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Point stability versus orbital stability (or instability): remarks on policy implications in classical growth cycle models

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Abstract

The dynamical richness exhibited by several recent extensions of the celebrated Goodwin's model give rise to remarkable consequences concerning the management of economic policies within the frame of dynamical macroeconomic models. The present paper tries to supply an economically based interpretation of such rich dynamics and to sketch out the way in which policy should be reconsidered in such cases. With this goal in mind the notion of policy of stabilisation is revisited through a comparison between the two notions of stability, namely point stability and orbital stability, which coexist within these dynamical frameworks.

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1. Introduction

In the proliferation of studies on growth arisen in last years not very often critical dynamical aspects have been sufficiently stressed, especially in terms of the implications they have for the management of economic policies. Recent work of the authors on the joint role of time lags and other relevant economic assumptions within the frame of the classical Goodwin's growth cycle, whose relevance as a theoretical frame for the analsyis of business cycle has been recently reconfirmed by two Nobel Prizes as R. Solow (1990) and R. Stone (1990), displayed some quite intriguing or even paradoxical effects, especially for what concerns policy implications. Many among such models do exhibit the appearance of persistent periodic behaviours (limit cycles) through Hopf bifurcation mechanisms. But apart the obvious intrinsic interest of such results, there are some consequences which really can not be forgotten by the ingenuous policy maker of such special worlds, due to the not completely self-evident nature of the parameter windows within which the various dynamical behaviours appear. It is the aim of the present paper to sketch some of these consequences following a typical policy-maker point of view, starting from the concept of stabilization policy and systematically comparing policy actions at the light of the two basic notion of stability which coexist in the models considered here: point stability, the notion typically utilized in elementary macroeconomic models, and orbital stability.

The present work is organised as follows. The second paragraph is devoted to a presentation of the reference models, which are given by several types of extension of the basic Goodwin's model. In the third section the most relevant results, in particular the appearance of periodic behaviours through Hopf's bifurcations, are presented together with a detailed interpretation of their actual dynamical economic meaning. These considerations are reformulated, under the form of "policy remarks", in the policy oriented framework of the last section.

2. The reference frame: time lags in Goodwin's type models

To enlight the question we will consider two extensions of Goodwin's original model, that we will call model A and B. The first one (A) is a model proposed by Farkas and Kotsis (1992), and used to give a "Goodwin's type" explanation to hungarian observed business cycles, which embodies a lagged Phillips mechanisms (i.e. a time lag in the wage reaction to employment changes) and a quite ad hoc logistic assumption on the dynamics of employment. Such a model has recently been quoted as one of the most relevant developments of Goodwin's principles (Flaschel and Groh 1995). The second (B) is a model recently proposed by the authors which spouses the same "lagged Phillips" mechanism with a more behavioural assumption on the employment rate dynamics (Fanti and Manfredi 1996b). The assumption of lagged wages is strongly supported in the economic theory: for instance it can be interpreted as a representation of the Keynesian assumption of "sticky wages".

The structure of these two models is quite similar. In terms of classical Goodwin's parameters they look as the following systems of integro-differential equations (definitions for the various variables and parameters are listed in the appendix with reference to the original Goodwin's formulation):

Model A

$$\begin{cases} \dfrac{\dot{V}}{V} = -(\alpha+\gamma) + \rho\int_{-\infty}^{t} U(\tau)G(t-\tau)d\tau \\ \dfrac{\dot{U}}{U} = (m-n-\alpha)(1-\dfrac{U}{x}) - mV \end{cases}$$

Model B

$$\begin{cases} \dfrac{\dot{V}}{V} = -(\alpha+\gamma) + \rho\int_{-\infty}^{t} U(\tau)G(t-\tau)d\tau \\ \dfrac{\dot{U}}{U} = (m-\alpha) - nU - mV \end{cases} \quad (2.1)$$

Both models can be studied, provided the structure of the time delay, embodied in the weight functions G, be of the Erlangian form, by, reconducing them to suitable ODE systems of higher order through the so called "linear trick" (Mc Donald 1978). This permits to study the conditions for the existence of periodic orbits through the usual Hopf's bifurcations machinery.

In particular we will contrast the results of models A and B against those provided by two other models, which constitute natural reference benchmark. The first model is the unlagged counterpart of model B, ie the following simple extension of Goodwin's model to take into account of a typical labour supply effect (Manfredi and Fanti 1996):

$$\begin{cases} \dfrac{\dot{V}}{V} = -(\alpha+\gamma)+\rho U \\ \dfrac{\dot{U}}{U} = (m-\alpha)-nU-mV \end{cases} \qquad (2.2a,b)$$

In (2.2) the growth rate of the labour supply (assumed constant in Goodwin's model) is endogenously determined according to a positive relation with employment, as stated by various labour market theories. A justification of the nU term in equation (2.2b) can be given by defining the labour supply $L_S(t)$ as: $L_S(t)=s(t)N(t)$, where s is the rate of offering labour and N the total population and then by assuming that a)the population is growing at a constant rate (or even stationary) and b) the rate of change of s is modeled, along Mincer's (1966) assumption of "discouraged worker" (DW), with the relation:

$$\frac{\dot{s}}{s} = -\delta + nU \quad \delta, n > 0 \qquad (2.3)$$

(we have embodied for simplicity the extra-constants in the term (m-α)).

Th effect on the original Goodwin's model of the (negative) quadratic employment term due to the DW assumption (we would call it a "density dependence" (DD) effect following the special terminology of ecological models), is strongly stabilizing. In particular, as it is well known from the basic Lotka-Volterra's theory, the effect of the DW term is that of giving rise to an asymptotically stable equilibrium which will be reached in the long run through damped oscillations (in some cases, when the DW effect is very strong, it could even sterilize the cycle, by giving rise to to a non oscillatory adjustment, but this fact seem less probable with reasonable values of the typical economic parameters).

The second benchmark model is the following Goodwin's extension (already analysed by V.Volterra in his 1926 pioneeristic work) enlarged to take account of lagged effects in the Phillips mechanism:

$$\begin{cases} \frac{\dot{V}}{V} = -(\alpha + \gamma) + \rho \int_{-\infty}^{t} U(\tau) G(t-\tau) d\tau \\ \frac{\dot{U}}{U} = (m - n - \alpha) - mV \end{cases} \qquad (2.4)$$

which shows how the effect of the time lag on the positive equilibrium of the model is sharply destabilizing. Model (1.4) is a prototypical example, in the scientific literature, of the so called "generally destabilizing"[1] role of lags (May 1973, Mc Donald 1978, Farkas 1995). A quite appealing fact regarding previous formulations, is that the introduction of lags preserves the structure of the stationary states of the underlying basic model, so making, from such point of view, the various models considered, lagged or not, fully comparable.

2. Limit cycles

Even the simplest assumptions on the structure of time lags, that of an exponentially fading memory G of parameter a (so with mean lag T=1/a), gives rise to persistent periodic behaviours in models A and B. It

[1] In effect even within the frame of Goodwin's growth cycle theory it is possible to ascertain, in some cases, also a stabilizing role of lags. This happens for instance in presence of a time delay in the adjustment of prices to wages changes. Chiarella (1990) has argued, notwithstanding, that periodic orbits can appear in a (hopefully positive) neighbourhood of a "zero lag". Without entering the details of such results we point out their completely lacking of any practical relevance: even if such limit cycles could effectively exist, the bifurcating lag would anyway be too little to be economically meaningful.

is possible to show that they collapse, respectively, in the following three-dimensional ODE systems:

Model A

$$\begin{cases} \dot{S} = a(U-S) \\ \dfrac{\dot{V}}{V} = -(\alpha+\gamma)+\rho S \\ \dfrac{\dot{U}}{U} = (m-n-\alpha)(1-\dfrac{U}{x}) - mV \end{cases}$$

Model B

$$\begin{cases} \dot{S} = a(U-S) \\ \dfrac{\dot{V}}{V} = -(\alpha+\gamma)+\rho S \\ \dfrac{\dot{U}}{U} = (m-\alpha) - nU - mV \end{cases} \qquad (3.1a,b)$$

Both models have two less interesting equilibria (a zero equilibrium and an equilibrium in which the product is completely distributed to profits) and one positive equilibrium, let us call it E_1, which in general under few suitable assumptions is fully meaningful from the economic point of view. By studying the properties of E_1 through Hopf bifurcations, it is possible to prove that models A,B will possess unique periodic orbits in suitable neighbourhoods of the bifurcation point. Such results introduced by Farkas (1984) for model A, are formalized in theorems 3.1, 3.2 and 3.3 in Farkas and Kotsis (1992), or theorem 7.3.1 in Farkas (1995) (and used in Fanti and Manfredi, 1996b, for model B), to which we explicitly refer. It is particularly informative to display the content of such bifurcation theorems by stressing the relations between the crucial parameters under inquiry, which in the case of model B are the mean lag (through its reciprocal a) and the reaction of labour supply to employment, given by n. Their meaning can be rephrased as follows:

a)The E_1 equilibrium loses its stability, coeteris paribus, when the time delay is large, ie when the parameter a is below a given threshold, whose form is represented by the downward sloping Φ line of fig. 1. Such a line, which defines the bifurcation relationship (as implicit in the Routh-Hurwicz condition) can be represented in the form:

$$a_0 = \Phi(n;\Theta) \qquad (3.2)$$

where Θ denotes the vector of (usually all) the remaining parameters of the model.

b)When the lost of stability takes place for strictly positive Φ values the system undergoes a Hopf's bifurcation, giving rise to periodic orbits in a neighbourhood of the bifurcation curve. This happens for values of n ($n<n_1$) which are not too large (fig. 1).

c)These periodic orbits will be locally asymptotically stable or unstable depending on wheter the n parameter exceeds or not a specific threshold value n_d. To be precise, every time in which the bifurcation curve is located above the "threshold line" R(n) (given by a straight line through the origin in fig. 1), then, it is possible to find a $\delta>0$ such that for each a in the neighbourhood $(a_0-\delta,a_0)$ the system will have a periodic orbit, which is unique (at least in a neighborhood of the equilibrium point E_1), and locally asymptotically stable (LAS). Viceversa, when the bifurcation curve is below R(n), then periodic orbits, which appear in a right neighborhood $(a_0,a_0+\delta)$ of the bifurcation line, are unique and unstable. It is to be noticed that these last stability remarks, due to Farkas (1984), are especially remarkable. In fact it is often stressed, in the literature on the applications of dynamical systems, the huge algebraic complexity which is needed to ascertain stability properties of limit cycles arising from Hopf bifurcations in dimensions higher than two (the "good luck" wish by Marsden and McCracken (1976) to unaware performers of such calculations is not metaphoric!). In the present case Farkas's theorems provide modelers with an especially useful result for a quite important class of mathematical models.

Looking more closely at fig. 1 we may deep the dynamical content of such results:

a)There is a window within the range R_A of admissible values of n ($n<n_0$) in which the DW effect is so strongly stabilizing to completely sterilize the destabilizing role played by the time lag. In this window, corresponding to the region ($n_1<n<n_0$) in which the Φ curve is negative, E_1 will always be LAS independently on the amplitude of the lag.

b)Viceversa there is a window ($0<n<n_1$), corresponding to the positive restriction of the $\Phi(n)$ curve, in which the DW effect is less important. This gives rise to a dynamical balancement between the two

reversed forces, ie the destabilizing one due to the time lag and the stabilizing one due to the DW effect, which permits to the system to undergo transitions from (local) stability to (local) instability trough Hopf bifurcations. In particular we will have instability in that part of the (n,a) plane which is below the bifurcation curve Φ(n), so generically corresponding to "large" lags, another common place in the literature. The Hopf's bifurcation theorems proven by Farkas permit to ascertain the stability properties of the given periodic orbits:

b1)There is a supercritical (" R_{Super} ") region ($0<n<n_d$), characterized by a low DW effect, in which the emerging periodic orbits will be (locally) stable. This fact can be explained in this way: the low DW effect makes in some sense weak the attraction effect of the equilibrium E_1: weak enough to permit to orbits which are very close to it to escape (so eventually tending to a limit cicle which surrounds the equilbrium), due to the destabilizing effect of the lag. At the same time, since the DW effect will always be important when "we are far from the equilibrium"; the stability of the periodic orbits implies that for low n such effect will always prevail on the escaping effect of the lag. In other terms this will anyway guarantee the invariance of some appropriate region of the phase plane.

b2)There is also a subcritical region $R_{Sub.}$, characterized by intermediate values of n ($n_d<n<n_1$), in which limit cycles are unstable, so reversing the effects discussed in b1).

Among these results there are some which appear to us particularly central for their economic interpretation and policy implications.

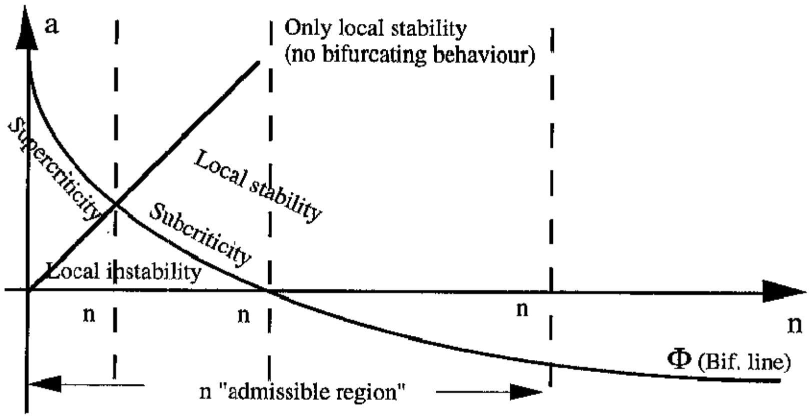

Fig. 1 Structure of dynamical windows for models A,B

By comparing our basic models (2.1) with their reference benchmark (models (2.2) and (2.3)) we can observe the following for what concerns the endogenization of the supply of labour through a discouragement effect. When no delayed mechanism is operating in Goodwin's model then this effect is simply stabilizing: this is nothing but the usual density dependent effect in Lotka-Volterra's ecologies. Viceversa, when delayed effects are operating (for instance in the wage formation process, but our work has shown that such results are much more general, see Fanti and Manfredi 1996), then the DE effect can give rise to an important and not imediately self-evident effects. When it is very strong it can completely annihilate the destabilizing role of the time lag, so generating a "normal" equilibrium dynamics. Ond the other side, if it is not very large, it can give rise to persistent oscillations. The economic dynamics interpretation is quite appealing: the generation of persistent oscillations needs a balancement between delays (not too long) and quadratic damping effects: *these latter must be "weak enough" so to prevent the asymptotical attractivity by the equilibrium and, at the same time, strong enough to permit an attraction effect, this time due to the limit cycle.*

4. Policy implications

Within the traditional dictionary of economic policy inside the frame of dynamical macroeconomic models, there is a concept, that of *policy of stabilization*, which is strongly based on the notion of stability which is typical of elementary dynamical macroeconomic systems, namely *point stability*.

But most typical macroeconomic dynamical models are much less reacher than those considered here, in the sense that they do not display the possibility of stable periodic behaviours. Furthermore we have seen that such periodic behaviours can appear in correspondence of parametric windows, the locations of which is not necessarily self-evident. For this reason, it is interesting to try contrast the two notions, point stability and orbital stability, which very often coexist in quite reasonable economic formulations as those examined here, for instance in terms of how they can modify the management of a policy of stabilization (we advert the reader that present considerations are always "local", since this is the nature of the limit cycles emerging from Hopf's bifurcation. We will not enter into more global dynamical properties). With this goal in mind let us assume that the n parameter be a *control* of economic policy, ie that be in some way manageable by the authorities in view of stabilization policies (it is useful for the moment to avoid to reason in terms of the specific nature of n, trying to think in general terms), while the a parameter be given (at some a* value) and not manageable (of course we could exchange the role of the two parameters without changing the substance of the problem): this means that the "space of possible policies" in the (a,n) plane is the horizontal line at a* level.

Let us first of all remember how would operate a policy maker who bases his decisions only on the "point stability" notion (let us call him a "pure Routh-Hurwicz Policy maker: RHpm"). Our policy maker only knows the form of the Φ curve, but not that of the R line and not even δ. His aim will be to obtain the best possible econometric estimates of the Φ curve and, once known it, to move rightward as most as he can do, to stay as far as possible from the "instability area" (notice that the fact of raising n while einforcing stability raises at the same time the rate of growth of the economics is specific to our model and we will regard it as purely incidental). In particular he will try to determine which economic parameters can affect the position of the Φ curve and its sensitivity to variations of such parameters. In a situation of high uncertainty about such parameters a good rule seems to be (see fig. 1) *to move rightward as much as possible the control parameter, and possibly to stay at the right of* n_l.[2]

Problems faced by the more theoretically involved policy-maker are more subtle and illustrate at the same time the risks in which would have incurred the unaware RHpm. Let us so turn our attention to fig. 2, which exhibit a quite rudimentary representation of the results of the previous section, with the addition of the stability and instability stripes (denoted respectively by S_{Super} and R_{Sub}) of the periodic orbits surrounding the bifurcation curve. Be such stripes of amplitude δ>0 (of course such δ is a small number: we represent it through a " big stripe" just for clarify the reasoning).

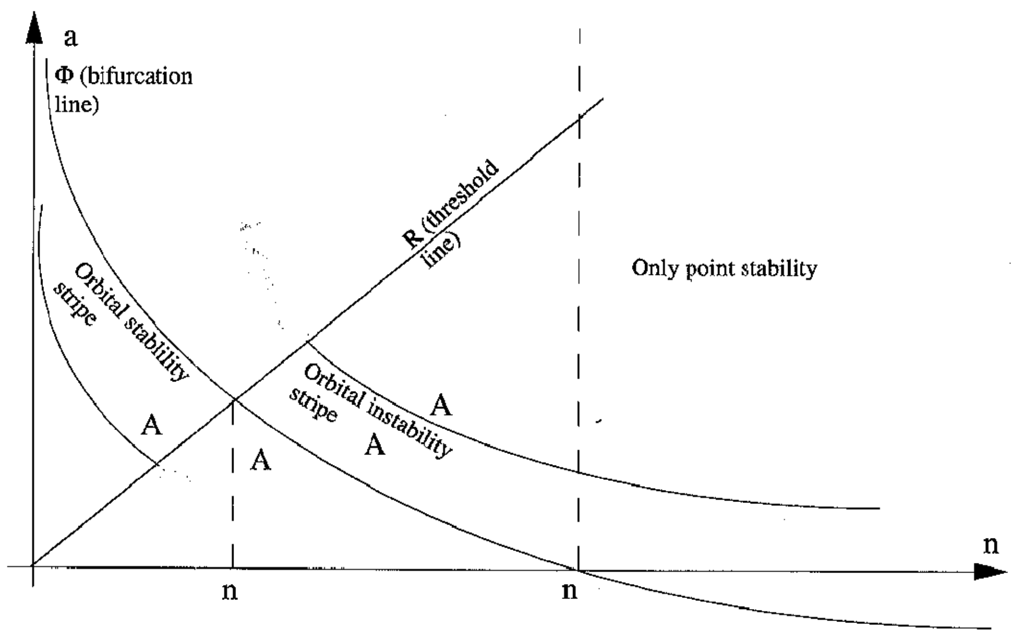

Fig. 2. Possible movements toward stability in models A,B

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[2] Of course a very general prescription could be to try, by managing other relevant parameters, to push Φ downward, so to reduce the amplitude of the instability area. In this first work we will suppose, for simplicity, that these parameters are not manageable.

The management of such a more rich dynamical world is much more difficult, due to the complex dynamics of the various dynamical windows, the location of which critically depends on the forms of the bifurcation curve and the threshold line, which in turn depend critically on all the parameters of the system. But let us suppose, for simplicity, that all other relevant parameters do not vary. Clearly even if the recipe given for the "pure Routh-Hurwicz world" can still be considered valid, there can be situations, due to the presence of the orbital stripes, in which the best policy does not necessarily consist in raising n. It is clear that these situations are relevant when the system is close to the bifurcation threshold Φ. The following remarks, referred to possible concrete dynamical situations, just try to capture the heuristic of the Hopf's world (It would be possible to construct a more fine taxonomy but we avoid it for reason of space).

Remark 1 ("quieta non movere"): be the system within the within the S_{Super} stripe (say in A_1). In this case the system is experiencing orbital stability. Unless there is the possibility to make a very large rightward movement of n (so perhaps a very costly policy) we really do not have any convenience in trying achieve point stability. In fact "wrong" rightward policies could have the undesidered effect could of carrying the system directly in the instability area (in the "Bermuda triangle", given by the shaded area in fig. 2), or, in case, drive it in the S_{Sub} stripe. What actually could happen in this region (characterised by periodic instability) depends on the state the system inherits while reaching it: an initial condition external to the unstable periodic orbit would again give rise to instability.

Remark 2: let us consider the reversed situation: the system actually be in A2, in the "Bermuda triangle", experiencing an instability behaviour. A "Routh-Hurwicz" stabilizing behaviour would suggest, as the only escape to such potentially dramatic situation, to increase n. But this is not in fact the only possibility: stability, namely of the orbital type, can also be achieved through a completely different policy, by reducing n rather than raising it. We do not have at present clearcut ways to compare the two alternatives, but we remember again, in favour of orbital stability, that a not sufficiently strong rightward policy would carry again the system in the S_{Sub} stripe. In other words: if we are not sure, by increasing n, to go beyond the S_{Sub} stripe, then it could be worth of consideration the completely opposite policy, through diminishing n.

The possible danger due policy measures not sufficiently intensive seems furthermore suggest the following:

Remark 3 ("Pay attention to small steps policies!"). Let us suppose the system be in A3, within the S_{Sub} stripe. In this domain we could have LAS of the equilibrium, in the case we are sufficiently close to it, but we could also have instability,for initial conditions sufficiently far from it. If policy should operate only through decreasing n for exogenous (for instance political) reasons, then it is clear that small step policies could even be very dangerous as the economic system could be driven to the region of instability. In this case a "courageous" policy, ie much more intensive in quantitative terms (so, in some cases, unpopular!) addressed to reach the stripe of orbital stability would be preferable.

5. Conclusions

The policy considerations presented in previous pages are of course to be intended, rather than define precise rules, as merely suggestive of the difficulties which are posed by some examples of nonlinear world. A quite encouraging fact is that the special behaviours considered in the present work, with a decreasing shape of the bifurcation relation shown in fig. 1 and the abovementioned sequence of "parametric windows, are not at all special, but rather seem generic to a quite broad class of mathematical models, ie, at least for what concerns our work, the entire class of basic Lotka-Volterra-Goodwin models extended to take into account the possibility of exponentially distributed lags (so giving rise to third order ODE systems) in some of the basic economic relationships. For instance, apart the two aforementioned models this result is typical of other models studied by the authors, for instance

Goodwin' type models embodying lags in the profit-investments chain and population effects, or efficiency wages behaviours.

A particularly interesting problem seems to be the role of the periodic orbits stability stripes. The amplitude of the stripes we have represented in fig. 2 is purely indicative of the qualitative nature of the problem. But clearly it would be critical for the policy maker to know how large is δ. While it has been possible (Farkas 1984), even if cumbersome, to determine the actual stability properties of models A,B, their policy meaning is completely unuseful if we do not know the amplitude of δ. On what δ depends? We expect that δ be the sup of a complex function of all the relevant models parameters:

$$\delta = \delta(\alpha, \gamma, \rho, n, m, ..)$$

In virtue of this fact it seems crucial to us to estimate the such relation and its sensitivity to changes in models parameter: this is the really challenge to government econometricians!

Finally: those presented here are only the simplest example of how nonlinear models can make economic policy difficult. Other models examined by the authors in recent works (Fanti and Manfredi 1996) can exhibit still more complicated bifurcation relationships, giving so rise, as a consequence, to also more complicated problems for the policy maker.

Appendix: Goodwin's model

The original Goodwin's formulation rests on the following system of two ODE's describing the dynamics of the labour share on total income (V) and the rate of employment (U):

$$\begin{aligned} \frac{\dot{V}}{V} &= -(\alpha+\gamma)+\rho U \\ \frac{\dot{U}}{U} &= (m-\alpha)-nU-mV \end{aligned} \qquad (A.1)$$

where:

$U=L/N$; L=actual employment level; N=labour supply

$V=wL/Q$ w=real wage rate; Q=national product

$Q=aL$; a=average productivity of labour

In particular a and N are assumed to be in steady growth with respective rates α and n:

$$a(t)=a_0e^{\alpha t} \qquad N(t)=N_0e^{at}$$

Finally, in model A x represents the maximal employment level reachable by the economics, which is, by assumption, bounded between zero and one.

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