{\|x_k-x_0\|^2}=\frac{1}{2}\alpha^T[v^T H_F(x_0)v]$.

Proof Let $\{v_k\}\subset\Re^n$ be the sequence such that $v_k=\frac{x_k-x_0}{\|x_k-x_0\|}$ and consider the following second order Taylor expansion for F:

$$F(x_k)=F(x_0)+\|x_k-x_0\|\,J_F(x_0)v_k+\frac{1}{2}\|x_k-x_0\|^2\,[v_k{}^T H_F(x_0)v_k+\sigma(x_k,x_0)],$$

where $\lim_{k\to+\infty}\sigma(x_k,x_0)=0$; since $\alpha^T J_F(x_0)=0$ we then have:

$$\alpha^T[F(x_k)-F(x_0)]=\|x_k-x_0\|\,\alpha^T J_F(x_0)v_k+\frac{1}{2}\|x_k-x_0\|^2\,\alpha^T[v_k{}^T H_F(x_0)v_k+\sigma(x_k,x_0)]=$$
$$=\frac{1}{2}\|x_k-x_0\|^2\,\alpha^T[v_k{}^T H_F(x_0)v_k+\sigma(x_k,x_0)],$$

so that it results:

$$\lim_{k\to+\infty}\frac{\alpha^T[F(x_k)-F(x_0)]}{\|x_k-x_0\|^2}=\lim_{k\to+\infty}\frac{1}{2}\alpha^T[v_k{}^T H_F(x_0)v_k+\sigma(x_k,x_0)]=\frac{1}{2}\alpha^T[v^T H_F(x_0)v].$$ ♦

At last, we remind the following necessary and sufficient optimality condition stated in [8].

Theorem 2.3 Consider problem P. The point $x_0\in S$ is a local efficient point if and only if the following condition holds:

$\forall t\in T_1\cap Cl(H)$, $\|t\|=1$, and $\forall\{x_k\}\subset X\setminus\{x_0\}$, $x_k\to x_0$, such that $t=\lim_{k\to+\infty}\frac{F(x_k)-F(x_0)}{\|F(x_k)-F(x_0)\|}$, there exists an integer $\bar{k}\geq 0$ such that:

$$F(x_k)\notin F(x_0)+H \quad \forall k>\bar{k}.$$

3. Necessary optimality conditions and second order regularity conditions

In [8], the following necessary optimality condition has been established.

Theorem 3.1 If $x_0\in S$ is a local efficient point for problem P then we have:

$$T_1\cap Int(H)=\emptyset. \tag{3.1}$$

The characterization (2.1) of the tangent cone T_1 allows us to obtain the following necessary optimality condition [10] for a vector optimization problem.

Theorem 3.2 If $x_0 \in S$ is a local efficient point for problem P and f is twice differentiable, then we have:

$$(K_L+\bar{K}_Q)\cap \mathrm{Int}(H)=\emptyset. \tag{3.2}$$

Let us note that the second order necessary optimality condition (3.2), expressed in the image space, holds without any requirements of a second order constraint qualification.

In order to study the existence of an hyperplane which separates $(K_L+\bar{K}_Q)$ and Int(H), we introduce the following notations: $\alpha^{\perp}=\{z\in\Re^{s+m}: \alpha^T z=0\}$, $\alpha^{\underline{\perp}}=\{z\in\Re^{s+m}: \alpha^T z\leq 0\}$, and $\alpha^{\underline{\underline{\perp}}}=\{z\in\Re^{s+m}: \alpha^T z<0\}$, where $\alpha\in\Re^{s+m}$.

Let us note that since K_L is a linear manifold, any separation hyperplane necessarily contains K_L ($K_L\subseteq\alpha^{\perp}$). Let us consider now the class of hyperplanes containing K_L; the following questions arise:

Q1) every hyperplane containing K_L separates $(K_L+\bar{K}_Q)$ and Int(H);

Q2) there exists at least one hyperplane containing K_L which separates $(K_L+\bar{K}_Q)$ and Int(H).

The previous questions Q1) and Q2) can be reformulated in the following way:

$$\forall\alpha\geq 0 \text{ such that } K_L\subseteq\alpha^{\perp} \text{ it results } K_Q\subseteq\alpha^{\underline{\perp}}, \tag{3.3}$$

$$\exists\alpha\geq 0 \text{ such that } K_L\subseteq\alpha^{\perp} \text{ and } K_Q\subseteq\alpha^{\underline{\perp}}. \tag{3.4}$$

The following Examples 3.1 and 3.2 point out the importance of carry on jointly the study of both the previous conditions; Example 3.1 shows that it is possible that conditions (3.4) holds while (3.3) is not verified; Example 3.3 shows that both (3.3) and (3.4) can be verified.

Example 3.1 Consider the problem P: $\{\max f(x,y)=x^2+y^3, -x^2-y^2\geq 0\}$ and the feasible point (0,0). Since (0,0) is the only feasible point then it is also a local efficient point; it results also that $K_L=\{(0,0)\}$ and that:

$$K_Q=\{(x,y)\in\Re^2: (x,y)=\lambda(1,-1)+\mu(0,-1),\ \lambda,\mu\geq 0,\ \lambda+\mu\neq 0\}.$$

The vector $\alpha=(1,2)^T$ is such that $K_L\subseteq\alpha^{\perp}$ and $K_Q\subseteq\alpha^{\underline{\perp}}$ so that (3.4) holds; the vector $\alpha=(2,1)^T$ is such that $K_L\subseteq\alpha^{\perp}$ but $K_Q\not\subseteq\alpha^{\underline{\perp}}$ so that (3.3) does not hold.

Example 3.2 Consider the problem P: $\{\max f(x,y)=-x^2-2y^2,\ -x^2-y^2\geq 0\}$ and the feasible point (0,0). Since (0,0) is the only feasible point then it is also a local efficient point; it results also that $K_L=\{(0,0)\}$ and that:

$$K_Q=\{(x,y)\in\Re^2: (x,y)=\lambda(-1,-1)+\mu(-2,-1),\ \lambda,\mu\geq 0,\ \lambda+\mu\neq 0\}.$$

Each $\alpha\geq 0$ is such that $K_L\subseteq\alpha^{\perp}$ and $K_Q\subseteq\alpha^{\perp}_{-}$ so that both (3.3) and (3.4) hold.

In order to point out the importance of the above questions, we specify conditions (3.3) and (3.4) in the decision space.

It is easy to verify that condition $K_L\subseteq\alpha^{\perp}$, with $\alpha\geq 0$, is equivalent to $\alpha^T J_F(x_0)=0$, that is the Fritz-John condition; condition (3.3) is equivalent to the following proposition (P1):

(P1) The Hessian matrix of the Lagrangean function, for every multiplier vector $\alpha\geq 0$ such that $\alpha^T J_F(x_0)=0$, is negative semidefinite on the linear manifold $W=\{d\in\Re^n: J_F(x_0)d=0\}$,

while condition (3.4) is equivalent to proposition (P2):

(P2) Among all multiplier vectors $\alpha\geq 0$ such that $\alpha^T J_F(x_0)=0$, there exists one of them such that the Hessian matrix of the Lagrangean function is negative semidefinite on the linear manifold $W=\{d\in\Re^n: J_F(x_0)d=0\}$,

where the Lagrangean function is defined as follows:

$$\mathcal{L}(x,\alpha)=\sum_{i=1}^{s}\alpha_i f_i(x)+\sum_{i=s+1}^{s+m}\alpha_i g_i(x),\ \alpha\in\Re^{s+m}. \tag{3.5}$$

The image space approach let us to characterize conditions (3.3) and (3.4).

Theorem 3.3 Consider the twice differentiable problem P; the following conditions are equivalent:

i) $\forall\alpha\geq 0$ such that $K_L\subseteq\alpha^{\perp}$ it results $K_Q\subseteq\alpha^{\perp}_{-}$,

ii) $K_Q\subseteq K_L\text{-}Cl(H)$. (3.6)

Proof i)⇒ii) Suppose ab absurdo that $q\in K_Q$, $q\notin K_L\text{-}Cl(H)$. By means of a separation theorem, $\exists\alpha\in\Re^{s+m}$ such that $\alpha^T t\leq 0\ \forall t\in K_L\text{-}Cl(H)$ and $\alpha^T q>0$.

Since $K_L\subseteq K_L\text{-}Cl(H)$ is a linear manifold it follows immediatly that $K_L\subseteq\alpha^{\perp}$; since also $-Cl(H)\subseteq K_L\text{-}Cl(H)$ then $\alpha^T(-h)\leq 0\ \forall h\in Cl(H)$ so that $\alpha^T h\geq 0\ \forall h\in Cl(H)$; it then follows that $\alpha\geq 0$. We then have that $\alpha\geq 0$ is such that $K_L\subseteq\alpha^{\perp}$ and $\alpha^T q>0$, $q\in K_Q$, and this contradicts the hypothesis.

ii)⇒i) Let $\alpha\geq 0$ such that $K_L\subseteq\alpha^{\perp}$; by the hypothesis, $\forall q\in K_Q\ \exists l\in K_L$ and $\exists h\in Cl(H)$ such that $q=l-h$. It then results $\alpha^T q=\alpha^T l-\alpha^T h\leq 0$ so that $K_Q\subseteq\alpha^{\perp}_{-}$. ♦

Theorem 3.4 Consider the twice differentiable problem P; the following conditions are equivalent:

i) $\exists\alpha\geq 0$ such that $K_L\subseteq\alpha^{\perp}$ and $K_Q\subseteq\alpha^{\perp}_{-}$,

ii) $Co(K_L+\bar{K}_Q)\cap Int(H)=\varnothing$. (3.7)

Proof i)⇒ii) Suppose ab absurdo that $t\in(K_L+Co(K_Q))\cap Int(H)\neq\varnothing$.
Then $\exists j\geq 0$, $\exists l_1,\ldots,l_j\in K_L$, $\exists\mu_1,\ldots,\mu_j>0$, $\exists k\geq 0$, $\exists q_1,\ldots,q_k\in K_Q$, $\exists\lambda_1,\ldots,\lambda_k>0$ such that $\sum_{i=1}^{j}\mu_i+\sum_{i=1}^{k}\lambda_i=1$ and $t=\sum_{i=1}^{j}\mu_i l_i+\sum_{i=1}^{k}\lambda_i q_i$; we then have $\alpha^T t=\sum_{i=1}^{j}\mu_i\alpha^T l_i+\sum_{i=1}^{k}\lambda_i\alpha^T q_i\leq 0$, and this is absurd since being $t\in Int(H)$ and $\alpha\geq 0$ it is $\alpha^T t>0$.

ii)⇒i) By using a known separation theorem, $\exists\alpha\geq 0$ such that $\alpha^T t\leq 0$ $\forall t\in Co(K_L+\bar{K}_Q)$; in particular since $K_L\subseteq Co(K_L+\bar{K}_Q)$ it is $\alpha^T J_F(x_0)v\leq 0$ $\forall v\in\Re^n$ so that $\alpha^T J_F(x_0)=0$, in other words $K_L\subseteq\alpha^{\perp}$. Since also $K_L\subseteq Co(K_L+\bar{K}_Q)$ then $\alpha^T q\leq 0$ $\forall q\in K_Q$ so that $K_Q\subseteq\alpha^{\perp}_{-}$. ♦

Remark 3.1 Let us note that it is possible that $Co(K_L+\bar{K}_Q)\cap Int(H)=\varnothing$ even if $K_Q\not\subseteq K_L\text{-}Cl(H)$, as we can verify in Example 3.1; note also that it can be both $K_Q\subseteq K_L\text{-}Cl(H)$ and $Co(K_L+\bar{K}_Q)\cap Int(H)=\varnothing$, as it is shown in Example 3.2.

In order to guarantee the validity of conditions (3.3) and (3.4) we need of further assumptions; in the scalar case (s=1), such other assumptions are referred to as second order constraints qualification; note that in the image space such a name is not so proper since these assumptions involve both the objective functions and the constraints.
This allows us to suggest the following kinds of regularity conditions.

Definition 3.1 A condition $\mathcal{R}$ will be referred to as a second order regularity condition in the image space if the following logical implication holds:

$$(K_L+\bar{K}_Q)\cap Int(H)=\varnothing \Rightarrow Co(K_L+\bar{K}_Q)\cap Int(H)=\varnothing. \quad (3.8)$$

Definition 3.2 A condition $\mathcal{R}s$ will be referred to as a second order strict regularity condition in the image space if the following logical implication holds:

$$(K_L+\bar{K}_Q)\cap Int(H)=\varnothing \Rightarrow K_Q\subseteq K_L\text{-}Cl(H). \quad (3.9)$$

By means of the previous Definition 3.1 and the above Theorems 3.2, 3.3 and 3.4, we have that, given a local efficient point $x_0 \in S$ for problem P, if a second order regularity condition $\mathcal{R}$ holds then:

$$\exists \alpha \geq 0 \text{ such that } K_L \subseteq \alpha^{\perp} \text{ and } K_Q \subseteq \alpha^{\perp}_{-},$$

while if a second order strict regularity condition $\mathcal{R}s$ holds then:

$$\forall \alpha \geq 0 \text{ such that } K_L \subseteq \alpha^{\perp} \text{ it results } K_Q \subseteq \alpha^{\perp}_{-}.$$

Remark 3.2 It is easy to see that a strict regularity condition is also a regularity condition: suppose $(K_L+\bar{K}_Q) \cap \mathrm{Int}(H)=\emptyset$, it results $K_L\text{-}\mathrm{Cl}(H) \cap \mathrm{Int}(H)=\emptyset$ so that, being $K_Q \subseteq K_L\text{-}\mathrm{Cl}(H)$, $\mathrm{Co}(K_L+\bar{K}_Q) \subseteq K_L\text{-}\mathrm{Cl}(H)$ and $\mathrm{Co}(K_L+\bar{K}_Q) \cap \mathrm{Int}(H)=\emptyset$.

The following theorems show some second order regularity conditions.

Theorem 3.5 Consider problem P; then the following conditions are second order regularity conditions:

i) $(K_L+\bar{K}_Q)$ is a convex cone,

ii) $\bar{K}_Q$ is a convex cone,

iii) $\dim(K_L)=n-1$.

Proof i) The thesis follows since $\mathrm{Co}(K_L+\bar{K}_Q)$ is the smallest convex cone containing $(K_L+\bar{K}_Q)$, so that $(K_L+\bar{K}_Q)=\mathrm{Co}(K_L+\bar{K}_Q)$.

ii) If $\bar{K}_Q$ is a convex cone then $(K_L+\bar{K}_Q)$ is also a convex cone so that the thesis follows from i).

iii) Since $\mathrm{rank}[J_F(x_0)]=\dim(K_L)=n-1$ we have that there exists a vector $d \in \Re^n$ such that $\{v \in \Re^n;\ J_F(x_0)v=0\}=\{v \in \Re^n;\ v=\lambda d,\ \lambda \in \Re\}$, this implies that $\bar{K}_Q$ is an halfline and in particular is a convex cone, the thesis then follows from ii). ♦

Note that that the conditions expressed in Theorem 3.5 do not imply that $K_Q \subseteq K_L\text{-}\mathrm{Cl}(H)$, so that these conditions are not strict regularity conditions, as it can be shown by means of Example 3.1 where $(K_L+\bar{K}_Q)$ is a convex cone but $K_Q \not\subseteq K_L\text{-}\mathrm{Cl}(H)$.

Remark 3.3 Note that in the scalar case (s=1), if the Kuhn-Tucker conditions hold then condition iii) is equivalent to require that the gradients of the constraints in x_0 are linearly independent.

Theorem 3.6 Consider problem P; then the following conditions are second order strict regularity conditions:

i) $K_Q \subseteq K_L$,

ii) $K_Q = \emptyset$,

iii) $\dim(K_L) = n$,

iv) $\dim(K_L) = s+m-1$.

Proof i), ii) The thesis follows since i) and ii) imply that $K_Q \subseteq K_L\text{-}Cl(H)$.

iii) Since $\text{rank}[J_F(x_0)] = \dim(K_L) = n$ then $J_F(x_0)v \neq 0$ $\forall v \in \Re^n$, $v \neq 0$, that is to say that $K_Q = \emptyset$, so that the proof follows from ii).

iv) Let $(K_L + \bar{K}_Q) \cap \text{Int}(H) = \emptyset$ and suppose ab absurdo that $\exists q \in K_Q$, $q \notin K_L\text{-}Cl(H)$; then $q \notin K_L$, $\dim(K_L + \bar{K}_Q) = s+m$ and $\text{Int}(H) \subset (K_L + \bar{K}_Q)$ which is absurd. ♦

The given approach in the image space allows us to state the following second order strict regularity condition given in the decision space, which generalizes the one suggested in [10] as regards to the scalar case.

Condition 1 $\forall v \in \Re^n$, $\|v\|=1$, such that $J_F(x_0)v=0$ $\exists \{x_k\} \subset X \setminus \{x_0\}$, $x_k \to x_0$, and $\exists j \in \{1,\ldots,s\}$ such that $v = \lim_{k\to+\infty} \frac{x_k - x_0}{\|x_k - x_0\|}$, $g(x_k)=g(x_0)$ and $f_i(x_k)=f_i(x_0)$ $\forall i \in \{1,\ldots,s\}$, $i \neq j$.

Theorem 3.7 If $x_0 \in S$ is a local efficient point for problem P and Condition 1 holds then: $\forall \alpha \geq 0$ such that $K_L \subseteq \alpha^\perp$ it results $K_Q \subseteq \alpha^\perp_-$.

Proof Suppose ab absurdo that $\exists \alpha \geq 0$ such that $K_L \subseteq \alpha^\perp$, that is to say $\alpha^T J_F(x_0) = 0$, and $\exists q \in K_Q$, $q = v^T H_F(x_0) v$, such that $q \notin \alpha^\perp_-$ so that $\alpha^T[v^T H_F(x_0) v] > 0$. Since $v^T H_F(x_0) v \in K_Q$ then $J_F(x_0)v = 0$ so that for Condition 1 there exists a sequence $\{x_k\} \subset X \setminus \{x_0\}$, $x_k \to x_0$, such that $v = \lim_{k\to+\infty} \frac{x_k - x_0}{\|x_k - x_0\|}$, $f_i(x_k) = f_i(x_0)$ $\forall i \in \{1,\ldots,s\}$, $i \neq j$, and $g(x_k) = g(x_0)$. By means of Theorem 2.2 we have that:

$$0 < \frac{1}{2}\alpha^T[v^T H_F(x_0) v] = \lim_{k\to+\infty} \frac{\alpha^T[F(x_k) - F(x_0)]}{\|x_k - x_0\|^2} = \lim_{k\to+\infty} \frac{\alpha_j{}^T[f_j(x_k) - f_j(x_0)]}{\|x_k - x_0\|^2};$$

for a known limit theorem, it then follows that there exists an integer $\bar{k} \geq 0$ such that $\alpha_j{}^T[f_j(x_k) - f_j(x_0)] > 0$ $\forall k > \bar{k}$, since $\alpha_j \geq 0$ we then have that $f_j(x_k) > f_j(x_0)$ $\forall k > \bar{k}$ and this is absurd since $x_0 \in S$ is a local efficient point for problem P. ♦

At last, let us note that the necessary optimality conditions obtained by means of the regularity conditions, can be expressed equivalently in the decision space as follows.

Theorem 3.8 Let $x_0 \in S$ be a local efficient point for the twice differentiable problem P, if a second order regularity condition $\mathcal{R}$ holds then (P3) holds:

(P3) $\exists\alpha\geq 0$ such that $\alpha^T J_F(x_0)=0$ and:

$$\alpha^T[v^T H_F(x_0)v]\leq 0 \quad \forall v\in\{v\in\Re^n:\ v\neq 0 \text{ and } J_F(x_0)v=0\}\ ,$$

while if a second order strict regularity condition $\mathcal{R}s$ holds then (P4) holds:

(P4) $\forall\alpha\geq 0$ such that $\alpha^T J_F(x_0)=0$ it results:

$$\alpha^T[v^T H_F(x_0)v]\leq 0 \quad \forall v\in\{v\in\Re^n:\ v\neq 0 \text{ and } J_F(x_0)v=0\}\ .$$

Remark 3.4 Let us note that, in the scalar case (s=1) and when Condition 1 is chosen as a second order regularity condition, the first part of Theorem 3.8 collapses to the second order necessary optimality condition stated in [10], which is more general than the one suggested in [15].

4. Sufficient second order optimality conditions in the image space

As it has been shown in [8], the necessary optimality condition $T_1\cap \text{Int}(H)=\emptyset$ is not also a sufficient optimality condition; by means of the necessary and sufficient optimality condition expressed in Theorem 2.3, we have to study the behaviour of F(x) with respects to vectors $t\in T_1\cap \text{Cl}(H)$, $t\neq 0$. In the first part of this section, we will consider the case $T_1\cap \text{Cl}(H)\neq\{0\}$, while in the former part we will study the case $T_1\cap \text{Cl}(H)=\{0\}$.

With this aim, firstly note that the necessary second order optimality condition stated in the previous section:

$$\exists\alpha\geq 0 \text{ such that } K_L\subseteq\alpha^{\perp} \text{ and } K_Q\subseteq\alpha^{\perp}_{\geq}, \tag{4.1}$$

is not also a sufficient optimality conditions, as it is pointed out by means of the following example.

Example 4.1 Consider the problem P: $\{\max f(x,y)=x-y^2,\ -y^2\geq 0\}$ and the feasible point (0,0); this point is not a local efficient point being x uncostrained. It results that $K_L=\{(x,y)\in\Re^2: y=0\}$ and $K_Q=\{(x,y)\in\Re^2: (x,y)=\lambda(-1,-1),\ \lambda>0\}$, the vector $\alpha=(0,1)^T$ is such that $K_L\subseteq\alpha^{\perp}$ and $K_Q\subseteq\alpha^{\perp}_{\geq}$ so that (4.1) holds.

Note also that Example 4.1 shows that neither the following stronger condition:

$$\exists\alpha\geq 0 \text{ such that } K_L\subseteq\alpha^{\perp} \text{ and } K_Q\subseteq\alpha^{\perp}_{=}, \qquad (4.2)$$

is a sufficient optimality conditions. The following theorem gives the structure of the problem when (4.2) is assumed and points out that when (4.2) holds then it is possible that $T_1\cap Cl(H)\neq\{0\}$.

Theorem 4.1 Let $x_0\in S$ be a feasible point for problem P.
If condition (4.2) holds then the following properties are verified:

i) $T_1=(K_L+\bar{K}_Q)$ with $K_Q\cap K_L=\emptyset$,

ii) $Co(T_1)\cap Int(H)=\emptyset$,

iii) $Co(T_1)\cap Fr(H)\subseteq K_L$.

Proof i) Since $K_Q\subseteq\alpha^{\perp}_{=}$ then $K_Q\cap K_L=\emptyset$ so that $T_1=(K_L+\bar{K}_Q)$.

ii) Suppose ab absurdo that $t\in Co(K_L+K_Q)\cap Int(H)$; then $\exists j\geq 0$, $\exists l_1,\dots,l_j\in K_L$, $\exists\mu_1,\dots,\mu_j>0$, $\exists k\geq 0$, $\exists q_1,\dots,q_k\in K_Q$, $\exists\lambda_1,\dots,\lambda_k>0$ such that $\sum_{i=1}^{j}\mu_i+\sum_{i=1}^{k}\lambda_i=1$ and $t=\sum_{i=1}^{j}\mu_i l_i+\sum_{i=1}^{k}\lambda_i q_i$; we then have $\alpha^T t=\sum_{i=1}^{j}\mu_i\alpha^T l_i+\sum_{i=1}^{k}\lambda_i\alpha^T q_i<0$, and this is absurd since being $t\in Int(H)$ and $\alpha\geq 0$ it is $\alpha^T t>0$.

iii) By means of ii), we just have to prove that $Co(T_1)\cap Cl(H)\subseteq K_L$. The proof of this inclusion is similar to the one given for ii). ♦

As we have seen, there could exist an hyperplane which separates $(K_L+\bar{K}_Q)$ with Cl(H) containing a face of Fr(H).

The above properties show that in order to state sufficient second order optimality conditions in the image space, we have to introduce sets containing K_Q , such as the following cone:

$$K_Q^*=\{\ t\in\Re^{s+m}:\ t=v^T H_F(x_0)v,\ J_F(x_0)v\geqq 0,\ J_F(x_0)v\not>0,\ v\in\Re^n,\ v\neq 0\ \}\ .$$

The previous cone can be stated, just like K_Q , independently from the multiplier vector $\alpha\geq 0$; given a particular vector $\alpha\geq 0$ some more cones can be introduced. Suppose $\alpha\geq 0$ to be in the form $\alpha=(\alpha_P,\alpha_N)^T=(\alpha_P,0)^T$, with $\alpha_P>0$, eventually by a reordering of the components $F_i:X\rightarrow\Re$ of the vector function $F:X\rightarrow\Re^{s+m}$; by means of the same reordering also the Jacobian matrix of F at x_0 and the vector of the Hessian matrices of the functions F_i can be considered in the form $J_F(x_0)=\begin{bmatrix} J_{FP}(x_0) \\ J_{FN}(x_0) \end{bmatrix}$ and $H_F(x_0)=\begin{bmatrix} H_{FP}(x_0) \\ H_{FN}(x_0) \end{bmatrix}$.

This reordering let us to define the other following sets containing K_Q :

$$K_Q^{\alpha}=\{\ t\in\Re^{s+m}:\ t=v^TH_F(x_0)v,\ J_{FP}(x_0)v=0,\ J_{FN}(x_0)v\geqq 0,\ v\in\Re^n,\ v\neq 0\ \},$$

$$K_{\alpha}^{*}=\{\ t\in\Re^{s+m}:\ t=v^TH_F(x_0)v,\ J_{FP}(x_0)v=0,\ v\in\Re^n,\ v\neq 0\ \}.$$

It is important to note that $K_Q^{\alpha}=K_Q^{*}\cap K_{\alpha}^{*}$ and that $K_Q\subseteq K_Q^{\alpha}\subseteq K_Q^{*}$ and $K_Q^{\alpha}\subseteq K_{\alpha}^{*}$ (3). As it results clear from Theorem 4.1 and from the necessary and sufficient optimality condition expressed in Theorem 2.3, in order to state sufficient optimality conditions involving a semipositive vector α, we are interested to find conditions implying that $\forall t\in K_L\cap Fr(H)$, $\|t\|=1$, and $\forall\{x_k\}\subset X\setminus\{x_0\}$, $x_k\to x_0$, such that $t=\lim_{k\to+\infty}\frac{F(x_k)-F(x_0)}{\|F(x_k)-F(x_0)\|}$, $\exists\bar{k}\geq 0$ such that $F(x_k)\notin F(x_0)+H\ \forall k>\bar{k}$.

Theorem 4.2 If the following condition (4.3) holds:

$$\exists\alpha\geq 0 \text{ such that } K_L\subseteq\alpha^{\perp} \text{ and } K_Q^{\alpha}\subseteq\alpha_{=}^{\perp}, \tag{4.3}$$

then $x_0\in S$ is a local efficient point for problem P.

Proof Since $K_Q\subseteq K_Q^{\alpha}$ then condition (4.3) implies that $T_1\cap Cl(H)\subseteq K_L$. By means of Theorem 2.3 and 4.1 we just have to prove that given a vector $t\in K_L$, $\|t\|=1$, and a sequence $\{x_k\}\subset X\setminus\{x_0\}$, $x_k\to x_0$, such that $t=\lim_{k\to+\infty}\frac{F(x_k)-F(x_0)}{\|F(x_k)-F(x_0)\|}$, we have $F(x_k)\notin F(x_0)+H\ \forall k>\bar{k}$, $\bar{k}\geq 0$. Let now be $\{v_k\}\subset\Re^n$ be the sequence such that $v_k=\frac{x_k-x_0}{\|x_k-x_0\|}$ and note that we can assume, without loss of generality, that the sequence $\{v_k\}$ is convergent to a vector $v\in\Re^n$. If $J_F(x_0)\neq 0$ then:

$$t=\lim_{k\to+\infty}\frac{F(x_k)-F(x_0)}{\|F(x_k)-F(x_0)\|}=\lim_{k\to+\infty}\frac{F(x_k)-F(x_0)}{\|x_k-x_0\|}\lim_{k\to+\infty}\frac{\|x_k-x_0\|}{\|F(x_k)-F(x_0)\|}=\frac{J_F(x_0)v}{\|J_F(x_0)v\|};$$

since $t\in Fr(H)$ then $J_F(x_0)v\in Fr(H)$ so that $J_F(x_0)v\geq 0$, $J_F(x_0)v\not> 0$.

In general (for $J_F(x_0)\neq 0$ or $J_F(x_0)=0$) we then have that $J_F(x_0)v\geqq 0$, $J_F(x_0)v\not> 0$. This condition implies, for the hypotesis, that $J_{FP}(x_0)v=0$; infact being $\alpha_P>0$, if ab absurdo $J_{FP}(x_0)v\geq 0$, then we would have $\alpha_P{}^TJ_{FP}(x_0)v>0$ and this is absurd since $0=\alpha^TJ_F(x_0)=\alpha_P{}^TJ_{FP}(x_0)$. Being $J_{FP}(x_0)v=0$ and $J_{FN}(x_0)v\geqq 0$ we then have that $b=v^TH_F(x_0)v\in K_Q^{\alpha}$ and that $\alpha^T[v^TH_F(x_0)v]<0$; by means of Theorem 2.2 it then results that $\lim_{k\to+\infty}\frac{\alpha^T[F(x_k)-F(x_0)]}{\|x_k-x_0\|^2}<0$. For a known limit theorem this result

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[3] Note also that $0\in\bar{K}_Q$ while not necessarily we have $0\in K_Q^{\alpha}$, $0\in K_Q^{*}$ or $0\in K_{\alpha}^{*}$.

implies that there exists an integer $\bar{k}\geq 0$ such that $\frac{\alpha^T[F(x_k)-F(x_0)]}{||x_k-x_0||^2}<0$ $\forall k>\bar{k}$ so that $F(x_k)\notin F(x_0)+Cl(H)$ $\forall k>\bar{k}$; the proof is then complete. ♦

By means of the previous theorem we can state directly the following results.

Corollary 4.1 If the following condition (4.4) holds:

$$\exists\alpha\geq 0 \text{ such that } K_L\subseteq\alpha^{\perp} \text{ and } K_Q^*\subseteq\alpha_{=}^{\perp}, \tag{4.4}$$

then $x_0\in S$ is a local efficient point for problem P.

Corollary 4.2 If the following condition (4.5) holds:

$$\exists\alpha\geq 0 \text{ such that } K_L\subseteq\alpha^{\perp} \text{ and } K_\alpha^*\subseteq\alpha_{=}^{\perp}, \tag{4.5}$$

then $x_0\in S$ is a local efficient point for problem P.

Note that in Corollary 4.1 the cone K_Q^* can be determined independently from the chosen multiplier vector $\alpha\geq 0$; note also that Corollary 4.2 generalizes to the vector case a sufficient second order optimality condition regarding to scalar optimization problems [15]; note also that the following Example 4.2 points out that condition (4.5) is stronger than condition (4.3).

Example 4.2 Consider the problem P: $\{\max f(x,y,z)=x+y-x^2-y^2+z^2,\ -x-y\geq 0,\ z\geq 0,\ -z\geq 0\}$ and the feasible point (0,0,0) which is also optimum. Consider the multiplier vector $\alpha=(1,1,0,0)^T$; it results that $K_L=\{a(1,-1,0,0)^T+b(0,0,1,-1)^T,\ a,b\in\Re\}$, $K_Q=K_Q^\alpha=K_Q^*=\{a(-1,0,0,0)^T,\ a>0\}$ and $K_\alpha^*=\{a(-1,0,0,0)^T,\ a\in\Re\}$, so that condition (4.3) holds while condition (4.5) is not verified.

So far, we have studied the case $T_1\cap Cl(H)\neq\{0\}$; from now on we want to state sufficient optimality conditions such that $T_1\cap Cl(H)=\{0\}$.

Firstly note that the existence of a separation hyperplane within $(K_L+\bar{K}_Q)$ and Cl(H) does not guarantee the optimality of $x_0\in S$, as it is pointed out by means of the following Example 4.3.

Example 4.3 Consider the problem P: $\{\max f(x,y)=x^3+y^3,\ x^3+y^3\geq 0\}$ and the feasible point (0,0). It results that $K_L=K_Q=\{0\}$ so that, even if (0,0) is not a local efficient point, there exists a separation hyperplane within $(K_L+\bar{K}_Q)$ and Cl(H).

Example 4.3 points out that we need of further assumptions, in addiction to the separation between $(K_L+\bar{K}_Q)$ and Cl(H), in order to achieve a sufficient second order optimality condition, as it is stressed in the next theorem.

Theorem 4.3 If both the two following conditions hold then $x_0 \in S$ is a local efficient point for problem P:

i) $Co(T_1)=Co(K_L+\bar{K}_Q)$,

ii) $\exists \alpha>0$ such that $K_L \subseteq \alpha^{\perp}$ and $K_Q \subseteq \alpha^{\perp}_{-}$.

Proof We firstly prove that hypothesis ii) implies that $Co(K_L+\bar{K}_Q) \cap Cl(H)=\{0\}$. Suppose ab absurdo that there exists $t \in (K_L+Co(K_Q)) \cap Cl(H)$, $t \neq 0$.

Then $\exists j \geq 0$, $\exists l_1,\dots,l_j \in K_L$, $\exists \mu_1,\dots,\mu_j>0$, $\exists k \geq 0$, $\exists q_1,\dots,q_k \in K_Q$, $\exists \lambda_1,\dots,\lambda_k>0$ such that $\sum_{i=1}^{j} \mu_i + \sum_{i=1}^{k} \lambda_i = 1$ and $t=\sum_{i=1}^{j} \mu_i l_i + \sum_{i=1}^{k} \lambda_i q_i$; we then have $\alpha^T t = \sum_{i=1}^{j} \mu_i \alpha^T l_i + \sum_{i=1}^{k} \lambda_i \alpha^T q_i \leq 0$, and this is absurd since being $t \in Cl(H)$, $t \neq 0$, and $\alpha>0$ it is $\alpha^T t>0$.

By means of hypothesis i) we then have that $Co(T_1) \cap Cl(H)=\{0\}$ so that the thesis follows directly from the necessary and sufficient optimality condition given in Theorem 2.3. ♦

Corollary 4.3 If both the two following conditions hold then $x_0 \in S$ is a local efficient point for problem P:

i) $0 \notin (K_L+K_Q)$,

ii) $\exists \alpha>0$ such that $K_L \subseteq \alpha^{\perp}$ and $K_Q \subseteq \alpha^{\perp}_{-}$.

Proof The thesis follows from Theorem 4.3 and Theorem 2.1. ♦

Corollary 4.4 If the following condition (4.6) holds:

$$\exists \alpha>0 \text{ such that } K_L \subseteq \alpha^{\perp} \text{ and } K_Q \subseteq \alpha^{\perp}_{=}, \tag{4.6}$$

then $x_0 \in S$ is a local efficient point for problem P.

Proof Since $K_Q \subseteq \alpha^{\perp}_{=}$ then $0 \notin (K_L+K_Q)$, so that the thesis follows directly from Theorem 4.3 and Theorem 2.1. ♦

Corollary 4.4 generalizes a similar result given for the scalar case and stated in the decision space [15]; let us note also that Corollary 4.4 can follow directly from Theorem 4.2, since when $\alpha>0$ it is $K_Q^{\alpha}=K_Q$.

At last we point out that the image space allows us to state sufficient second order optimality conditions without using any separation theorem.

Theorem 4.4 If the following condition (4.7) holds:

$$(K_L+\bar{K}_Q)\cap Cl(H)=\{0\} \text{ and } 0\notin(K_L+K_Q), \tag{4.7}$$

then $x_0\in S$ is a local efficient point for problem P.

Proof Since $0\notin(K_L+K_Q)$ then $T_1=(K_L+\bar{K}_Q)$, so that $T_1\cap Cl(H)=\{0\}$ and the thesis follows directly from Theorem 2.3. ♦

Corollary 4.5 If the following condition (4.8) holds:

$$(K_L+\bar{K}_Q)\cap Cl(H)=\{0\} \text{ and } K_Q\cap K_L=\emptyset, \tag{4.8}$$

then $x_0\in S$ is a local efficient point for problem P.

Note that in general when $(K_L+\bar{K}_Q)\cap Cl(H)=\{0\}$ then every condition implying that $T_1=(K_L+\bar{K}_Q)$ gives a sufficient second order optimality condition.

5. The unconstrained case

Consider now the following unconstrained vector problem:

$$\bar{P}: \begin{cases} \max f(x) \\ x\in X \end{cases},$$

where $X\subseteq\Re^n$ is an open set, $f:X\to\Re^s$ is a twice differentiable function and $x_0\in X$. Point $x_0\in X$ is said to be a *local strict efficient point* if there exists a suitable neighbourhood $I\subseteq X$ of x_0 such that:

$$\nexists y\in I,\ y\neq x, \text{ such that } f(y)\geq f(x_0).$$

Theorem 5.1 Consider problem $\bar{P}$ and suppose $x_0\in X$ to be a local efficient point. If the following condition (5.1) holds:

$$0\notin(K_L+K_Q), \tag{5.1}$$

then $x_0\in X$ is a strict local efficient point.

Proof Suppose ab absurdo that $x_0\in X$ is not a strict local efficient point; then there exists a sequence $\{x_k\}\subset X\setminus\{x_0\}$, $x_k\to x_0$, such that $f(x_k)=f(x_0)$; it then results that $\lim\limits_{k\to+\infty}\frac{f(x_k)-f(x_0)}{\|x_k-x_0\|^2}=0$ and this implies, by means of Lemma 2.1, that $0\in(K_L+K_Q)$, which contradicts the hypothesis. ♦

As an application of Theorem 5.1 consider the following Example 5.1.

Example 5.1 Consider the problem $\overline{P}$: $\{\max f(x,y)=(x-y^2, -x), (x,y)\in\Re^2\}$ and the feasible point (0,0). It is easy to verify that $K_L=\{(x,y)\in\Re^2: x+y=0\}$ and $K_Q=\{(x,y)\in\Re^2: x<0 \text{ and } y=0\}$ so that $0\notin(K_L+K_Q)$; since (0,0) is a local efficient point it then follows that it is also a strict local efficient point, as it can be verified directly since $f(x,y)\geqq 0$ only for $(x,y)=(0,0)$.

Let us note that by means of the proof given in Theorem 5.1 it follows immediately that when alternate efficient solutions occur, necessarily we have $0\in(K_L+K_Q)$, as it can be seen in Example 5.2.

Example 5.2 Consider the problem $\overline{P}$: $\{\max f(x,y)=(x, -x), (x,y)\in\Re^2\}$ and the feasible point (0,0); this point is a local efficient point but not a local strict efficient point since all the points of the kind $(0,y)$, $y\in\Re$, are local efficient points. It is easy to verify that $K_L=\{(x,y)\in\Re^2: x+y=0\}$ and that $K_Q=\{(0,0)\}$ so that $0\in(K_L+K_Q)$.

The following Example 5.3 points out finally that x_0 may be a strict local efficient point even if $0\in(K_L+K_Q)$; in other words, condition (5.1) is a sufficient but not necessary condition in order that a local efficient point is also a strict local efficient point.

Example 5.3 Consider the problem $\overline{P}$: $\{\max f(x,y)=(-x^4, -x^4-y^4), (x,y)\in\Re^2\}$ and the feasible point (0,0). It is easy to verify that (0,0) is a strict local efficient point and that $K_L=K_Q=\{(0,0)\}$, it then results $0\in(K_L+K_Q)$.

Note that condition (5.1) implies also that $T_1=(K_L+\overline{K}_Q)$, so that the following second order sufficient strict optimality conditions can be obtained by means of the results stated in the previous section.

Theorem 5.2 If at least one of the following conditions hold then $x_0\in X$ is a local strict efficient point for problem $\overline{P}$:

i) $\exists\alpha\geq 0$ such that $K_L\subseteq\alpha^{\perp}$ and $K_Q^{\alpha}\subseteq\alpha^{\perp}_{\leqq}$, (5.3)

ii) $\exists\alpha\geq 0$ such that $K_L\subseteq\alpha^{\perp}$ and $K_Q^{*}\subseteq\alpha^{\perp}_{\leqq}$, (5.4)

iii) $\exists\alpha\geq 0$ such that $K_L\subseteq\alpha^{\perp}$ and $K_{\alpha}^{*}\subseteq\alpha^{\perp}_{\leqq}$, (5.5)

iv) $\exists\alpha>0$ such that $K_L\subseteq\alpha^{\perp}$ and $K_Q\subseteq\alpha^{\perp}_{\leqq}$, (5.6)

v) $0\notin(K_L+K_Q)$ and $\exists\alpha>0$ such that $K_L\subseteq\alpha^{\perp}$ and $K_Q\subseteq\alpha^{\perp}_{<}$, (5.7)

vi) $(K_L+\bar{K}_Q)\cap Cl(H)=\{0\}$ and $0\notin(K_L+K_Q)$ (5.8)

vii) $(K_L+\bar{K}_Q)\cap Cl(H)=\{0\}$ and $K_Q\cap K_L=\emptyset$, (5.9)

Let us note that the sufficiency of condition (5.5) can be alternatively proved by means of condition (5.6).

Theorem 5.3 If condition (5.5) holds then $x_0\in X$ is a local strict efficient point for problem $\bar{P}$.

Proof Consider the vector α in the form $\alpha=(\alpha_P,\alpha_N)^T=(\alpha_P,0)^T$, with $\alpha_P>0$, and suppose the function f to be reordered in the same manner, $f=(f_P,f_N)$; consider also the following uncostrained multiobjective problem P_α :

$$P_\alpha: \begin{cases} \max f_P(x) \\ x\in X \end{cases} .$$

By the hypotesis we have that $\alpha_P>0$, $\alpha_P{}^T J_{fP}(x_0)=\alpha^T J_f(x_0)=0$ and $\alpha_P{}^T[v^T H_{fP}(x_0)v]=\alpha^T[v^T H_f(x_0)v]<0$ $\forall v\in\Re^n$, $v\neq 0$, such that $J_{fP}(x_0)v=0$; this implies for condition (5.6) that $x_0\in X$ is a local strict efficient point for problem P_α, that is to say that $\exists I\subseteq\Re^n$, neighbourhood of x_0, such that $\nexists y\in I\cap X$, $y\neq x_0$, such that $f_P(y)\geqq f_P(x_0)$. As a consequence, we have that $\nexists y\in I\cap X$, $y\neq x_0$, such that $f(y)\geqq f(x_0)$ so that $x_0\in X$ is a local strict efficient point also for problem $\bar{P}$. ♦

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