Report n. 126

Second Order Optimality Conditions

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Pisa, october 1997
Questa ricerca è stata finanziata dal Ministero dell'Università e della Ricerca scientifica (Ex fondi 40%).

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Abstract. The aim of the paper is to establish some new second order optimality conditions by means of suitable second order tangent sets.
Key words. Optimality conditions, second order conditions.

1 Introduction

In these last years a new look inside second order optimality conditions has been given by some authors; such conditions have been established without any requirement of constraint qualification unlike the classic second order optimality conditions [see 1-10].
Recently in [6] the authors have extended necessary second order optimality conditions obtained by Penot [9] in such a way that sufficient conditions can be obtained from necessary ones by replacing non strict inequalities by strict inequalities.
The main purpose of this paper is to refine the results given in [6] in order to obtain more general optimality conditions.

2. Statement of the problem

Consider the problem

$$P: \max f(x) : x \in S$$

where S is a subset of a finite dimensional real normed space X and f is twice differentiable real valued function defined on an open set including S.
Troughout the paper, the gradient and the Hessian of f at $x_0 \in S$ are denoted by $f'(x_0)$ and $f''(x_0)$ respectively; the gradients are considered to be row vectors, while vectors in X are considered to be column vectors and the transpose of a vector v is denoted by v'. For a vector d in X, the set of all non zero vectors in X that are orthogonal to d is denoted by $d^{\perp}$ and the set of all vectors which are orthogonal to $f'(x_0)$ is denoted by $\ker f'(x_0)$.
For a better understanding of the optimality conditions that we will establish in section 3, we recall some recent results .

As is well-known, by means of the Bouligand tangent cone to S at x_0, defined as $S'(x_0) = \{ d : \exists \{x_n\} \subset S, x_n \to x_0, \exists \alpha_n \to +\infty, \text{ with } \alpha_n (x_n - x_0) \to d \}$, it is possible to establish the following first order necessary and sufficient optimality conditions:

- if x_0 is a local maximizer of f over S, then $f'(x_0)d \le 0$ whenever $d \in S'(x_0)$;
- if $f'(x_0)d < 0$ whenever $0 \ne d \in S'(x_0)$, then x_0 is a local maximizer of f over S,

When x_0 is a critical point, we have the following second order optimality conditions:

a) if x_0 is a local maximizer of f over S, then $f'(x_0)d \le 0$ whenever $d \in S'(x_0)$ and if $d \in S'(x_0) \cap \ker f'(x_0)$, then $d' f''(x_0) d \le 0$;

b) if $f'(x_0)d \le 0$ whenever $d \in S'(x_0)$ and $d' f''(x_0) d < 0$ whenever $0 \ne d \in S'(x_0) \cap \ker f'(x_0)$, then x_0 is a local maximizer of f over S.

From now on, we assume that $f'(x_0) \ne 0$.

Unfortunately, a) and b) do not hold in general, as is shown in the following example.

Example 1

Consider problem P where $S=\{(x,x^2) : x \ge 0\}$, $f(x,y) = - x^2 + 2y$ and the point $x_0 = (0,0)$. It is easy to verify that $S'(x_0) = \{d=(d_1,0), d_1 \ge 0\}$, $f'(x_0)d = 0$ for all $d \in S'(x_0)$, $d' f''(x_0) d = - 2 d_1^2 < 0$ for all $0 \ne d \in S'(x_0)$, so that b) holds but x_0 is not a local maximizer of f over S. Furthermore, if $f(x,y) = x^2 - 2y$, x_0 is a local maximizer of f over S, but a) does not hold since $d' f''(x_0) d = 2 d_1^2 > 0$ when $d_1 \ne 0$.

Example 1 points out that $S'(x_0)$ is not an appropriate set in establishing second order optimality conditions, so that some authors [6-10] have introduced new sets. Penot in [9] gives the following definition:

Definition 1

The second order tangent set to S at x_0 in the direction d is the set defined as
$$S''(x_0,d) = \{w: \exists w_n \to w, \exists t_n \to 0^+, \text{ such that } x_n = x_0+t_n d+ \tfrac{1}{2} t_n^2 w_n \in S\}.$$

By means of $S''(x_0,d)$, the following second order necessary optimality condition is established [9] :

If x_0 is a local maximizer of f over S, then the following conditions hold:

i) $f'(x_0)d \le 0$ whenever $d \in S'(x_0)$.

ii) if $d \in S'(x_0) \cap \ker f'(x_0)$, then $f'(x_0) w + d' f''(x_0) d \le 0$ for all $w \in S''(x_0,d)$.

Replacing the inequality in ii) by strict inequality, a sufficient optimality condition can not be obtained (see [6]) ; for such a reason in [6] the following second order tangent set is defined:

Definition 2

Let k be a nonnegative real number and d a vector in X. The second order tangent set to S at x_0 in the direction d is the set defined as

$$T''_k(S,x_0,d) = \{w: \exists \{x_n\} \subset S,\ x_n \to x_0,\ \exists\ \alpha_n \to +\infty,\ \exists\ \beta_n \to +\infty, \text{ with } \frac{\beta_n}{\alpha_n} \to k,\ \alpha_n (x_n-x_0) \to d \text{ and } \beta_n [\alpha_n(x_n-x_0)-d] \to w\}.$$

Remark 1

Let us note that when k = 2, the set $T''_2(S,x_0,d)$ coincides with $S''(x_0,d)$

In [6] the following second order optimality conditions are established.

Theorem 1

If x_0 is a local maximizer of f over S, then the following conditions hold:

i) $f'(x_0)d \le 0$ whenever $d \in S'(x_0)$.

ii) If $0 \neq d \in S'(x_0) \cap \ker f'(x_0)$, then

(a) $f'(x_0)\, w + d'\, f''(x_0)\, d \le 0$ whenever $w \in T''_2(S,x_0,d) \cap d^{\perp}$

(b) $d'\, f''(x_0)\, d \le 0$ whenever $0 \in T''_2(S,x_0,d)$

(c) $f'(x_0)\, w \le 0$ whenever $w \in T''_0(S,x_0,d) \cap d^{\perp}$.

Theorem 2

Suppose that a point x_0 satisfies the following conditions :

i) $f'(x_0)d \le 0$ whenever $d \in S'(x_0)$.

ii) If $d \in S'(x_0) \cap \ker f'(x_0)$ with $\| d \| = 1$, then

(a) $f'(x_0)\, w + d'\, f''(x_0)\, d < 0$ whenever $w \in T''_2(S,x_0,d) \cap d^{\perp}$

(b) $d'\, f''(x_0)\, d < 0$ whenever $0 \in T''_2(S,x_0,d)$

(c) $f'(x_0)\, w < 0$ whenever $w \in T''_0(S,x_0,d) \cap d^{\perp}$.

Then x_0 is a local maximizer of f over S.

Let us note that (a), (b) of ii) in Theorem 2 do not ensure the optimality of the point x_0 as it is pointed out in Example 2 which stresses also the rilevance of condition (c).

Example 2

Consider problem P where $f(x,y) = - y$, $S = \{(x,y) : y = x\sqrt{x}\ , x \geq 0\}$ and the point $x_0= (0,0)$. It is easy to verify that $S'(x_0) = \{t\,(1,0), t \geq 0\}$. Let $d=(1,0)$; we have $f'(x_0)d= 0$, $T_2''(S,x_0,d)=\emptyset$, $T_0''(S,x_0,d) \cap d^{\perp} = \{w=(0,w_2), w_2>0\}$.

Since $f'(x_0)w = - w_2 < 0$ whenever $w \in T_0''(S, x_0, d) \cap d^{\perp}$, condition (c) is satisfied and x_0 is a local maximizer of f over S.

Let us note that if for a vector z in X we denote by $\hat{z}$ the projection of z on the hyperplane orthogonal to $f'(x_0)$ and by z^* the projection of z on the line $[f'(x_0)]$ generated by $f'(x_0)$, we have $z = \hat{z} + z^*$ and $f'(x_0)z = f'(x_0)z^*$. As a consequence, denoting by $P_2''(S,x_0,d)$ and $P_0''(S,x_0,d)$ the projections of $T_2''(S,x_0,d)$, and $T_0''(S,x_0,d)$ on the line $[f'(x_0)]$, respectively, theorems 1 and 2 can be reformulated as follows:

Theorem 1*

If x_0 is a local maximizer of f over S, then the following conditions hold:

i) $f'(x_0)d \leq 0$ whenever $d\in S'(x_0)$.

ii) If $0 \neq d \in S'(x_0)\cap \ker f'(x_0)$, then

(a) $f'(x_0)\, w + d'\, f''(x_0)\, d \leq 0$ whenever $w \in P_2''(S,x_0,d)$

(b) $f'(x_0)\, w \leq 0$ whenever $w \in P_0''(S,x_0,d)$.

Theorem 2*

Suppose that a point x_0 satisfies the following conditions :

i) $f'(x_0)d \leq 0$ whenever $d\in S'(x_0)$.

ii) If $d \in S'(x_0)\cap \ker f'(x_0)$ with $\| d \| = 1$, then

(a) $f'(x_0)\, w + d'\, f''(x_0)\, d < 0$ whenever $w \in P_2''(S,x_0,d)$

(b) $f'(x_0)\, w < 0$ whenever $w \in P_0''(S,x_0,d)$.

Then x_0 is a local maximizer of f over S.

3 New second order optimality conditions

From now on and without loss of generality, we will assume that any bounded sequence is a convergent sequence (substituting the sequence with a suitable subsequence, if necessary).

In this section we will establish new second order optimality conditions by refining condition ii) of theorems 1* and 2*.

With this aim we introduce new second order tangent sets as follows:

$$T_2^*(S,x_0,d) = \{w : \exists \{x_n\} \subset S, x_n \to x_0, \exists \alpha_n \to +\infty \text{ with } \alpha_n (x_n-x_0) \to d, \; 2\alpha_n [\alpha_n (x_n - x_0) - d]^* \to w\}.$$

$$T_0^*(S,x_0,d) = \{v : \exists \{x_n\} \subset S, x_n \to x_0, \exists \alpha_n \to +\infty, \exists \beta_n \to +\infty \text{ with } \frac{\beta_n}{\alpha_n} \to 0 \text{ such that } \alpha_n (x_n-x_0) \to d \text{ and } \beta_n [\alpha_n(x_n-x_0) - d]^* \to v\}.$$

The following theorems hold.

Theorem 3

If x_0 is a local maximizer of f over S, then the following conditions hold:

i) $f'(x_0)d \leq 0$ whenever $d \in S'(x_0)$.

ii) If $0 \neq d \in S'(x_0) \cap \ker f'(x_0)$, then

(a) $f'(x_0) w + d' f''(x_0) d \leq 0$ whenever $w \in T_2^*(S,x_0,d)$

(b) $f'(x_0) w \leq 0$ whenever $w \in T_0^*(S,x_0,d)$.

<u>Proof</u>.

Condition i) is very known.

Let $d \in S'(x_0) \cap \ker f'(x_0)$. Then there are sequences $\{x_n\}$, $\{\alpha_n\}$ such that $x_n \in S$, $\alpha_n > 0$, $\alpha_n (x_n - x_0) \to d$.

Let v_n be defined by $v_n = \alpha_n (x_n - x_0) - d$ so that $x_n = x_0 + \frac{d}{\alpha_n} + \frac{v_n}{\alpha_n}$.

Applying a second order Taylor expansion using the mean value theorem, we have:

$$\alpha_n(f(x_n) - f(x_0)) = f'(x_0) (v_n)^* + \frac{1}{2\alpha_n}(d+v_n)' f''(x_0+\delta_n(x_n - x_0)) (d+v_n) \quad (1)$$

where $0 \leq \delta_n \leq 1$ for every n.

Consider the sequence $\{\alpha_n(v_n)^*\}$; the following exaustive cases occur:

1) $\{\alpha_n(v_n)^*\}$ is a convergent sequence, that is $2\alpha_n(v_n)^* \to w$

2) $\{\alpha_n(v_n)^*\}$ is a divergent sequence in norm, that is $\alpha_n \| (v_n)^* \| \to +\infty$

In the first case $w \in T_2^*(S, x_0, d)$ and from (1) we have

$$2\alpha_n^2 (f(x_n) - f(x_0)) = f'(x_0)[2\alpha_n(v_n)^*] + (d+v_n)' f''(x_0+\delta_n(x_n - x_0))(d+v_n)$$

so that $2\alpha_n^2 (f(x_n) - f(x_0)) \to f'(x_0)w + d' f''(x_0) d$. If x_0 is a local maximizer of f over S , then $2\alpha_n^2 (f(x_n) - f(x_0)) \le 0$ and therefore ii) (a) holds.

In the second case consider any sequence $\{\beta_n\}$ of positive number such that $\beta_n(v_n)^* \to w$ (for instance $\beta_n = \frac{1}{\| (v_n)^* \|}$). Since $\frac{\beta_n}{\alpha_n} = \frac{\beta_n \| (v_n)^* \|}{\alpha_n \| (v_n)^* \|}$ obviously it results $\frac{\beta_n}{\alpha_n} \to 0$ so that $w \in T_0^*(S, x_0, d)$. From (1) we have:

$$\beta_n\alpha_n(f(x_n) - f(x_0)) = f'(x_0)\beta_n(v_n)^* + \frac{\beta_n}{2\alpha_n}(d+v_n)' f''(x_0+\delta_n(x_n - x_0))(d+v_n)$$

so that $\beta_n\alpha_n(f(x_n) - f(x_0)) \to f'(x_0)w$. If x_0 is a local maximizer of f over S , then $\beta_n\alpha_n(f(x_n) - f(x_0)) \le 0$ and therefore ii) (b) holds. ♦

Theorem 4

Suppose that a point x_0 satisfies the following conditions :

i) $f'(x_0)d \le 0$ whenever $d \in S'(x_0)$.

ii) If $d \in S'(x_0) \cap \ker f'(x_0)$ with $\| d \| = 1$, then

(a) $f'(x_0)w + d' f''(x_0) d < 0$ whenever $w \in T_2^*(S,x_0,d)$

(b) $f'(x_0)w < 0$ whenever $0 \neq w \in T_0^*(S,x_0,d)$.

Then x_0 is a local maximizer of f over S.

Proof.

Suppose that x_0 is not a local maximizer of f over S. Then there exists a feasible sequence $\{x_n\}$ converging to x_0 such that $f(x_n) > f(x_0)$ for all n.

Define $\alpha_n = \frac{1}{\|x_n - x_0\|}$; obviously we have $\alpha_n(x_n - x_0) \to d \in S'(x_0)$ with $\| d \| = 1$. Since $\alpha_n(f(x_n) - f(x_0))$ is positive for all n, it is easy to prove, by means of first order Taylor's expansion, that i) is not violated only if $f'(x_0)d = 0$. Therefore let us assume that $d \in \ker f'(x_0)$.

If $\{2\alpha_n(v_n)^*\}$ is a convergent sequence, with $2\alpha_n(v_n)^* \to w \in T_2^*(S, x_0, d)$, from (1) we have $2\alpha_n^2 (f(x_n) - f(x_0)) \to f'(x_0)w + d' f''(x_0) d \ge 0$, which contradicts ii) (a).

If $\alpha_n \| (v_n)^* \| \to +\infty$, setting $\beta_n = \frac{1}{\| (v_n)^* \|}$. we have $\beta_n \to +\infty$ and $\frac{\beta_n}{\alpha_n} = \frac{1}{\alpha_n \| (v_n)^* \|} \to 0$ so that $\beta_n (v_n)^* \to w \in T_0^*(S, x_0, d)$ with $\| w \| = 1$.

From (1) we have

$$\frac{\alpha_n}{\| (v_n)^* \|} (f(x_n) - f(x_0)) =$$

$$= f'(x_0) \frac{(v_n)^*}{\| (v_n)^* \|} + \frac{1}{2\alpha_n \| (v_n)^* \|} (d+v_n)' f''(x_0+\delta_n(x_n - x_0)) (d+v_n)$$

so that $\frac{\alpha_n}{\| (v_n)^* \|} (f(x_n) - f(x_0)) \to f'(x_0) w \geq 0$ and this contradicts ii) (b). ♦

Remark 2

It is easy to prove that

$P''_2(S, x_0, d) \subset T^*_2(S, x_0, d)$ and $P''_0(S, x_0, d) \subset T^*_0(S, x_0, d)$.

Example 3 points out that these inclusions are proper and, consequently, the necessary optimality condition stated in Theorem 3 is more general than that given in Theorem 1 or 1*. Furthermore, the sufficient optimality condition in Theorem 4 is more handable than the one in Theorem 2 or 2*(see examples 3,4), since it can happen that $\ker f'(x_0) \cap (T''_0(S,x_0,d) \cap d^{\perp}) \neq \varnothing$, or equivalently $0 \in P''_0(S,x_0,d)$, while we have $f'(x_0)w \neq 0$ whenever $0 \neq w \in T^*_0(S, x_0, d)$.

Example 3

Consider problem P where $f(x,y,z) = x^2-3z$, $S=\{(x,y,z) : y = x\sqrt{x},\ z = x^2,\ x \geq 0\}$ and the point $x_0 = (0,0,0)$. It is easy to verify that $S'(x_0) = \{t(1,0,0) ; t \geq 0\}$.

Let $d = (1,0,0)$; by means of simple calculations we find $f'(x_0)d = 0$,

$T''_2(S, x_0, d) = P''_2(S, x_0, d) = \varnothing \subset T^*_2(S, x_0, d) = \{w: w=(0,0,2)\}$,

$P''_0(S, x_0, d) = T^*_0(S, x_0, d) = \{(0,0,0)\}$.

Let us note that ii) (b) of Theorem 2* is not satisfied, so that Theorem 2* cannot be invoked to establish the optimality of x_0; on the contrary, ii) of Theorem 4 is satisfied since $T^*_0(S, x_0, d) \setminus \{(0,0,0)\} = \varnothing$ and $f'(x_0)w + d'f''(x_0)d = -6 + 2 < 0$, so that x_0 is a local maximizer of f over S.

Example 4

Consider problem P where $S=\{(x,y,z) : y = x\sqrt{x},\ z = x\sqrt[4]{x^3},\ x \geq 0\}$,

$f(x,y,z) = x^2-3z$, and the point $x_0 = (0,0,0)$. We have $S'(x_0) = \{t(1,0,0) ; t \geq 0\}$. Let $d= (1,0,0)$; by means of simple calculations we find $f'(x_0)d = 0$,

$\overset{''}{T_2}(S, x_0, d) = \overset{''}{P_2}(S, x_0, d) = \varnothing = \overset{*}{T_2}(S, x_0, d)$, $\overset{''}{P_0}(S, x_0, d) = \{(0,0,0)\}$.

$\overset{*}{T_0}(S, x_0, d) \setminus \{(0,0,0)\} = \{t(0,0,1) : t > 0\}$.

Once again ii) (b) of Theorem 2* is not satisfied, so that Theorem 2* cannot be invoked to test the optimality of x_0 ; on the contrary, ii) of Theorem 4 is satisfied since $\overset{*}{T_2}(S,x_0,d)=\varnothing$ and $f'(x_0)w=-t < 0$, whenever $0\neq w\in \overset{*}{T_0}(S, x_0, d)$, so that x_0 is a local maximizer of f over S.

4 Particular cases

In the previous section we have pointed out the role of condition ii) (b) in Theorem 4 in establishing a sufficient optimality condition. In this section we will prove that such a condition is superflous when S is a convex set and this allows to achieve some classic results. With this aim we establish the following Lemma.

Lemma 1

Consider problem P where S is a convex set. Suppose that a point x_0 satisfies the following condition :

i) $f'(x_0)d \leq 0$ whenever $d\in S'(x_0)$.

Then $f'(x_0)\, w \leq 0$ whenever $w \in \overset{''}{T_k}(S, x_0, d)$.

Proof.

$w \in \overset{''}{T_k}(S, x_0, d)$ implies the existence of a feasible sequence $\{x_n\}$ with $x_n \to x_0$ and the existence of positive sequences of real numbers $\alpha_n \to +\infty$, $\beta_n \to +\infty$ such that $\dfrac{\beta_n}{\alpha_n} \to k$, $\alpha_n (x_n-x_0) \to d$ and $\beta_n v_n \to w$ where $v_n = \alpha_n(x_n - x_0) - d$. Define $d_n = \alpha_n (x_n - x_0)$. Since $f'(x_0)\, d = 0$, it results $f'(x_0)\, v_n = f'(x_0)\, d_n = \alpha_n\, f'(x_0)(x_n - x_0)$. On the other hand, the convexity of S implies that $x_n - x_0$ is a feasible direction and thus $x_n - x_0 \in S'(x_0)$, so that $f'(x_0)(x_n - x_0) \leq 0$. As a consequence $f'(x_0)\, \beta_n v_n \to f'(x_0)\, w \leq 0$.

The proof is complete. ◆

The following theorem holds.

Theorem 5
Consider problem P where S is a convex set. Suppose that a point x_0 satisfies the following conditions :
i) $f'(x_0)d \leq 0$ whenever $d \in S'(x_0)$.
ii) If $d \in S'(x_0) \cap \ker f'(x_0)$ with $\| d \| = 1$, then
$f'(x_0)\, w + d' f''(x_0)\, d < 0$ whenever $w \in \overset{*}{T}_2(S, x_0, d)$.
Then x_0 is a local maximizer of f over S.

Proof.
Taking into account Theorem 4, we must prove that , if $d \in S'(x_0) \cap \ker f'(x_0)$ with $\| d \| = 1$, the convexity of S implies $f'(x_0)\, w < 0$ whenever $0 \neq w \in \overset{*}{T}_0(S, x_0, d)$. Taking into account the proof given in Lemma 1, we have $f'(x_0)\, (v_n)^* = f'(x_0)\, v_n \leq 0$. As a consequence $f'(x_0)\, \beta_n (v_n)^* \rightarrow f'(x_0)\, w \leq 0$ and , taking into account that $0 \neq w \in \overset{*}{T}_0(S, x_0, d) \subset [f'(x_0)]$, necessarily we have $f'(x_0)\, w < 0$.
The proof is complete. ♦

Remark 3
The proof given in Lemma 1 points out that it results $f'(x_0)\, w \leq 0$ whenever $w \in \overset{*}{T}_2(S, x_0, d)$, so that replacing ii) with
ii)* If $d \in S'(x_0) \cap \ker f'(x_0)$ with $\|d\|=1$, then $d' f''(x_0) d < 0$
we obtain again a sufficient optimality condition for a convex set which is more restrictive than ii). With this regard, it is sufficient to consider problem P where $S=\{(x,y) : y \geq x^2\}$, $f(x,y) = x^2 - 2y$ and the point $x_0=(0,0)$. It can be verified that ii)* does not hold, while ii) is satisfied.

At last we will prove the following theorem which establishes the equivalence between ii) and ii)* for closed local cone S with vertex at x_0 (that is S is the intersection of a closed cone with vertex at x_0 and a neighbourhood of x_0).

Theorem 6
Consider problem P where S is a closed local cone with vertex at x_0. Suppose that a point x_0 satisfies the following condition :
i) $f'(x_0)d \leq 0$ whenever $d \in S'(x_0)$.
Then ii) and ii)* are equivalent.

Proof.

We have $f'(x_0)\, w \leq 0$ whenever $w \in \overset{*}{T_2}(S, x_0, d)$ (see Remark 3), so that ii)* implies ii). On the other hand, any $d \in S'(x_0)$ is a feasible direction; choosing $x_n = x_0 + \frac{d}{n}$, $\alpha_n = n$, we have $2\alpha_n(\alpha_n (x_n - x_0) - d) = 0$ for all n, so that $0 \in \overset{*}{T_2}(S, x_0, d)$ and ii) implies ii)*. ♦

As a consequence of Theorems 5 and 6 we obtain the following known result:

Theorem 7

Consider problem P where S is a polyhedral set. A vertex x_0 is a local maximizer of f over S if the following condition is satisfied:

$f'(x_0)d \leq 0$ for all directions starting from x_0 and $d' f''(x_0)\, d < 0$ if $f'(x_0)d=0$.

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