Report n. 120

Gestation lags and efficiency wage mechanisms in a Goodwin-type growth model

Piero Manfredi e Luciano Fanti

Pisa, June 1997

Gestation lags and efficiency wage mechanisms in a Goodwin-type growth model

Piero Manfredi e Luciano Fanti***

Abstract

The present paper investigates the joint effects on the classical Goodwin's growth cycle model of two different assumptions: the efficiency wage hypothesis on one side and the assumption of the existence of a time lag in the profits-investments chain (i.e.: a gestation lag in the investment plans) on the other side. It is possible, as the main result of the work, to show the existence, within special parametric windows of the two crucial parameters of the model, namely the average time delay in investments and the reaction of productivity of labour to wages changes, of persistent oscillations, through the machinery of Hopf's bifurcations.

Key words: Goodwin-type models, efficiency wage hypothesis, gestation lags, Hopf's bifurcation, limit cycles

*(to whom correspondence should be addressed)
Dipartimento di Statistica e Matematica Applicata all'Economia,
Facoltà di Economia dell'Università di Pisa, Via Ridolfi 10, 56124 Pisa - Italy
e-mail: manfredi@ec.unipi.it

**Dipartimento di Scienze Economiche
Facoltà di Economia dell'Università di Pisa, Via Ridolfi 10, 56124 Pisa - Italy

JEL Classification: E3, J0

Gestation lags and efficiency wage mechanisms in a Goodwin-type growth model

1. Introduction

The present paper aims to investigate the role of time delays in the chain between profit and investments in a Goodwin-type framework (or Lotka-Volterra-Goodwin: LVG since now on), by combining the assumption of a gestation lag in the investment plans with the existence of an efficiency wage behaviour.

The implications of the former assumption, which postulates the existence of a delay between the investment decision and the operativeness of new productive capacity, have been extensively studied in several contexts in the macroeconomic literature, for instance in the "traditional" macroeconomic analysis (Kalecki 1975) but also in "microeconomically funded" formulations such as the seminal paper of the "Real business cycle theory" school by Kydland e Prescott[1] (1982). Some of its implications within Goodwin-type settings have been studied in Fanti and Manfredi (1997a), who were able to show that the gestation lag alone has, as is typical, a purely destabilizing effect on Goodwin's growth cycle, and that furthermore, persistent periodic behaviours can appear when other mechanisms, such as quadratic labour supply effects are considered.

The latter hypothesis, e.g. the efficiency wage hypothesis, even if very popular, has up to now received little attention, at least to our knowledge, as regards to its

[1] We will obviously make the assumption that any involved short term adjustment needed to bridge the consequent gap between current savings (i.e.: profits) and current investments, takes place essentially through inventories changes, without giving rise to further effects, such as changes in the level of the output of the economy. These assumptions are needed to maintain the coherency with the basic Goodwin frame, in which the output-capital ratio is constant by assumption, and the rate of growth of the output is always determined by the investment rate. Possible changes in prices due to disequilibrium situations will not be considered in the present paper just because the model is a "real model". This amounts to assume that wages are perfectly indexed with respect to prices changes. Hence, for instance, if the current investment should exceed the actual saving, the corresponding change in prices would not effect at all the income distribution.

macrodynamical implications. Limiting ourselves to Goodwin-type frames (Goodwin 1967), Chooi (1995) has studied, by means of local analysis the effects of the introduction of a very general wage dependent effort function e=e(w). He was able to give a full local picture of the dynamics around the positive equilibrium of the system; in particular he was able to show, by means of local bifurcation analysis, that when the elasticity of the effort with respect to wage is negative in a neighborhood of the positive equilibrium, then persistent periodic dynamics can appear through Hopf's bifurcations.

Fanti and Manfredi (1995) have studied the effects on Goodwin's model of a somewhat simpler "diminishing returns" relationship between the effort of the worker and the wage level within what they called the Gew model ("Goodwin and the efficiency wages hypothesis"). Thanks to the simplicity of their assumption, which modifies the basic Goodwin's model into a dissipative predator-prey formulation, they could obtain a global picture of the effects of efficiency wage. They also tried to deep the possible consequences of the EW hypothesis, by conjecturing the possibility that the capitalists realize the "existence" of their diminishing returns effort function, and so try to behave optimally, and studied the consequences of this assumption as well. This investigation has also been completed with normative considerations (Manfredi and Fanti 1996a). Fanti and Manfredi (1997c) have then deeped the study of the effects of the EW hypothesis in Goodwin's frames, by studying its effects in conjunction with those due to the existence of sticky wages, i.e. of lagged Phillips effects, showing the modalities through which the EW mechanism can be responsible for the generation of limit cycles, through, again, Hopf's bifurcations.

As previously mentioned the aim of the present paper is to provide an investigation of the combined effects of the EW mechanism and of the gestation lag within a Goodwin-type frame, with particular attention to steady state considerations and to the possible existence of persistent periodic behaviours. Indeed, the main motivation for the investigation of the joint effects of these peculiar economic forces lies in the fact that, as already shown in the aforementioned papers by the same authors

(1995 and 1997a), they play fully antithetical roles on the basic positive equilibrium of Goodwin's model. In fact, the efficiency-wage has a stabilizing effect (even strongly stabilizing!), whereas the gestation lag is basically destabilizing. As it is well known from the classical theory of time delays (May 1973), the combination of such "contradictory" forces is a good premise for the generation of limit cycles.

The present paper is organized as follows. In the second section the two variants of the classical Goodwin's model which constitute the motivations for the present work, the Gew model (Fanti and Manfredi 1995) and the Prolag model (the variant of Goodwin's model embodying a time delay in the chain between profits and investments, Fanti and Manfredi 1997), are reviewed and combined in a more general model (Prgew) embodying both such effects. The analysis of equilibria, basic static considerations plus local stability analysis, is performed in the third section, which contains also a discussion of the reciprocal differences, in steady state, among the three models aforementioned. The fourth section attacks the problem of the existence of persistent periodic behaviours through the Hopf's bifurcation machinery. The bifurcation curve in the relevant parameter space is given together with a conjecture on the location of the regions of supercritical/subcritical behaviour of the involved limit cycles. The conjecture is fully confirmed by numerical simulations. The nucleus of the results of the paper are synthesized in the conclusions.

Goodwin's model, efficiency wages and the "gestation lag" hypothesis

The very well known Goodwin's growth cycle model is defined by following system of two ordinary differential equations:

$$\begin{cases} \dfrac{\dot{V}}{V} = -(\alpha+\gamma)+\rho U \\ \dfrac{\dot{U}}{U} = (m-\alpha-n)-mV \end{cases} \qquad (2.1)$$

In (2.1) U=U(t) is the employment rate level at time t, defined as the ratio between the effective employment L(t) and the supply of labour N(t) (this latter can in case be estimated by means of the total population), and V(t) is the distributional share of the wage earners, given by the ratio V=wL/Q, where w is the real wage rate and Q is the total product per unit of time. Moreover V can be expressed as: V=w/A, where A is the average productivity of labour : A=Q/L.

The model (2.1) is easily derived by introducing within the following "dynamical identities":

$$\begin{cases} \frac{\dot{V}}{V} = \frac{\dot{w}}{w} - \frac{\dot{A}}{A} \\ \frac{\dot{U}}{U} = \frac{\dot{Q}}{Q} - \left(\frac{\dot{A}}{A} + \frac{\dot{N}}{N}\right) \end{cases} \qquad (2.2)$$

(expressing the relationships V=w/A and U=L/N in terms of their rates of growth over time) the following basic Goodwin's assumptions:

i) the labour market is driven by the following linear Phillips relationship:

$$\frac{\dot{w}}{w} = -\gamma + \rho U \quad (\gamma > 0,\ \rho > 0; \rho > \gamma) \qquad (2.3)$$

where γ, ρ are characteristic parameters of the labour market, reflecting respectively the ability of the working class in defend its wage in situation of "zero" employment (γ) and the speed of reaction of the growth rate of wage to changing employment (ρ).

ii) the accumulation"rules" are such that: a) the wage earners do not save; b) the profits are entirely reinvested; c) the technology is Leontief-type and the capital/output ratio 1/m (m>0) is constant over time

These accumulation rules give rise to the following equation for the rate of growth of output:

$$\frac{\dot{Q}}{Q} = m(1-V) \qquad (2.4)$$

iii) a constant exogenous growth of the supply of labour N (at the rate $n>0$) and of the average productivity of labour A (at the rate $\alpha>0$)

By assuming that the coefficients in (2.1): $A=\alpha+\gamma, B=\rho, C=m-\alpha-n, D=m$, are all positive (and costant) and that, moreover it holds: $\rho > \alpha + \gamma$, it is easy to see that model (2.1) has a unique non-zero equilibrium with coordinates:

$$(U^*, V^*) = \left(\frac{\alpha+\gamma}{\rho}, \frac{m-\alpha-n}{m}\right) \qquad (2.5)$$

which is also economically meaningful.

As it is well known system (2.1) is, under the given assumptions, a predator-prey model of the Lotka-Volterra-type, so exhibiting the typical conservative oscillatory behaviour, and which has, by definition of the U,V variables, a dynamics bounded by the feasible set $T=[0,1]x[0,1]$.

In the present work we will follow the formulation of the "efficiency wage hypothesis" given in Fanti and Manfredi (1995), in which the productivity of labour depends on the worker's effort, which in turn depends on the wage level:

$$A = A(t, w(t)) = k(t)w(t)^{b(t)} \qquad (2.6)$$

In the last expression k(t) is the exogenous component of productivity growth, which, for simplicity, will be assumed to grow at the constant rate α, while *b* is a parameter measuring the elasticity of productivity to wage changes, which will be assumed constant and bounded in the set (0,1). The assumption (2.6), despite being very simple[2], seems to be able to supply, in comparison with more general formulations, such as Chooi 1995, to more clearcut dynamical results (Manfredi and Fanti 1996a).

[2] For instance it is at all clear that productivity can not grow unbounded as wages increase, as results from (2.7).

Writing (2.6) in dynamical terms we get:

$$\frac{\dot{A}(t)}{A(t)} = \frac{\dot{k}(t)}{k(t)} + b\frac{\dot{w}(t)}{w(t)} = \alpha + b\frac{\dot{w}(t)}{w(t)} \qquad (2.7)$$

which shows that the rate of growth of productivity is defined as the sum of two components: i) a constant one, given by exogenous dynamics of technical progress, like in Goodwin's model, ii) a component proportional to the endogenous dynamics of the wage rate.

By combining (2.7) with the Phillips relationship (2.3), we obtain the final equations of the GEW ("Goodwin and Efficiency Wage") system (Fanti and Manfredi 1995)

$$\begin{cases} \frac{\dot{V}}{V} = -B_1 + B_2 U \\ \frac{\dot{U}}{U} = C_1 - C_2 U - C_3 V \end{cases} \qquad (2.8)$$

where:

$$\begin{array}{l} B_1 = \alpha + (1-b)\gamma > 0; \quad B_2 = (1-b)\rho > 0 \\ C_1 = m + b\gamma - \alpha - n > 0; \quad C_2 = b\rho > 0; \quad C_3 = m > 0 \end{array} \qquad (2.9)$$

The only coefficient in (2.8) the sign of which is not not restricted a priori is C_1; but since the quantity (m-α-n) is assumed to be positive in the Goodwin's model, by mantaining such an assumption in model (2.8), the C_1 coefficient will results positive even more so. Therefore the system (2.8) is a classical quadratic model of predator-prey interaction à la Lotka-Volterra, with logistic growth of the prey population (the rate of employment) in absence of the predator (the wage share), whose dynamical properties are well known since Volterra 1926. The specific role of the efficiency wage parameter in driving the dynamics of the GEW model is investigated in Fanti and Manfredi (1995), while the long term consequences of the GEW hypothesis on Goodwin-type economies together with welfare considerations are developed in Manfredi and Fanti (1996a).

In Fanti and Manfredi (1997a), the introduction of an exponentially distributed gestation lag between the decision and the realization of the investment plans has been considered. The final model is given by the following system of three ordinary differential equations:

$$\begin{aligned} \frac{\dot{S}}{S} &= a\left(\frac{(1-V)}{S} - 1\right) \\ \frac{\dot{V}}{V} &= -(\alpha + \gamma) + \rho U \\ \frac{\dot{U}}{U} &= \frac{mS - (\alpha + n)(1 + mS)}{1 + mS} \end{aligned} \qquad (2.10)$$

where the dynamical variable S is defined as:

$$S(t) = \int_{-\infty}^{t} (1 - V(\tau))\Pi(t - \tau)d\tau \quad (2.11)$$

and the function Π is the exponentially distributed weight of the time-delay:

$$\Pi(s) = ae^{-as} \qquad a > 0 \qquad (2.12)$$

The specific formulation (2.10), (2.11) and (2.12) has been developed in Fanti and Manfredi (1997a) and derives from the application of the so called "linear chain trick" (McDonald 1978), to the integro-differential system embodying the exponentially weighted gestation lag.[3] As previously pointed out the main effect of the gestation lag is that of destabilizing, at least locally, the positive Goodwin's equilibrium, breaking of course the conservative peace of LVG oscillations

The direct superimposition of the formulations (2.8) and (2.10) embedding the two effects due, respectively, to the efficiency wage and the gestation lag hypothesis, takes to the following system (the Prgew system)

[3] The linear "trick" performs the transformation from a "reducible" integro-differential system with Erlangian-type time delays to the corresponding higher order ordinary differential equations system.

$$\begin{cases} \frac{\dot{S}}{S} = a(\frac{(1-V)}{S} - 1) \\ \frac{\dot{V}}{V} = -[\alpha + (1-b)\gamma] + (1-b)\rho U \\ \frac{\dot{U}}{U} = \frac{mS}{1+mS} - [\alpha + b(-\gamma + \rho U)] - n \end{cases} \qquad (2.13)$$

The main reason why to consider the two combined effects in system (2.13) lies in the possible dynamical reachness generated by such combined effects, which are completely antithetical for what concerns their local effects on the positive equilibrium of Goodwin's model (one, the EW mechanism, is fully stabilizing; the other one, the gestation lag, is fully destabilizing). In effect, as underlined in classical work about the role of time delays in dynamical systems (May, 1973), the joint effect of such antitethical forces can bring about to important local effects such as the appearance of persistent periodic oscillations, mainly detectable through the machinery of Hopf's bifurcations.

3. Equilibria and local stability

It is imediate to check that model (2.13) has, apart the zero equilibrium (E_0), an equilibrium with zero wage share (E_2) of coordinates

$$\left(U^{**} = \frac{b\gamma - g}{b\rho} + \frac{m}{b\rho(1+m)},\ V^{**} = 0,\ S^{**} = 1 \right) \qquad (3.1)$$

and a positive equilibrium (E_1) of coordinates

$$U^* = \frac{\gamma}{\rho} + \frac{\alpha}{\rho(1-b)} \quad V^* = 1 - S^* \quad S^* = \frac{g - nb}{m[1 - b - (g - nb)]} \qquad (3.2)$$

where $g=\alpha+n$ denotes the average growth rate of the product in Goodwin's model. As expected, the PRGEW model combines the equilibrium features of the basic GEW and PROLAG model.

For what concerns the "zero labour share" equilibrium E_2, we briefly notice that it appeared in the Gew model as a direct consequence of the efficiency wage hypothesis and it is, clearly, preserved under the more general formulation (2.13). In Fanti and Manfredi (1995) the economic properties of the "zero labour share" equilibrium within the Gew model were carefully discussed. Since in this paper we are mainly concerned with the possibility of the appearance of periodic behaviours, we will essentially concentrate on the positive equilibrium E_1[4].

For what regards the equilibrium E_1 we notice first that the equilibrium level of the employment rate is meaningful provided:

$$b < 1 - \frac{\alpha}{\rho - \gamma} = S_0 \qquad (3.3)$$

and it is identical to the corresponding equilibrium level of the Gew model (an obvious fact as the equations of the labour share are the same in the two models). Moreover, the condition $0<S^*<1$, which is necessary and sufficient to have an economically meaningful equilibrium of the wage share, gives the conditions:

[4] Since the system preserves positivity (the positive orthant is left invariant under the flow of the system) no periodic behaviours which surrounds the zero labour share equilibrium are definitively possible.

$$S^* > 0 \rightarrow \frac{g - nb}{m[1 - b - (g - nb)]} > 0 \rightarrow b < \frac{1 - (\alpha + n)}{1 - n} = S_1$$

$$S^* < 1 \rightarrow \alpha + n(1 - b) < m[1 - b - (g - nb)] \rightarrow \frac{\alpha}{1 - b} + n < m\left[1 - (\frac{\alpha}{1 - b} + n)\right] \qquad (3.4a,b)$$

The conditions (3.3), (3.4a) and (3.4b) do not appear restrictive, at least with "normal" (non patological) values of the involved economic parameters. In particular it is to be notice that (3.4b) depends on the quantity:

$$G = \frac{\alpha}{1 - b} + n \qquad (3.5)$$

The quantity (3.5) is the long term rate of growth of the Gew model when the positive equilibrium of the GEW system is stable. It is easily seen that (3.5) would also be the long term rate of growth of the model (2.13) corresponding to the E_1 equilibrium, in the event the positive equilibrium E_1 be the long term outcome of the model itself (i.e. when the E_1 equilibrium is, at least, locally stable):

$$\left(\frac{\dot{Q}}{Q}\right)_{E_1}^{Pr\,Gew} = \left(\frac{\dot{U}}{U} + \frac{\dot{a}}{a} + n\right)_{E_1} = \alpha + b\left(\frac{\dot{w}}{w}\right)_{E_1} + n = \alpha + b(-\gamma + \rho U^*) + n = \frac{\alpha}{1 - b} + n = G$$

By using (3.5), (3.4b) can be written as:

$$G < m[1 - G] \qquad (3.6)$$

from which the following condition on the output -capital ratio follows:

$$m > G/(1-G) \;\rightarrow\; m > \frac{\frac{\alpha}{1-b}+n}{1-\left[\frac{\alpha}{1-b}+n\right]} = \frac{\alpha+n(1-b)}{(1-b)(1-n)-\alpha} \qquad (3.7)$$

When rewritten in terms of b (3.7) looks as:

$$b < 1-\frac{\alpha(1+m)}{m(1-n)-n} = S_2 \qquad (3.7')$$

It is easy to show that $S_2<S_1$. The thresholds S_0 and S_2 define the set of values of the b parameter which are compatible with an economic meaningful (i.e.: strictly positive) equilibrium: this is the set $0<b<b_0$, where $b_0=\min(S_0,S_2)$[5].

It is in particular convenient to express the E_1 equilibrium levels of the state variables V and S in terms of the corresponding long term rate of growth:

$$S^* = \frac{G}{m(1-G)} \;;\; V^* = 1-\frac{G}{m(1-G)} \qquad (3.8)$$

It appears of some interest to compare the positive equilibrium (E1) levels of the labour share V across the full set of variants to Goodwin's model considered by the authors (table 1): the GEW model, the Prolag model and the Prgew model. As seen in more detail elsewhere, both the gestation lag (in the peculiar formulation utilised) and the efficiency wage mechanism tend to depress the labour share compared to the basic Goodwin's model.

---

[5]Using the same values of Fanti and Manfredi 1995 (α=n=0.03, ρ=2,γ=1, m=0.5 or m=0.33), we get:

$$a)\, m=0.5:\; S_0 = 1-\frac{\alpha}{\rho-\gamma} = 0.97 \;;\; S_2 = 1-\frac{\alpha(1+m)}{m(1-n)-n} \cong 0.90$$

$$b)\, m=0.33:\; S_0 = 0.97;\; S_2 \cong 0.97$$

Tab. 1: *"Evolution" of the E_1 equilibrium level of the wage share V, of the corresponding rate of growth, and of the employment level of the economy across the three variants of the basic Goodwin model: the Prolag model, the Gew model, the Prgew model. Corresponding values in Goodwin's model are added for reference*

	Lvg (Goodwin)	Prolag	Gew	Prgew
Wage share	$1-\frac{g}{m}$	$1-\frac{g}{m(1-g)}$	$1-\frac{G}{m}$	$1-\frac{G}{m(1-G)}$
Rate of growth of the output	g=α+n	g=α+n	$G=\frac{\alpha}{1-b}+n$	$G=\frac{\alpha}{1-b}+n$
Employment level	$\frac{\alpha+\gamma}{\rho}$	$\frac{\alpha+\gamma}{\rho}$	$\frac{\gamma}{\rho}+\frac{\alpha}{\rho(1-b)}$	$\frac{\gamma}{\rho}+\frac{\alpha}{\rho(1-b)}$

It is so easy to check that the equilibrium wage share of the PRGEW model is smaller than the corresponding levels of all the other models considered, since it cumulates two effects of the same sign (both negative): one due to the gestation lag and the other one due to efficiency wage hypothesis. The loss suffered by wage share, which amounts to:

$$\frac{1}{m}\left[\frac{G}{m(1-G)}-g\right] \qquad (3.9)$$

is then decomposable through the two joint effects, with an interesting economic interpretation. On the one side the gestation lag weakens the wages in the long term through a subtle (and somewhat sneaky) mechanism by which the employment is weakened in the short run (but not in the long), by preventing it from being able to efficiently support the working class in its fight in defense of the wage. On the other side the efficiency wage hypothesis induces a mechanism of automatic protection of the productivity and then of the profits, during the phases of high wage growth, which has various and articulated effects (Manfredi and Fanti 1995)

a) the damping to the variations of the wage share brings about a stabilizing effect on the economy;

b) the reduction of the predatory effect of the wage share on the employment gives rise, in the long period, to a higher employment;

c) the gain in terms of rate of growth of the output of the economy due to the existence of a second source of growth of the productivity which is, namely, the growth of the real wage. This gain has a dynamical effect of long period consisting in a loss of wage earners in the distributive sphere.

As regards the local stability analysis of the positive (E_1) equilibrium of model (2.13), we have the following jacobian matrix:

$$J = \begin{bmatrix} \frac{\partial \dot{S}}{\partial S} = -a & \frac{\partial \dot{S}}{\partial V} = -a & 0 \\ 0 & \frac{\partial \dot{V}}{\partial V} = f(U) & \frac{\partial \dot{V}}{\partial U} = V \cdot f'(U) \\ \frac{\partial \dot{U}}{\partial S} = U \cdot \frac{\partial g(U,S)}{\partial S} & 0 & \frac{\partial \dot{U}}{\partial U} = g(U,S) + U \cdot \frac{\partial g(U,S)}{\partial U} \end{bmatrix}_{E_1} = \begin{bmatrix} -a & -a & 0 \\ 0 & 0 & \rho(1-b)V^* \\ m(1-G)^2 U^* & 0 & -b\rho U^* \end{bmatrix}$$

The corresponding characteristic equation:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \qquad (3.10)$$

has the coefficients:

$$a_1 = a + b\rho U^* \quad a_2 = ab\rho U^* \quad a_3 = am\rho(1-b)(1-G)^2 U^* V^* \qquad (3.10')$$

which are all strictly positive. Hence Descartes rules of coefficients implies that there can be no positive real roots: the real roots of (3.10), will always be negative, a fact which would automatically imply local stability if all the three roots of (3.10) were real.

By resorting to the well-known Routh-Hurwicz criterion, which gives a necessary and sufficient condition in order that all the eigenvalues have negative real parts (i.e. in order that the E_1 equilibrium be locally stable), we have the condition:

$$a_1a_2 - a_3 = (a + b\rho U^*)ab\rho U^* - am\rho(1-b)(1-G)^2 U^* V^* > 0 \qquad (3.11)$$

from which it follows the inequality:

$$(a + b\rho U^*)b - (1-b)(1-G)[m(1-G) - G] > 0 \qquad (3.12)$$

It is natural to investigate the stability boundaries provided by the Routh-Hurwicz criterion by centering the discussion around the two main parameters introduced by the Prgew formulation: the *a* parameter which tunes the shape of the delaying kernel, and the *b* parameter which synthesizes the intensity of the efficiency wage mechanism operating in the economy. Coherently with this program let us rewrite (3.12) as:

$$K(a,b) = (a + b\rho U^*)b - H(b) > 0 \qquad (3.12')$$

where:

$$H(b) = (1-b)[1-G(b)][m(1-G(b)) - G(b)] \qquad (3.13)$$

where G has been rewritten as an explicit function of b: G=G(b). It is easy to check that, over the set of the feasible values of b ($0<b<b_0$), the function H(b) is a positive decreasing function of *b*, being defined as a product of positive and decreasing functions of the same argument. Hence, the function K(a,b) is an increasing function of both its "basic" arguments, *a* and *b*. It follows that, roughly, to guarantee that the positive equilibrium E_1 of the Prgew system be locally asymptotically stable, we have to ensure that i) the *a* parameter is relatively large (or correspondingly the average delay 1/a is little) and/or ii) the *b* parameter, measuring the relative elasticity of the productivity to wage, is high. Both these results were expected on the basis of our previous knowledge on the behaviour of the economic forces acting on the two systems subsumed in the Prgew model.

4. Appearance of persistent periodic behaviours

The story told until now implies that by weakening the conditions favourable to the stability, that is by decreasing either the values of the parameters a,b, a local lost of stability may take place. If this local lost of stability should take place through a motion of complex eigenvalues of (3.10) crossing the imaginary axis, then periodic behaviours would appear through the mechanism of Hopf's bifurcation.

Since this is indeed the case, we try now to precisely state the conditions under which Hopf's bifurcation occurs. It is convenient to rewrite the Routh-Hurwicz inequality (3.11) as:

$$a > \frac{H(b)}{b} - b\rho U^* = \Phi(b) \qquad (4.1)$$

The condition (4.1) has to be considered only over the set $(0,b_0)$ of the feasible values of b. We can notice that:

a)if the second member of (4.1) is negative then (4.1) itself always holds, in virtue of the nature of the parameter *a*, which is always positive by definition. Hence no loss of stability is possible;

b) if the second member of (4.1) is positive then (4.1) becomes non trivial and it will be satisfied for values of the *a* parameter which lie, in the (a,b) plane, above the curve $\Phi(b)$ defined by the equation:

$$a = \Phi(b) \qquad (4.2)$$

Hence every time the curve $\Phi(b)$ is crossed through a movement from above to below, the local stability of the E_1 equilibrium is lost, and unstable behaviours occur. It is easy to check that this depends on the crossing of the imaginary axis by the two complex eigenvalues of the characteristic equation (3.10), so giving rise to a Hopf bifurcation. In fact, since all the coefficients of (3.10) are kept positive, then the Routh-Hurwicz determinant which assesses stability:

$$\Delta_2 = a_1 a_2 - a_3$$

can change its sign only as a consequence of a change in sign in the real parts of the complex eigenvalues of the (3.10) itself. This is easily seen thanks to Orlando's formula (Lorenz 1993):

$$a_1 a_2 - a_3 = 0 \quad \Rightarrow \quad (\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3) = 0 \qquad (4.3)$$

where λ_i are the eigenvalues of (3.10). In fact (4.2) holds only in the following events: i)(3.10) has a zero eigenvalue with algebraic multeplicity two; ii)(3.10) has two real (and hence in fact three) eigenvalues, having the same absolute value but opposite sign, iii)(3.10) has a couple of imaginary roots. Since the first two cases are excluded by assumption (i)because $a_3>0$, ii)because the coefficients of the characteristic polynomial are all strictly positive) then the only possible case is the third one.

This result is also practical since, once ascertained that this is the only possible case, it can be used to detect the involved couple of pure imaginary eigenvalues $\pm\omega i$, by simply solving the characteristic equation under the condition:

$$\Delta_2 = a_1 a_2 - a_3 = 0 \qquad (4.4)$$

This gives the imaginary coefficient ω as the solution of the equation $\omega^2 = a_2$, a result useful to estimate the amplitude of the period of the involved periodic behaviours which appear as a consequence of the Hopf's bifurcation.

To complete the proof of the existence of a Hopf's bifurcation, a further condition is needed, which amounts to show that the bifurcating couple of complex eigenvalues cross the imaginary axis with nonzero speed. This is easily proven and, just for ease of exposition, we postpone the detailed proof to the appendix.

The previous considerations thereby show that the portion of the Φ(b) at which the *a* parameter takes strictly positive values curve constitutes the (Hopf's) bifurcation curve of the E_1 equilibrium of the Prgew system. We can easily verify that:

a) $\lim_{b \to 0} \Phi(b) = +\infty$ b) $\Phi(b_0) = \frac{H(b_0)}{b_0} - \rho U^* b_0 = -\rho U^* b_0 < 0$

c) $\Phi'(b) = \frac{H'(b)b - H(b)}{b^2} - \rho U^* < 0 \quad \text{as: } H < 0, H' < 0$

The shape of the curve Φ(b) is then of the type depicted in fig.1. Let us denote in particular with b_1 the value in correspondence of which the curve Φ(b) crosses the b axis.

Fig. 1. The form of the Hopf's bifurcation curve a=Φ(b) in the (a,b) domain

The Hopf's bifurcation problem just examined exhibits a bifurcation curve, the one described through (4.1) and (4.2) and depicted in fig. 1, which seems typical of a larger class of problems (see Farkàs and Kotsis, 1992; and Fanti and Manfredi, 1996b). The main result, as regards the two main parameters of this work, that is the lag parameter *a* (reciprocal of the average delay *T*), embedded in the delaying kernel, and the efficiency wage parameter *b*, is syntethized in the following:

Proposition: within the feasible set of the efficiency wage parameter ($b<b_0$), the positive equilibrium (E_1) of the Prgew system (2.15) is always locally asymptotically stable (LAS) for $b_1<b<b_0$. For $0<b<b_1$ E_1 is LAS for values of *a* which lie above the line of equation a=Φ(b). When, starting from a combination of parameter values in correspondence of which E_1 is LAS, the *a* parameter is decreased so that the line a=Φ

(b) is crossed, then the Prgew system undergoes a Hopf bifurcation, with the appearance of local limit cycle surrounding the E_1 equilibrium[6].

The authors were not able to provide a complete investigation of the stability properties of the limit cycles mentioned in the previous proposition. Nevertheless, following the fundamentals contributions by Farkas and coworkers (1984, 1992, 1995) who were first, to our knowledge, in providing a full central manifold analysis of the stability of Hopf's bifurcations in some basic Lotka-Volterra-Goodwin models with time delays, it seemed natural to us to formulate, concerning the location of the so called supercritical and subcritical regions, i.e. the regions of local stability/instability of the involved limit cycles (see also Fanti and Manfredi (1996b)), the following conjecture.

The sense of our conjecture (which is represented in fig.2), which is motivated by the local qualitative similarities of the system (2.15) with other models having known properties, such as those studied in Farkas (1984), Farkas and Kotsis (1992), Fanti and Manfredi (1997b), is that there should exist a critical threshold value of b, indicated with b_d, $0<b_d<b_1$, which splits the region $(0,b_1)$ in the two regions of "supercriticity" and "subcriticity" of the periodical orbits. More precisely: if the crossing of the curve $a=\Phi(b)$ takes place in the region $0<b<b_d$, then the corresponding periodic orbits are locally asynptotically stable, and conversely if the crossing takes place in the region $b_d<b<b_1$, where the corresponding periodic orbits are locally unstable (fig.2).

Fig. 2 Boundaries of stability of the positive equilibrium of the Prgew system and conjectured location of the regions of supercriticity and subcriticity

[6] The parameter combination chosen here is not the only possible way to express Hopf's theorem. For instance it is completely equivalent to base the discussion on the reciprocal of *a*, $T=1/a$, rather than *a*. This would have the advantage of working directly with the average delay as a bifurcation parameter, but in practical terms is, we repeat it, completely equivalent.

The exactness of our conjecture is confirmed by numerical experiments. In particular the amplitude of the subcritical region appears to be quite small. The fig. 3,4,5,6,7 report of numerical simulations of the Prgew system for a standard configuration of the set of economic parameters involved (γ=1, ρ=2, α=0.03,n=0.03, m=0.33) already used in past works of the authors (Fanti and Manfredi 1995, 1997a, 1997b), by varying the two crucial parameters a and b.

In particular, as it was expected, the Prgew system inherits, when the role of the delay is not too important, most of the main features of the Gew system: i)an asymptotically stable equilibrium with zero labour share (fig. 3) when the efficiency wage effect is very strong (i.e: b very large: close to one), ii)a monotonic convergence to the positive equilibrium (fig. 4) in an intermediate window of the b parameter, iii)convergence to the positive equilibrium (fig. 5) through damped oscillations when b is furtherly decreased. Moreover, when the role of the delay is increased through decreasing a, the more interesting dynamical effects discussed in this paper arise. Fig. 6 reports the shape of one of the limit cycle appearing through Hopf's bifurcation in the Prgew system. Finally fig. 7 shows to what "speed" the ray of such limit cycles expands as we consider parameter values which are more far from the bifurcation value. It is to be noticed that numerical simulation seem to suggest that when local stability (point or orbital) exists, this seems to be also global.

Fig.3 Prgew system: convergence to the equilibrium with zero labour share (b=.94, a=0.68)

Fig.4 Prgew system: monotonic convergence to the positive equilibrium (b=.55, a=0.68)

Fig.5 Prgew system: oscillatory convergence to the positive equilibrium (b=.30, a=0.68)

Fig.6 Prgew system: a stable limit cycle (b=0.2, a=0.68)

Fig. 7 Prgew system: expansion of the ray of the stable limit cycle (b=0.19, a=0.68) corresponding to a 0.01 increase in the distance between the b parameter and its bifurcation value (coeteris paribus)

The economic interpretation of the main results of this work is particularly appealing: it will exist a value b_1, beyond which the elasticity productivity-wage is so high to prevent in any case the appearance of the destabilizing effect due to the gestation lag. On the contrary, if b is not too high ($0<b<b_1$) then the stabilizing effect due to the efficiency wage is less important and tends to balance with the destabilizing effect due to the gestation lag. This balancement between opposite forces permits to the system to undergo transitions from stability to instability trough the mechanism of the Hopf bifurcation. In particular the region where unstable behaviours appear corresponds to the region distinguished generically by high average lags which is located below the bifurcation curve $\Phi(b)$. It is than useful to further distinguish:

b1) the "supercriticity" (R_{Super}) region ($0<b<b_d$), characterized by a very low elasticity productivity - wage, in which the attraction by the equilibrium is minimal: this allows for the stability of the periodic orbits which "emerge" from it.

b2) the "subcriticity" region (R_{Sub}), characterized by intermediate values of b ($b_d<b<b_1$), in which the periodical orbits are unstable due to the action of intermediate values of *b* with very high values of *a*.

In other words: the main aspect of the dynamics is obviously the trade-off between the two parameters, *a* and *b*, which have opposite effects on the stability features of the E_1 equilibrium. Thanks to the thresholds b_d, b_1, b_0, is, in particular, possible to partition the feasible region R_A of the *b* parameter with explicit reference to the behaviour of emerging periodical orbits, in the following sequence of dynamic windows: a) R_{super} where the periodic orbits (as they exist) are locally asintotically stable; b) R_{sub}, where the periodical orbits (as they exist) are unstable; c) R_S: where periodical orbits can not exist.

It is enlightening to contrast the results offered by the Prgew model with its non-delayed counterpart given by the Gew model, for what concerns in particular their effects on the basic Goodwin's model. The Gew hypothesis mainly introduces a stabilizing effects on Goodwin's conservative oscillations, due to the introduction of a density-dependent term in the Lotka-Volterra-Goodwin scheme.

Viceversa in the Prgew model, which, due to the introduction of the gestation lag, has a greater dimension, the stabilizing effect of efficiency wage mechanism "competes" with the destabilizing effect of the delay. This "fight" can be responsible of several important consequences which are not really self-evident. First of all, if the efficiency wage effect is very strong it can completely sterilize the destabilizing effect of the gestation lag, so generating a "simple" equilibrium dynamics. If on the contrary such effect is less intense, can give rise to a balancement with the destabilizing force of the delay, which can be responsible for the appearance of periodic behaviours. These oscillations will be persistent, i.e. stable within a special supercritical window. The dynamical interpretation is that to have stable limit cycles we need a "special" combination of the stabilizing effect due to the Gew hypothesis with the destabilizing effect played by the gestation lag: the former one has to be weak enough so to prevent the E_1 equilibrium to be "attractive" when excited by the destabilizing action of the delay, but also strong enough to a)prevent that the destabilizing action of the delay generates explosive oscillations and b)to generate, at the same time, an attraction effect towards an "attractor" different from the equilibrium itself: the limit cycle

5. Conclusions

This paper shows how persistent oscillations can arise in economies experiencing delays effects on the supply side, such as gestation lags in investment plans, when such effects are combined with efficiency wage behaviours. The nonlinear interaction between these two economic forces enriches the spectrum of Goodwin's possible dynamics and seems to have interesting consequences in terms of policies (Fanti and Manfredi 1996b).

It is, to our concern, important to stress the roles of a mechanism such as that of the efficiency wage, which appeared, when considered alone, purely stabilizing (or even sterilizing) with respect to Goodwin conservative fluctuations (Fanti and Manfredi 1995), but can viceversa be responsible of much reacher dynamical effects when it is counterbalanced by the existence, within the economy, of forces of oppositesign, of which the gestation lags constitute a traditional, and well known, instance.

References

Fanti L., Manfredi P. (1995), Efficiency Wage and Cyclical Growth (in italian), *Studi Economici*, 3, 79-114

Fanti L., Manfredi P. (1996b), Point stability versus orbital stability (or instability): remarks on policy implications in classical growth cycle models, submitted for publication to *Structural Change and Economic Dynamics*

Fanti L., Manfredi P. (1997a), Crescita con ciclo, ritardi nei piani di investimento ed effetti di popolazione (in italian), accepted for publication by *Rivista Internazionale di Scienze Sociali*

Fanti L., Manfredi P. (1997b), Vischiosità dei salari e dinamica occupazionale in modelli di crescita alla Goodwin (in italian), accepted for publication in *Economia Politica*

Fanti L., Manfredi P. (1997c), Efficiency wages and lagged Phillips mechanisms in Goodwin-type growth models, preprint

Farkas M. (1984), Stable Oscillations in a Predator-Prey Model with Time Lag, *Journal of Math. Analysis and App.*, 10, 175-188

Farkas M., Kotsis M. (1992), Modelling Prey-Predator and Wage-Employment Dynamics, in Feichtinger G. (Ed.), Dynamic Economic Models and Optimal Control, Elsevier, 513-526

Farkas M. (1995), Periodic Motions, Springer Verlag, New York Tokio

Gandolfo G., Metodi di Dinamica Economica, Isedi, Milano, 1973.

Goodwin R.M.,"A Growth Cycle", in Feinstein C.H. ed., Socialism, Capitalism and Economic Growth, Cambridge University Press, London, 1967.

Guckenheimer J., Holmes P. (1984), Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields, Springer Verlag, Newyork, Tokio, Berlin

Kalecki, Sulla dinamica dell'economia capitalistica, Torino, UTET, 1975

Kydland F., Prescott E. (1982), Time to build an aggegate fluctuation, *Econometrica*, November 1982

Lorenz H.W., Nonlinear dynamical economics and chaotic motion, Springer-Verlag, New York, Tokio, 1993

Manfredi and Fanti (1996a), Long-term effects of the efficiency wage hypothesis in Goodwin's type economies, submitted to *Journal of Economics*

Manfredi P., Fanti L.(1996b), Population Dynamics and Goodwin's Model, Preprint,

Marsden J., McCracken M. (1976), The Hopf bifurcation and its applications, Springer Verlag, New-York Tokio Berlin

McDonald N. (1978), Time Lags in Biological Systems, Lecture Notes Biomath. 29, Springer Verlag, New York, Tokio, Berlin

May R.M. (1973), Stability and Complexity in Model Ecosystems, Princeton University Press, Princeton

Appendix

As it is well known the Hopf's bifurcation theorem holds if, in correspondence of the bifurcating value of the parameter of interest, the following nonzero speed condition on the eigenvalues crossing the imaginary axis holds (Guckenheimer and Holmes 1984, Farkas 1995):

$$\left[\frac{d\,\mathrm{Re}(a)}{da}\right]_{a=a_0} \neq 0 \qquad \text{(A.1)}$$

where Re(a) denotes the real part of the involved eigenvalues, explicitly written as a function of the bifurcation parameter. As it is known (see for instance Marsden and McCracken 1976) given the characteristic polynomial:

$$P(\lambda) = \lambda^3 + A(a)\lambda^2 + B(a)\lambda + C(a) \qquad \text{(A.2)}$$

(the coefficients of which are to be regarded as explicitly depending on the chosen bifurcation parameter), the derivative of the real part of the bifurcating eigenvalues with respect to the bifurcation parameter is given by:

$$\left[\frac{d\,\mathrm{Re}(a)}{da}\right] = (-1)\frac{B'(a)A(a) + B(a)A'(a) - C'(a)}{2\left[A^2(a) + B(a)\right]} = (-1)\frac{\Theta(a)}{\Psi(a)} \qquad \text{(A.3)}$$

In the case of the Prgew model (see the characteristic equation (3.10)) it holds:

$$A(a) = a + b\rho U^*; \quad A'(a) = \frac{dA(a)}{da} = 1$$

$$B(a) = ab\rho U^* \quad ; \quad B'(a) = \frac{dB(a)}{da} = b\rho U^* \quad ;$$

$$C(a) = am\rho(1-b)(1-G)^2 U^* V^* \quad ; \quad C'(a) = \frac{dC(a)}{da} = m\rho(1-b)(1-G)^2 U^* V^* \qquad \text{(A.4)}$$

Since B(a)>0 is always positive, the sign of the derivative (3) only depends on the sign of the Θ(a) quantity appearing as numerator of (4). We have:

$$\Theta(a) = b\rho U^* \cdot (a + b\rho U^*) + ab\rho U^* - m\rho(1-b)(1-G)^2 U^* V^* \qquad \text{(A.5)}$$

and hence:

$$\begin{aligned} \Theta(a) &= \rho U^* \left\{ b \cdot (a + b\rho U^*) + ab - m(1-b)(1-G)^2 V^* \right\} = \\ &= \rho U^* \left\{ \left[(a + b\rho U^*) b - m(1-b)(1-G)^2 V^* \right] + ab \right\} = \\ &= \rho U^* \left\{ b\left[(a - \Phi(b)) \right] + ab \right\} \end{aligned} \qquad \text{(A.6)}$$

Inspection of (6) reveals that, when the chosen bifurcation parameter *a* takes its bifurcating value $a=a_0=\Phi(b)$, it will hold:

$$\Theta(a_0) = \rho U^* \left\{ b\left[(a_0 - \Phi(b)) \right] + a_0 b \right\} = \rho U^* a_0 b > 0 \qquad \text{(A.7)}$$

a quantity which is always positive. This definitively proves that:

$$\left[\frac{d\,\mathrm{Re}(a)}{da} \right]_{a=a_0} < 0$$

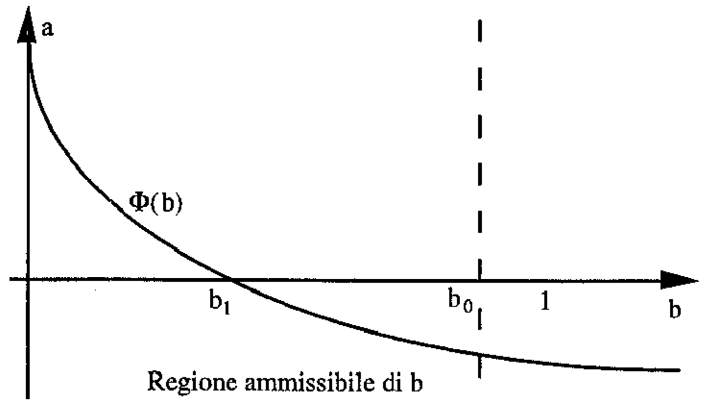

Fig. 1. The form of the Hopf's bifurcation curve a=Φ(b) in the (a,b) domain

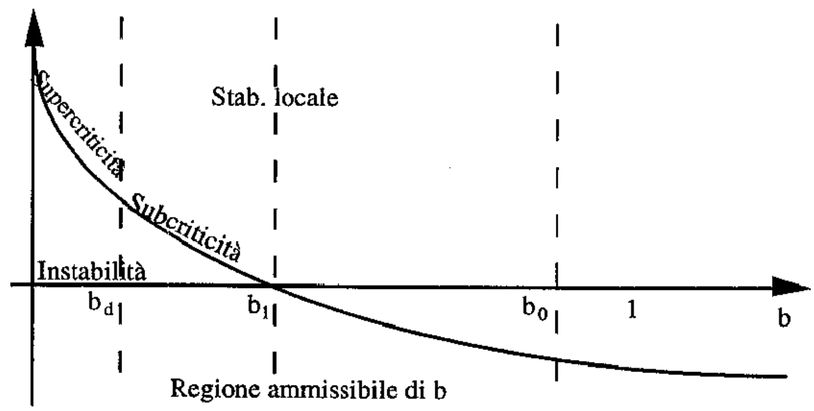

Fig. 2 Boundaries of stability of the positive equilibrium of the Prgew system and conjectured location of the regions of supercriticity and subcriticity

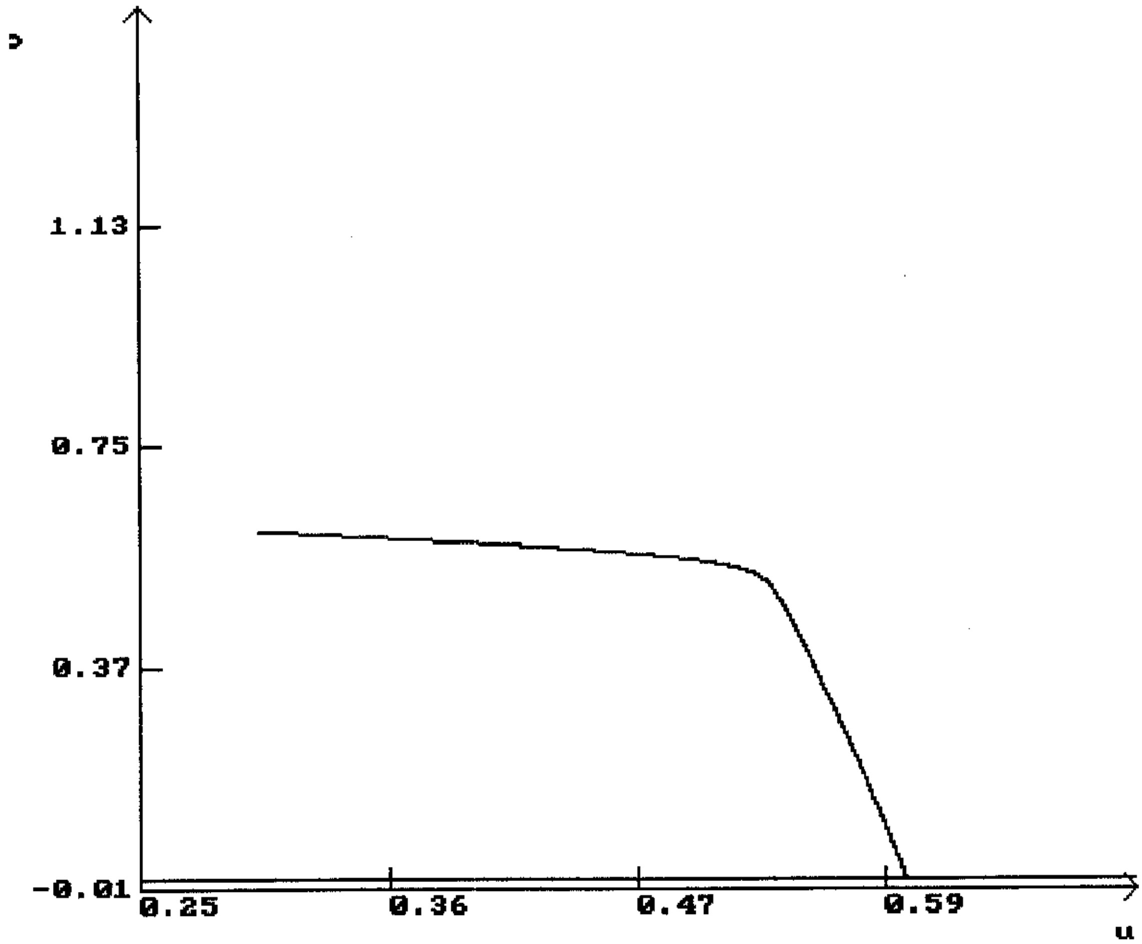

Fig. 3

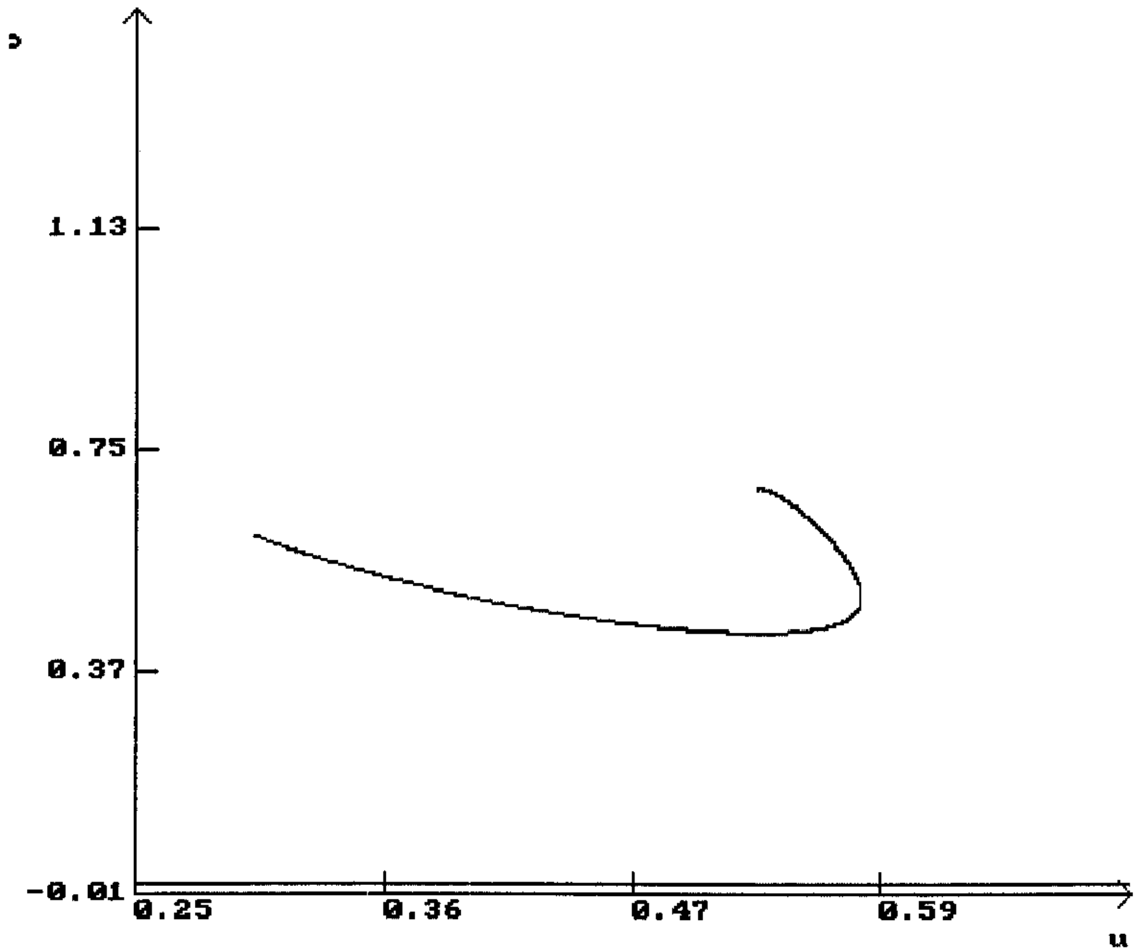

Fig. 4

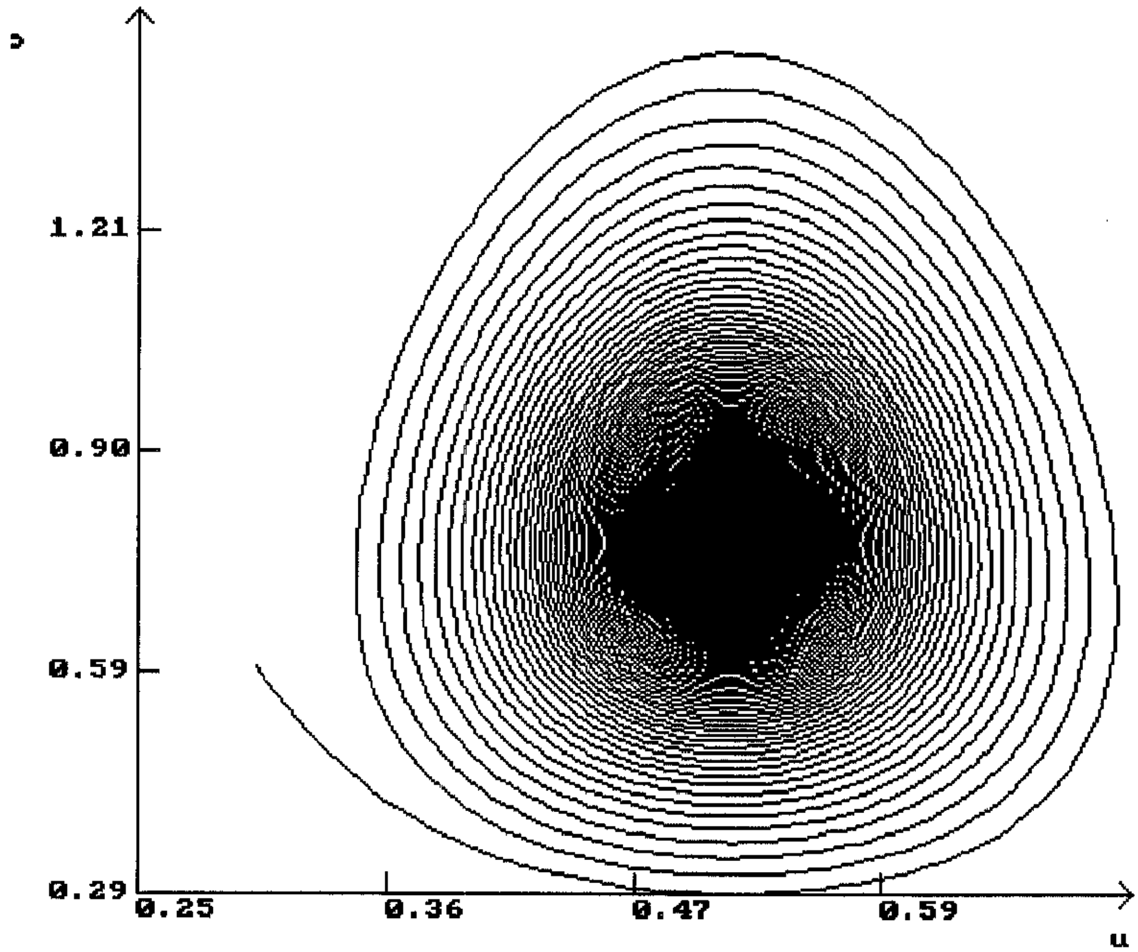

Fig. 5

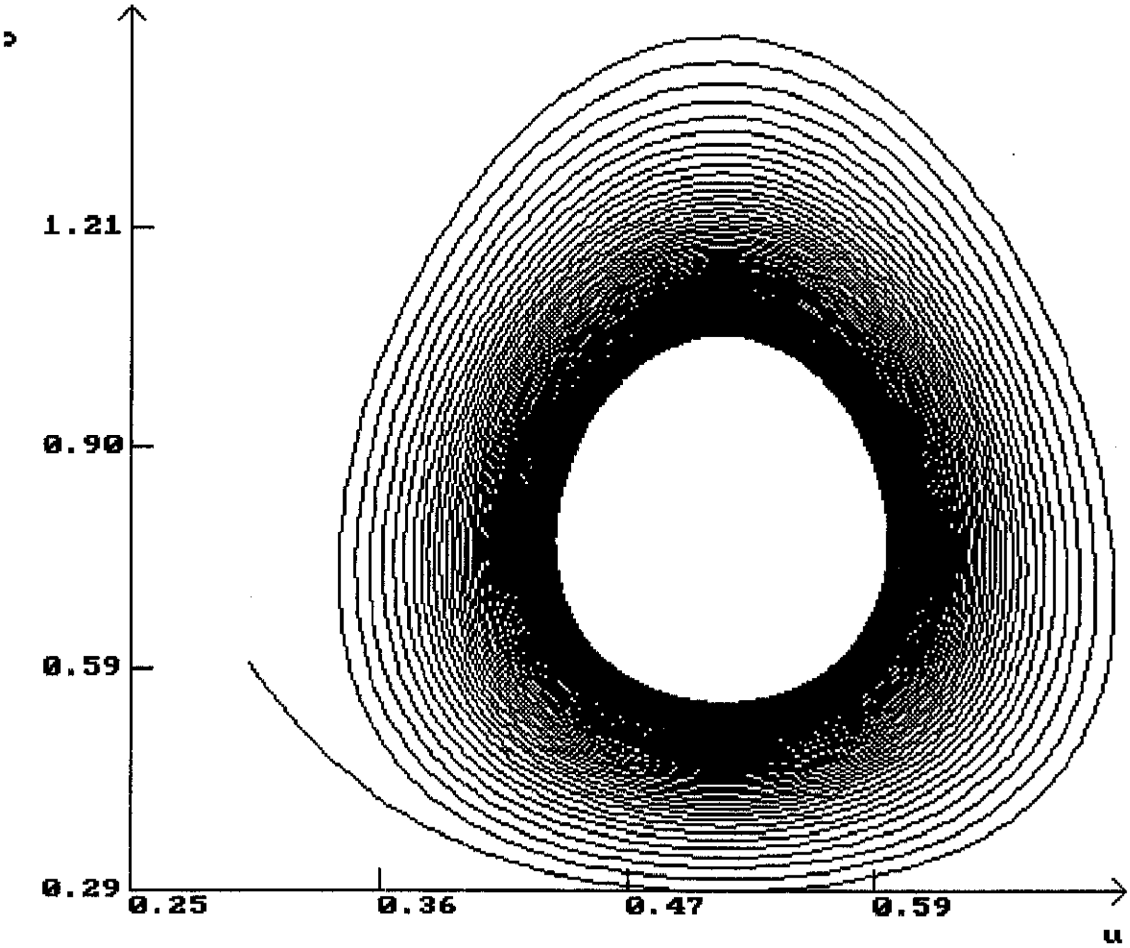

Fig. 6

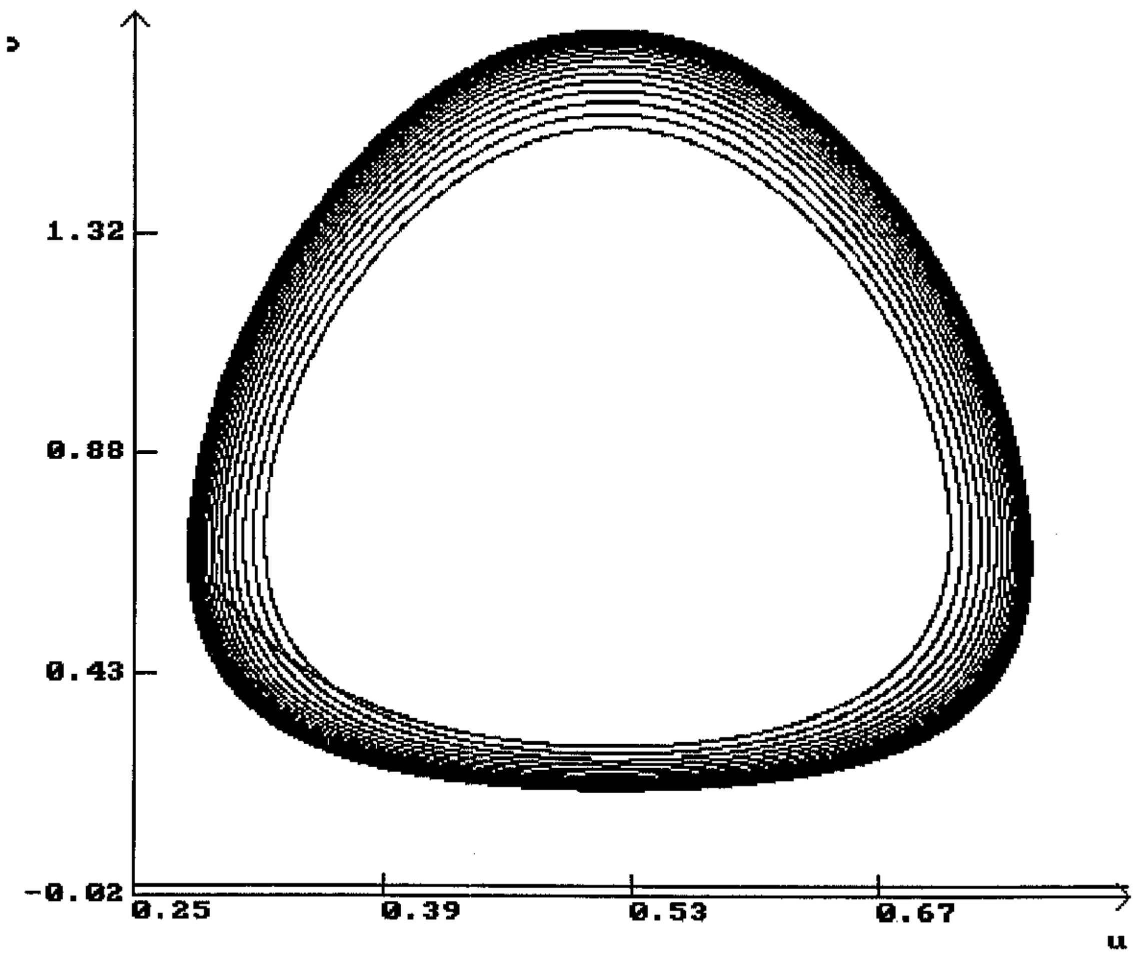

Fig. 7