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Abstract

This paper is devoted to the study of infinite horizon continuous time optimal control problems with incentive compatibility constraints. An incentive compatibility constraint is a constraint on the continuation of the payoff function at every time. We prove that the dynamic programming principle holds, the value function is a viscosity solution of the associated Hamilton-Jacobi-Bellman equation, and that it is the minimal supersolution satisfying certain boundary conditions. When the incentive compatibility constraint only depends on the present value of the state variable, we prove existence of optimal strategies, and we show that the problem is equivalent to a state constraints problem in an endogenous state region which depends on the data of the problem. Our analysis is useful to address second best pareto optimum-incentive compatibility constrained problems. Some economic examples are analyzed.

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1 Introduction

In this paper we study optimal control problems in continuous time/infinite horizon with *incentive compatibility constraints*. We consider a classical infinite horizon optimal control problem with a constraint on the continuation value of the plan at each time $t \geq 0$. The continuation value at each time t is constrained from below by a function of the state and of the control at time t. The constraint can be interpreted from an economic point of view as an outside option/incentive compatibility constraint.

This type of constraint gives rise to two different optimal control problems: optimal stopping problems and optimal control problems with incentive compatibility constraints. In the first case we have a *positive* perspective: we want to study the optimal behavior of an economic agent in a dynamic setting allowing him at any time in the future to stop the process and to exercise an outside option which gives him a reward which is a function of the state (termination payoff). In this setting the set of admissible controls includes those violating the incentive compatibility constraint. The agent is allowed to get the termination payoff. In the second case we have a *normative* perspective. The point of view is the one of a social planner who wants to characterize optimal contracts or second best solutions to dynamic problems under incentive compatibility constraints for the agents of the economy. The set of admissible controls does not include those violating the incentive compatibility constraint. The social planner looks for an optimal policy for the agents among the policies which do not include the termination of the process by the agents. The goal is the definition of a social contract taking into account the fact that the agents can decide in the future to go out of the contract. This event is prevented by including the incentive compatibility constraint.

Many economic problems can be fitted in the framework described above. They belong to the so called second best literature. There are two main classes of models where this type of problems arise. The first class comes from the study of differential games. The second class comes from policy making without full precommitment (time consistency problems).

It is well recognized that in differential games such as the exploitation of a common resource ([Benhabib and Radner, 1992, Dockner, Sorger 1996, Tornell and Velasco, 1992, Dutta and Sundaram 1993, Dutta and Sundaram 1993]), capital accumulation ([Fudenberg and Tirole 1983]), pollution, voluntary provision of a public good ([Dockner et al. 1996]), the outcome of the noncooperative interaction obtained as a subgame perfect equilibrium or as trigger strategy equilibrium may be Pareto inefficient, i.e., the reward for the agent is smaller than the one obtained by a representative agent under perfect competition (the so called *tragedy of commons*). This result leads to the problem of designing contracts which are efficient among the subgame perfect equilibria (see [Rustichini, 1992, Benhabib and Rustichini, 1996]) and to the problem of designing contracts yielding at any time in the future a utility level higher than the one obtained according to a specific subgame perfect equilibrium. This type of problems can be formalized as the maximization of the utility of the representative agent under the constraint that at every time in the future the continuation value of the consumption plan is greater than the utility obtained from the strategy of a subgame perfect equilibrium. Typically the constraint is given by the value function of a control problem without constraints.

The second class of models comes from optimal taxation problems in an intertemporal setting without full precommitment, or in great generality from the analysis of an economy where there is a private sector and the government. The problem is the definition of an optimal plan without full commitment at time zero (for instance a tax plan made up of taxes by the government and saving decisions by the private sector) in such a way that the private sector and the government do not have an incentive to deviate at each date in the future (decisions are taken sequentially without commitment), see [Chari and Kehoe 1990, Chari and Kehoe 1993, Marcet and Marimon, 1992, Marcet and Marimon, 1996] for some interesting examples.

The incentive compatibility constraint is endogenous to the model, it is usually given by a function only of the state and not of the control (in many cases it is a value function of the associated unconstrained problem). The outside option can also be interpreted in some cases as a policy variable (fixed costs, royalties, taxes) or as an exogenous opportunity (different investment opportunity). This type of constraints has been recently analyzed in discrete time in [Marcet and Marimon, 1996, Rustichini, 1998a, Rustichini, 1998b]. In the first two papers the problem has been studied by means of Lagrange multipliers, in the third one through dynamic programming. Our paper provides a dynamic programming solution to the problem in the continuous time case, some similarities between our characterization of the value function of the constrained problem and the one provided for the discrete time case in [Rustichini, 1998a] can be noted (see Remark 4.5).

The optimal control problem is a state constraints problem with infinite horizon, a discounted objective function and an additional constraint on the continuation value for the plan. Such a constraint concerns the future of the trajectory and gives rise to non standard technical problems. In this paper we first analyze a constraint described by a function of the control and of the state and then we restrict our attention to the case of a function depending only on the state. In the general case we prove that the Dynamic Programming Principle holds. This allows us to write the Hamilton-Jacobi-Bellman equation (HJB in the rest of the paper) associated with our problem and to prove, under suitable additional assumptions, that the value function of the constrained problem is a solution in the viscosity sense of that equation. We cannot obtain uniqueness of the solution of the HJB equation in general but we are able to characterize the value function as the minimal viscosity supersolution (satisfying suitable boundary conditions) of the associated HJB equation. Restricting our attention to the case where the incentive constraint only depends on the state variable, we prove a result about existence (and uniqueness) of optimal strategies and then we prove that the above problem is equivalent to a state constraints problem in a region E which is implicitely determined by the data of the problem. Some topological properties of E are discussed. This allows us to adapt known results and techniques on state constraints problems in order to study the properties of the optimal trajectories.

In Section 6 two economic examples with a linear state equation, concave objective function and an incentive compatibility constraint defined by a constant are analyzed. The first example is the optimal saving problem, the second is the firm's capital accumulation problem with adjustment costs. In both cases the value function is constrained from below by a positive constant. The analysis shows that the solution of the constrained problem depends on the value of the parameters of the model. In the first example, if the interest rate is larger than the discount rate then both the value function and the optimal policy of the constrained problem coincide with those of the unconstrained problem, provided that the initial stock of capital is large enough. If the opposite condition holds then the constrained value function is smaller than the unconstrained value function and the second best optimal control induces a smaller rate of consumption than the first best policy. The minimal stock of capital allowing existence of the second best policy is larger than in the first case. In the second example, as the constant describing the constraint goes up we observe four different parameter regions. For a small constant (first region) the constraint is not binding and therefore the unconstrained solution coincides with the constrained solution, for a higher constant (second region) the unconstrained solution is equal to the constrained solution but the initial stock of capital should be large enough to have a solution. As the constant is furthermore increased (third region) the investment rate is higher than the first best solution and an initial stock of capital larger than in the previous case is needed to have a solution. Finally when the constant is beyond a certain level (fourth region) the problem becomes ill posed for every initial stock of capital.

Summing up in the two examples we have that an incentive compatibility constraint has two effects: it restricts the state region for which a solution exists and it induces a higher rate of investment. The optimal policy foresees a stationary level of the state variable when the incentive constraint becomes binding.

The paper is organized as follows. In Section 2 we describe the problem. In Section 3 we show that for an incentive compatibility constraint described by a function of the state and of the control the Dynamic Programming Principle holds. In Section 4 we characterize the value function of the constrained problem as a viscosity solution of the HJB equation. In Section 5 we restrict our attention to an incentive compatibility constraint defined by a function depending only on the state. In Section 6 we analyze two economic examples.

2 The Problem

Let $\mathbf{C} \subset \mathbb{R}^d$ and let $\mathcal{C}$ be the set of all functions $c : \mathbb{R}^+ \mapsto \mathbf{C}$ that are measurable and locally integrable. Given $c \in \mathcal{C}$ consider the state equation

$$\begin{cases} \dot{x}(s) = f(x(s), c(s)); & s \geq 0 \\ x(0) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases} \tag{1}$$

where f is a function that is Lipschitz continuous in x, uniformly with respect to c. In the standard framework the positive real half-line describing the domain of the function c represents the time dimension of the problem. The function c is the control, $c(s) \in \mathbb{R}^d$ is the value of the control function at time s. The dimension of the state variable x is n while d is the dimension of the control variable. Let $x(t; x_0, c) \in \mathbb{R}^n$ denote the solution of (1) at time $t \geq 0$ given the control $c \in \mathcal{C}$ and the initial condition $x_0 \in \mathbb{R}^n$. The solution always exists given the assumptions specified above.

Given a set $A \subset \mathbb{R}^n$, define for every $x_0 \in A$ the set $\mathcal{C}_A(x_0)$ as the set of controls $c \in \mathcal{C}$ such that $x(t; x_0, c) \in A$ for every $t \geq 0$. A represents the state constraint, for example a positivity state constraint requires A to be the positive orthant in $\mathbb{R}^n$.

We consider two different control problems: an *unconstrained* and a *constrained* problem.

Let us first take a continuous objective function $f_0 : A \times \mathbf{C} \mapsto \mathbb{R}$. Given $x_0 \in \mathbb{R}^n$, the unconstrained problem is just a classical state constraints optimal control problem with infinite horizon: maximize the functional

$$J(x_0; c) = \int_0^{+\infty} e^{-\rho t} f_0(x(t; x_0, c), c(t))dt \tag{2}$$

over all controls $c \in \mathcal{C}_A(x_0)$ such that $J(x_0, c)$ is well defined, see e.g. [Hartl, Sethi and Vickson, 1995] for the maximum principle approach and [Soner, 1986, Capuzzo-Dolcetta and Lions, 1990, Cannarsa et al. 1991, Ishii and Koike, 1996, Soravia, 1997b] and [Bardi and Capuzzo-Dolcetta 1998, Ch. IV] for the dynamic programming approach to this kind of problems.

The unconstrained value function is defined as

$$V_u(x_0) \stackrel{def}{=} \sup_{c \in \mathcal{C}_A(x_0)} J(x_0; c),$$

$V_u(x_0) = -\infty$ when $\mathcal{C}_A(x_0) = \emptyset$. Under suitable controllability assumptions (e.g. the set A has a $C^{1,1}$ boundary and $\forall x_0$, $\exists$ c_0 such that $\langle f(x_0, c_0), n(x_0)\rangle < -\varepsilon$ where $n(x_0)$ is the outward normal vector at x_0 and $\varepsilon > 0$ is independent of x_0, see [Cannarsa et al. 1991, Remark 4.7]) we have that $\mathcal{C}_A(x_0) \neq \emptyset$ $\forall x_0 \in A$. We make the following assumption.

Assumption 2.1

(i) The sets A and $\mathbf{C}$ are closed and convex.

(ii) f is continuous and there exists a constant $M > 0$ such that

$$\begin{array}{ll} |f(x_1, c) - f(x_2, c)| \leq M|x_1 - x_2| & \forall x_1, x_2 \in A, \ \forall c \in \mathbf{C} \\ |f(x, c)| \leq M(1 + |x| + |c|) & \forall x \in A, \ \forall c \in \mathbf{C}. \end{array} \tag{3}$$

(iii) f_0 is continuous and uniformly continuous in x, uniformly in c.

To ensure that the value function V_u is always finite we also assume the following.

Assumption 2.2 *For every $x_0 \in A$ and every admissible control strategy $c \in \mathcal{C}_A(x_0)$ we have $|J(x_0; c)| \leq M(|x_0|)$ where M is a suitable nondecreasing function on $\mathbb{R}^+$.*

Only few results on state constraints optimal control problems are known without making further assumptions. We will not concentrate on this topic since we want to focus our attention on the incentive constrained problem defined below. Further assumptions will be needed, some of them, like Assumptions 2.4 (ii) and (iii) will also imply "good" properties for the unconstrained problem.

To begin with the constrained problem we define $\mathcal{C}(x_0)$ as the set of controls $c \in \mathcal{C}_A(x_0)$ such that the following constraint is satisfied for almost every $t \geq 0$:

$$\int_t^{+\infty} e^{-\rho s} f_0(x(s;x_0,c),c(s))ds \geq e^{-\rho t} D(x(t;x_0,c),c(t)), \tag{4}$$

where $D : A \times \mathbf{C} \mapsto \mathbb{R}$ is a suitable continuous function. The constraint (4) can be interpreted as an incentive compatibility and/or rationality constraint. The control plan should guarantee at each time t a residual payoff higher than a function of the state and of the control at time t. The function D can be interpreted as an *outside option.*

The constrained problem consists of maximizing the functional (2) over all controls $c \in \mathcal{C}(x_0)$.

Remark 2.3 Having decided to work with measurable control strategies, the above constraint (4) cannot be well defined for every time $t \geq 0$. This fact can be overcome by choosing a different set of control strategies: the one of functions $c : \mathbb{R} \to \mathbb{R}^d$ that are right continuous with left limit (RCLL in the following) at every time $t \geq 0$.

However, choosing this set of control strategies will render the problem difficult, since this set cannot be easily handled. For this reason we will always work with the measurable strategies merely pointing out the results that can be proved also in the RCLL case (e.g. the Dynamic Programming Principle). When D does not depend on c (see Section 5) there will be no reason to deal with RCLL control strategies and therefore we will consider only measurable admissible controls. ∎

The constrained value function is defined as

$$V(x_0) \stackrel{def}{=} \sup_{c \in \mathcal{C}(x_0)} J(x_0;c), \tag{5}$$

$V(x_0) = -\infty$ when $\mathcal{C}(x_0) = \emptyset$. Obviously $\mathcal{C}(x_0) \subset \mathcal{C}_A(x_0)$ so that we have

$$V(x_0) \leq V_u(x_0) \qquad \forall x_0 \in A.$$

Let us define the set $E \subset A$ as the set of all $x \in A$ such that $\mathcal{C}(x) \neq \emptyset$, i.e. the set of all $x \in A$ such that $V(x) > -\infty$. Setting $D_0(x) = \inf_{c \in \mathbf{C}} D(x,c)$, then

$$E \subset \{x \in A : V_u(x) \geq D_0(x)\} \subset A.$$

Both the unconstrained and the constrained problem can be rewritten by setting

$$w(t;x_0,c) = \int_t^{+\infty} e^{-\rho(s-t)} f_0(x(s;x_0,c),c(s))ds. \tag{6}$$

The function $w(\cdot;x_0,c)$ is then the unique solution of the problem

$$\begin{cases} \dot{w}(t) = \rho w(t) - f_0(x(t;x_0,c),c(t)) & t \geq 0 \\ \lim_{t \to +\infty} e^{-\rho t} w(t) = 0. \end{cases}$$

The unconstrained problem becomes:

Maximize $J(x_0, c) = w(0)$ *under the constraints*

$$\begin{cases} \dot{x}(t) = f(x(t), c(t)) & t \geq 0 \\ \dot{w}(t) = \rho w(t) - f_0(x(t), c(t)) & t \geq 0 \\ x(0) = x_0 \\ \lim_{t \to +\infty} e^{-\rho t} w(t) = 0 \\ x(t) \in A, \end{cases} \tag{7}$$

while the constrained problem is obtained by adding the requirement that

$$w(t) \geq D(x(t; x_0, c), c(t)) \qquad t \geq 0.$$

We observe that the above problem is not a standard state constraints problem since the new state variable w solves a backward equation with terminal condition at $t = +\infty$. This indicates the main difficulty we have to face when dealing with an incentive constraint: not only it is a non local constraint (which would be eliminated by adding the new state variable) but also a constraint on the future value of the plan. The above formulation is equivalent to the first one and will be useful in some proofs (see Section 5).

We now collect various technical assumptions that will be used in the paper. Here we denote by $B(x_0, r)$ the closed ball in $\mathbb{R}^n$ centered at x_0 with radius r and we say that a function $\omega : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a modulus if it is continuous, subadditive, nondecreasing, and such that $\lim_{a \to 0} \omega(a) = 0$. Moreover, given $\varepsilon > 0$ and $x \in E$ we say that a control strategy $c \in \mathcal{C}(x)$ (respectively in $\mathcal{C}_A(x)$) is ε-suboptimal for the constrained (respectively unconstrained) problem if $J(x; c) > V(x) - \varepsilon$ (respectively $J(x; c) > V_u(x) - \varepsilon$).

Assumption 2.4

(i) For every $x \in E$ *the set of pairs* $(f_0(x, \mathbf{C}), f(x, \mathbf{C}))$ *is closed and convex.*

(ii) For every $x_0 \in E$, $T > 0$ *there exists* $r > 0$ *and a modulus* ω *such that, for every* $y \in B(x_0, r) \cap E$ *there exists a control strategy* $c_y \in \mathcal{C}_E(y)$ *such that*

$$\int_{t_1}^{t_2} [|f_0(x(t; y, c_y), c_y(t))| + |f(x(t; y, c_y), c_y(t))|] \, dt \leq \omega(|t_1 - t_2|), \quad \forall\, 0 \leq t_1, t_2 \leq T, \tag{8}$$

$$\int_{t_1}^{+\infty} e^{-\rho t} |f_0(x(t; y, c_y), c_y(t))| dt \overset{t_1 \to +\infty}{\longrightarrow} 0 \quad \text{uniformly for } y \in B(x_0, r) \cap E. \tag{9}$$

(iii) For every $x_0 \in E$, $T > 0$ *and for every* $\varepsilon > 0$ *there exists* $r > 0$ *and a modulus* ω *such that, for every* $y \in B(x_0, r) \cap E$ *there exists an* ε*-suboptimal control strategy* $c_{y,\varepsilon} \in \mathcal{C}_E(y)$ *such that*

$$\int_{t_1}^{t_2} |f_0(x(t; y, c_{y,\varepsilon}), c_{y,\varepsilon}(t))| + |f(x(t; y, c_{y,\varepsilon}), c_{y,\varepsilon}(t))| \, dt \leq \omega(|t_1 - t_2|), \quad \forall\, 0 \leq t_1, t_2 \leq T, \tag{10}$$

$$\int_{t_1}^{+\infty} e^{-\rho t} |f_0(x(t; y, c_{y,\varepsilon}), c_{y,\varepsilon}(t))| dt \overset{t_1 \to +\infty}{\longrightarrow} 0 \quad \text{uniformly for } y \in B(x_0, r) \cap E. \tag{11}$$

(iv) *For every* $x_0 \in E$, $T > 0$ *and for every* $\varepsilon > 0$ *there exists* $r > 0$ *and a modulus* ω *such that, for every* $y \in B(x_0, r) \cap E$ *there exists an* ε*-suboptimal control strategy* $c_{y,\varepsilon} \in \mathcal{C}(y)$ *such that*

$$|x(t_1, y, c_{y,\varepsilon}) - x(t_2; y, c_{y,\varepsilon})| \leq \omega(|t_1 - t_2|) \qquad \forall t_1, t_2 \in [0, T]. \tag{12}$$

Remark 2.5 Assumptions (ii), (iii), (iv) are expressed in a quite complicated form. This is due to the fact that in some economic examples both the set of states and the control set are unbounded. In the bounded cases we could avoid or simplify such assumptions (see e.g. [Soravia, 1997b]). ∎

Remark 2.6 We observe that:

1. Assumption (ii) (8) implies that

$$\sup_{t \in [0,T], y \in B(x_0,r) \cap E} |x(t, y, c_y)| < +\infty. \tag{13}$$

2. Either of the conditions (iii)-(10) or (iv) implies that

$$\sup_{t \in [0,T], y \in B(x_0,r) \cap E} |x(t, y, c_{y,\varepsilon})| < +\infty. \tag{14}$$

3. Assumption (8) (respectively (10)) implies equintegrability with respect to $y \in B(x_0, r)$ (respectively with respect to $y \in B(x_0, r)$ and $\varepsilon > 0$) of the functions $x'(t; y, c_y)$, $w'(t; y, c_y)$ (respectively $x'(t; y, c_{y,\varepsilon})$, $w'(t; y, c_{y,\varepsilon})$). By equintegrability of a family of functions $\{d_i\}, i \in I$ defined in a subset O of $\mathbb{R}^n$ we mean that

$$\forall \sigma > 0,\ \exists \delta > 0 : \int_{O_1} d_i(t)dt < \sigma \ \text{ for every } \ i \in I,\ O_1 \subset O, measure(O_1) < \delta. \tag{15}$$

4. Since E is not known in general, we will be checking Assumption (i) for $x \in A$ (or for $V_u(x) \geq D_0(x)$), which will in particular imply it for $x \in E$.

∎

3 The Dynamic Programming Principle

In this Section we prove the Dynamic Programming Principle for the constrained problem. This is a standard result for the unconstrained problem. In the constrained case it is not obvious because of the constraint on the future value of the plan. We will prove the result by assuming in the whole section that Assumption 2.1 (ii)-(iii) is satisfied and that the control strategies are measurable, locally integrable and such that the functional J is well defined. By standard adaptations of the arguments below, it can be seen that the result also holds when f_0 is only continuous, and when we take RCLL control strategies.

We need two useful Lemmas.

Lemma 3.1 *Let Assumptions 2.1 (ii)-(iii) and 2.2 be satisfied. Given any* $T > 0$, $x_0 \in E$, $c \in \mathcal{C}(x_0)$, *the control* c_T *defined as* $c_T(s) = c(T+s)$ *with* $s > 0$ *belongs to* $\mathcal{C}(x(T; x_0, c))$.

Proof. Given a $c \in \mathcal{C}(x_0)$ we have to prove that

$$\int_t^{+\infty} e^{-\rho s} f_0(x(s; x(T; x_0, c), c_T), c_T(s))ds \geq e^{-\rho t} D(x(t; x(T; x_0, c), c_T), c_T(t)), \quad a.e.\ t \geq 0,$$

where $c_T(r) = c(T+r)$ for $r \geq 0$ and $x(r; x(T; x_0, c), c_T) = x(r+T; x_0, c)$. Therefore, the above inequality is equivalent to

$$\int_t^{+\infty} e^{-\rho s} f_0(x(s+T; x_0, c), c(T+s))ds \geq e^{-\rho t} D(x(t+T; x_0, c), c(T+t)), \quad a.e.\ t \geq 0$$

and, by the change of variable $\sigma = s + T$, to

$$\int_{t+T}^{+\infty} e^{-\rho\sigma} f_0(x(\sigma; x_0, c), c(\sigma))d\sigma \geq e^{-\rho(t+T)} D(x(t+T; x_0, c), c(T+t)), \quad a.e.\ t \geq 0$$

which follows from the fact that $c \in \mathcal{C}(x_0)$. This implies that $c_T \in \mathcal{C}(x(T; x_0, c))$. ∎

The above Lemma states that given an initial state point for which a control satisfying the constraint (4) exists then such a control restricted to the interval $[T, +\infty)$ for every $T > 0$ satisfies the constraint (4) starting from $x(T; x_0, c)$.

Lemma 3.2 *Let Assumptions 2.1 (ii)-(iii) and 2.2 be satisfied. Given* $x_0 \in E$, $T > 0$ *and* $c \in \mathcal{C}(x_0)$, *take a control trajectory* $\overline{c}_T \in \mathcal{C}(x(T; x_0, c))$ *and define the new control*

$$c_1(t) = \begin{cases} c(t) & t \in [0, T) \\ \overline{c}_T(t-T) & t \in [T, +\infty). \end{cases} \tag{16}$$

If $J(x(T; x_0, c); \overline{c}_T) \geq J(x(T; x_0, c); c)$, *then* $c_1 \in \mathcal{C}(x_0)$.

Proof. Given a $c \in \mathcal{C}(x_0)$ and the control c_1 defined in (16) we want to prove that

$$\int_t^{+\infty} e^{-\rho s} f_0(x(s; x_0, c_1), c_1(s))ds \geq e^{-\rho t} D(x(t; x_0, c_1), c_1(t)) \quad a.e.\ t \geq 0.$$

Let first $t \geq T$. Since $x(r; x_0, c_1) = x(r-T; x(T; x_0, c), \overline{c}_T)$ for $r \geq T$, then for every $t > T$ the latter inequality is equivalent to

$$\int_t^{+\infty} e^{-\rho s} f_0\left(x(s-T; x(T; x_0, c), \overline{c}_T), \overline{c}_T(s-T)\right) ds \geq e^{-\rho t} D(x(t-T, x(T; x_0, c), \overline{c}_T), \overline{c}_T(t-T))$$

and, by the change of variable $\sigma = s - T$, to

$$\int_{t-T}^{+\infty} e^{-\rho\sigma} f_0\left(x(\sigma; x(T; x_0, c), \overline{c}_T), \overline{c}_T(\sigma)\right) d\sigma \geq e^{-\rho(t-T)} D(x(t-T, x(T; x_0, c), \overline{c}_T), \overline{c}_T(t-T))$$

which follows from the fact that $\overline{c}_T \in \mathcal{C}(x(T; x, c))$. Therefore for every $t \geq T$ the constraint (4) is satisfied.

Let now $t \in [0, T)$. In this case

$$\int_t^{+\infty} e^{-\rho s} f_0(x(s; x_0, c_1), c_1(s)) ds = \int_t^{+\infty} e^{-\rho s} f_0(x(s; x_0, c), c(s)) ds$$

$$+ \left[\int_T^{+\infty} e^{-\rho s} f_0(x(s; x_0, c_1), c_1(s)) ds - \int_T^{+\infty} e^{-\rho s} f_0(x(s; x_0, c), c(s)) ds \right]. \qquad (17)$$

Let $J(x(T; x_0, c); \bar{c}_T) \geq J(x(T; x_0, c); c)$. Then it is easy to show that

$$\int_T^{+\infty} e^{-\rho s} f_0(x(s; x_0, c_1), c_1(s)) ds - \int_T^{+\infty} e^{-\rho s} f_0(x(s; x_0, c), c(s)) ds \geq 0. \qquad (18)$$

Following the argument we have employed in the first part of the proof we have that

$$\int_T^{+\infty} e^{-\rho s} f_0(x(s; x_0, c_1), c_1(s)) ds = \int_T^{+\infty} e^{-\rho s} f_0 \left(x(s - T; x(T; x_0, c), \bar{c}_T), \bar{c}_T(s - T) \right) ds$$

$$= e^{-\rho T} \int_0^{+\infty} e^{-\rho \sigma} f_0 \left(x(\sigma; x(T; x_0, c), \bar{c}_T), \bar{c}_T(\sigma) \right) d\sigma = e^{-\rho T} J(x(T; x_0, c); \bar{c}_T),$$

where $\sigma = s - T$. Similarly

$$\int_T^{+\infty} e^{-\rho s} f_0(x(s; x_0, c), c(s)) ds = e^{-\rho T} J(x(T; x_0, c); c).$$

Therefore the inequality in (18) is established. The claim now follows from the fact that $c \in \mathcal{C}(x_0)$, which implies that

$$\int_t^{+\infty} e^{-\rho s} f_0(x(s; x_0, c), c(s)) ds \geq e^{-\rho t} D(x(t; x_0, c), c(t)) \quad a.e.\ t \geq 0.$$

Since $c(t) = c_1(t)$ for $t \in [0, T)$ and $c \in C(x_0)$ it follows from (17) and (18) that

$$\int_t^{+\infty} e^{-\rho s} f_0(x(s; x, c_1), c_1(s)) ds \geq \int_t^{+\infty} e^{-\rho s} f_0(x(s; x, c), c(s)) ds \geq e^{-\rho t} D(x(t; x_0, c), c_1(t)).$$

Therefore for every $t \in [0, T)$ the constraint (4) is satisfied by the control c_1. ■

Theorem 3.3 *Let Assumptions 2.1 (ii)-(iii) and 2.2 be satisfied. Then for every $T > 0$ we have*

$$V(x_0) = \sup_{c \in \mathcal{C}(x_0)} J_T(x_0; c),$$

where

$$J_T(x_0; c) \stackrel{def}{=} \int_0^T e^{-\rho t} f_0(x(t; x_0, c), c(t)) dt + e^{-\rho T} V(x(T; x_0, c)).$$

Proof. First we prove that

$$V(x_0) \geq \sup_{c \in \mathcal{C}(x_0)} J_T(x_0; c).$$

Let $c \in \mathcal{C}(x_0)$. It is enough to prove that $J_T(x_0; c) \leq V(x_0)$, where

$$J_T(x_0; c) = \int_0^T e^{-\rho t} f_0(x(t; x_0, c), c(t))dt + e^{-\rho T} V(x(T; x_0, c))$$

From Lemma 3.1 it follows that $c_T \in \mathcal{C}(x(T; x_0, c))$. If this control is optimal then from the definition of the value function it follows that $J_T(x_0; c) = J(x_0; c) \leq V(x_0)$. If c_T is not optimal starting at $x(T; x_0, c)$ then for an arbitrary $\varepsilon > 0$ we can find a control $c_{\varepsilon,T} \in \mathcal{C}(x(T; x_0, c))$, (the so-called ε-suboptimal control), such that

$$V(x(T; x_0, c)) < \varepsilon + J(x(T; x_0, c); c_{\varepsilon,T}), \text{ and } J(x(T; x_0, c); c_{\varepsilon,T}) \geq J(x(T; x_0, c); c_T). \quad (19)$$

Then, by Lemma 3.2, the control c_ϵ defined as

$$c_\varepsilon(t) = \begin{cases} c(t) & t \in [0, T) \\ c_{\varepsilon,T}(t - T) & t \in [T, +\infty) \end{cases}$$

belongs to $\mathcal{C}(x_0)$ and we have

$$J_T(x_0; c) \leq J(x_0; c_\varepsilon) + e^{-\rho T}\varepsilon \leq V(x_0) + \varepsilon,$$

where the first inequality comes from (19) and the second from the definition of the value function. This can be proved for every $\varepsilon > 0$ and therefore $J_T(x_0; c) \leq V(x_0)$ for $x_0 \in \mathcal{C}(x_0)$.

To complete the proof we have to prove the opposite inequality. We observe that for every $\varepsilon > 0$ there exists $c'_\varepsilon \in \mathcal{C}(x_0)$ such that

$$V(x_0) < J(x_0; c'_\varepsilon) + \varepsilon$$

and, by Lemma 3.1,

$$J(x_0; c'_\varepsilon) = \int_0^T e^{-\rho t} f_0(x(t; x_0, c'_\varepsilon), c'_\varepsilon(t))dt + e^{-\rho T} J(x(T; x_0, c'_\varepsilon); c'_{\varepsilon T})$$

$$\leq \int_0^T e^{-\rho t} f_0(x(t; x_0, c'_\varepsilon), c'_\varepsilon(t))dt + e^{-\rho T} V(x(T; x_0, c'_\varepsilon)) = J_T(x_0; c'_\varepsilon).$$

Therefore

$$V(x_0) < J_T(x_0, c'_\varepsilon) + \varepsilon,$$

which implies that

$$V(x_0) \leq \sup_{c \in \mathcal{C}(x_0)} J_T(x_0; c).$$

∎

4 The Hamilton-Jacobi Equation

The dynamic programming principle proved in Section 3 allows us to conclude that the value function is a solution of the associated Hamilton-Jacobi equation in the sense of [Ishii and Koike, 1996]. The Ishii-Koike definition of solution is based on the earlier definition of Soner [Soner, 1986].The arguments employed below are similar to those used in [Ishii and Koike, 1996, Soner, 1986].

Define $\mathcal{A}(x)$ to be the set of all $c \in \mathbf{C}$ such that there exists $r > 0$ such that for every $y \in E \cap B(x, r)$ there exists $c(\cdot) \in \mathcal{C}(y)$ such that $c(t) = c$ for $t \in [0, r]$. We will assume that

$$\mathcal{A}(x) \neq \emptyset \tag{20}$$

for every $x \in E$. In simple words we can say that the set $\mathcal{A}(x)$ is closely related to the set of $c \in \mathbf{C}$ that are starting points of admissible strategies for the constrained problem and we can think of it as the set of istantaneous contol strategies that are effectively doable when we are at state x. The set E in Examples 6.1 and 6.2 satisfies this assumption.

Let

$$H(x,p) = \sup_{c \in \mathbf{C}} \{< f(x,c), p > + f_0(x,c)\}$$

and

$$H_{\text{in}}(x,p) = \sup_{c \in \mathcal{A}(x)} \{< f(x,c), p > + f_0(x,c).\}$$

Since $\mathbf{C}$ may be unbounded both Hamiltonians may take infinite values. However, being a supremum of continuous functions, H is lower-semicontinuous in p. Moreover, H is uniformly continuous in x, uniformly for bounded p such that $H(x,p)$ is finite.

We now give the Ishii-Koike definition of viscosity solution applied to this case (recall that the Ishii-Koike definition is a little stronger than the one introduced in a similar context in [Soravia, 1997b]). Let U^* and U_* denote respectively the upper- and the lower-semicontinuous envelope of $U : E \mapsto \mathbb{R}$.

Definition 4.1 *A locally bounded function $U : E \to \mathbb{R}$ is a viscosity subsolution (respectively supersolution) of the equation*

$$\rho U - H(x, DU) = 0 \tag{21}$$

in E if whenever $U^* - \varphi$ has a local maximum (respectively, $U_* - \varphi$ has a local minimum) at x relative to E, where $\varphi \in C^1(E)$, then

$$\rho U^*(x) - H(x, D\varphi(x)) \leq 0 \tag{22}$$

(respectively,

$$\rho U_*(x) - H_{\text{in}}(x, D\varphi(x)) \geq 0.) \tag{23}$$

A function U is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

In the above definition $\varphi \in C^1(E)$ means that there exists an open set Ω such that $E \subset \Omega$ and $\varphi \in C^1(\Omega)$.

Theorem 4.2 *Let $\mathcal{A}(x) \neq \emptyset$ for every $x \in E$, and let Assumptions 2.1-(ii)-(iii),2.2, and 2.4-(iv) be satisfied. Let H be upper-semicontinuous at every point (x,p) such that $H(x,p) \neq +\infty$. Then the value function V defined in (5) is a viscosity solution of (21) in E.*

Proof. Let $V^* - \varphi$ have a local maximum at $x_0 \in E$. We may assume that the maximum is 0. If $H(x_0, D\varphi(x_0)) = +\infty$ we are done. If not we will argue by contradiction. If (22) is not satisfied at x_0 then the continuity of H at $(x_0, D\varphi(x_0))$ implies that there exist $r, \delta > 0$ such that if $x \in E$, $|x - x_0| < r$ then

$$\rho\varphi(x) - H(x, D\varphi(x)) \geq \delta. \tag{24}$$

It follows that there exist $x_\epsilon \in E, x_\epsilon \to x_0$ as $\epsilon \to 0$, and almost optimal controls $c_\epsilon \in \mathcal{C}(x_\epsilon)$ as in Assumption 2.4-(iv) such that

$$\begin{aligned}\epsilon^2 &\geq -\int_0^\epsilon e^{-\rho t} f_0(x(t;x_\epsilon,c_\epsilon),c_\epsilon(t))dt - e^{-\rho\epsilon}\varphi(x(\epsilon,x_\epsilon,c_\epsilon)) + \varphi(x_\epsilon)\\ &= -\int_0^\epsilon e^{-\rho t}\Big(f_0(x(t;x_\epsilon,c_\epsilon),c_\epsilon(t)) - \rho\varphi(x(t,x_\epsilon,c_\epsilon))\\ &\quad + < f(x(t;x_\epsilon,c_\epsilon),c_\epsilon(t)), D\varphi(x(t;x_\epsilon,c_\epsilon)) > \Big)dt.\end{aligned}$$

Therefore for some $t < \epsilon$ we have

$$\rho\varphi(x(t,x_\epsilon,c_\epsilon)) - f_0(x(t;x_\epsilon,c_\epsilon),c_\epsilon(t)) - < f(x(t;x_\epsilon,c_\epsilon),c_\epsilon(t)), D\varphi(x(t;x_\epsilon,c_\epsilon)) > \leq 2\epsilon.$$

However this contradicts (24) since by Assumption 2.4-(iv) $|x(t;x_\epsilon,c_\epsilon) - x_0| < r$ for small ϵ.

Let now $V_* - \varphi$ have a local minimum (equal to 0) at $x_0 \in E$. Let $c \in \mathcal{A}(x_0)$. Let r be as in the definition of $\mathcal{A}(x_0)$. Let $c_x(\cdot) \in \mathcal{C}(x)$, for $x \in B(x_0, r)$, be such that $c_x(t) = c$ for $t \in [0,r]$, and $x(t;x,c) \in E$ for $t \in [0,r]$. For $0 < \epsilon \leq r$ let $x_\epsilon \in E \cap B(x_0,r)$ be such that

$$V(x_\epsilon) < \varphi(x_\epsilon) + \epsilon^2,$$

and $\lim_{\epsilon\to 0} x_\epsilon = x_0$. The dynamic programming principle gives

$$V(x_\epsilon) \geq \int_0^\epsilon e^{-\rho t} f_0(x(t;x_\epsilon,c_{x_\epsilon}),c_{x_\epsilon}(t))dt + e^{-\rho\epsilon}V(x(\epsilon,x_\epsilon,c_{x_\epsilon})).$$

Therefore

$$\begin{aligned}\epsilon^2 &\geq \int_0^\epsilon e^{-\rho t} f_0(x(t;x_\epsilon,c_{x_\epsilon}),c_{x_\epsilon}(t))dt + e^{-\rho\epsilon}\varphi(x(\epsilon,x_\epsilon,c_{x_\epsilon})) - \varphi(x_\epsilon)\\ &= \int_0^\epsilon e^{-\rho t}\Big(f_0(x(t;x_\epsilon,c_{x_\epsilon}),c_{x_\epsilon}(t)) - \rho\varphi(x(t,x_\epsilon,c_{x_\epsilon}))\\ &\quad + < f(x(t;x_\epsilon,c_{x_\epsilon}),c_{x_\epsilon}(t)), D\varphi(x(t;x_\epsilon,c_{x_\epsilon})) > \Big)dt.\end{aligned}$$

It follows that for some $y_\epsilon = x(t_\epsilon; x_\epsilon, c_{x_\epsilon})$ we have

$$2\epsilon \geq f_0(y_\epsilon, c) - \rho\varphi(y_\epsilon) + < f(y_\epsilon, c), D\varphi(y_\epsilon) > .$$

Moreover, since $c_{x_\epsilon}(t) = c$ for $t \in [0, r]$, we have $\lim_{\epsilon \to 0} y_\epsilon = x_0$. Thus, passing to the limit as $\epsilon \longrightarrow 0$, we obtain

$$\rho V_*(x_0) - < f(x_0, c), D\varphi(x_0) > - f_0(x_0, c) \geq 0.$$

Since c is an arbitrary element of $\mathcal{A}(x_0)$ the claim follows. ∎

Remark 4.3 Assumption 2.4-(iv) and the continuity of the Hamiltonian in Theorem 4.2 are only needed in the subsolution part. What we really need is that if φ is as in the proof, $H(x_0, D\varphi(x_0)) < +\infty$ and (22) is not satisfied at x_0, then $\rho\varphi(x(t; x_\epsilon, c_\epsilon)) - H(x(t; x, c_\epsilon), D\varphi(x(t; x, c_\epsilon))) \geq \delta$ for x in some neighborhood of x_0, ϵ^2-optimal controls c_ϵ, and $0 \leq t \leq \epsilon$. ∎

We now give a characterization of the value function.

Proposition 4.4 *Let u be a locally bounded and globally bounded from below, lower-semicontinuous viscosity supersolution of (21) in E. Let Assumptions 2.1 (ii) – (iii), and 2.2 be satisfied and let $H = H_{\text{in}}$ in* $\text{int}E$. *If $u \geq V$ on $E \setminus \text{int}E$ then $u \geq V$ in E.*

Proof. The proof generalizes, improves, and simplifies the proof of the first part of Theorem 2.1 in [Święch, 1996]. Let $c \in \mathcal{C}(\bar{x}), T > 0$, and let R be such that $\|x(\cdot; \bar{x}, c)\|_{L^\infty([0,T])} \leq R$. Denote $K = \|u\|_{L^\infty(E \cap B(0,R+1))}$. Let

$$u_\epsilon(x) = \inf_{y \in E \cap B(0,R+1)} \left\{ u(y) + \frac{|y - x|^2}{2\epsilon} \right\}$$

be the inf-convolution of u. It is Lipschitz continuous and semi-concave on $E \cup B(0, R)$, and $u_\epsilon \nearrow u$ pointwise. Let $E_{\epsilon_0} = \{x \in E \cap B(0, R) : \text{dist}\,(x, \partial E) > 2\sqrt{K\epsilon_0}\}$. Denote by x^+ a point such that

$$u_\epsilon(x) = u(x^+) + \frac{|x^+ - x|^2}{2\epsilon}.$$

The point x^+ may be not unique. If $\epsilon < \epsilon_0, 2\sqrt{K\epsilon} < 1$, and $x \in E_{\epsilon_0}$, then $x^+ \in \text{int}E$. We will be denoting by $D^+u_\epsilon(x)$ (respectively, $D^-u_\epsilon(x)$) the generalized superdifferential (respectively, subdifferential) of u_ϵ at x (see [Crandall et al. 1992]). We notice that $D^+u_\epsilon(x)$ is always nonempty since u_ϵ is semi-concave. It is rather standard to notice that if $p \in D^-u_\epsilon(x)$ then $p \in D^-u(x^+)$, $p = (x^+ - x)/\epsilon$ and thus

$$0 \leq \rho u(x^+) - H_{\text{in}}(x^+, p) = \rho u(x^+) - H(x^+, \frac{x^+ - x}{\epsilon}).$$

Therefore

$$0 \leq \rho u_\epsilon(x) - H(x, p) + \rho\frac{|x^+ - x|^2}{2\epsilon} + L\frac{|x^+ - x|^2}{\epsilon}.$$

Denote

$$\omega(x,\epsilon) = \sup_{x^+} \left\{ (\frac{\rho}{2} + L)\frac{|x^+ - x|^2}{\epsilon} \right\}.$$

It is easy to check that $\lim_{\epsilon\to 0}\omega(x,\epsilon) = 0$ (see [Crandall et al. 1992]) and that $\omega(\cdot,\epsilon)$ is upper-semicontinuous. Therefore u_ϵ satisfies

$$\rho u_\epsilon - H(x, Du_\epsilon) \geq -\omega(x,\epsilon).$$

Let now $p \in D^+u_\epsilon(x)$. Then

$$p = \lim_{n\to\infty} \sum_{i=1}^{n} \lambda_i^n p_i^n, \quad \text{where} \quad \sum_{i=1}^{n} \lambda_i^n = 1, p_i^n = Du_\epsilon(x_i^n), |x_i^n - x| \leq \frac{1}{n}.$$

Using convexity of H, and lower-semicontinuity of H and $-\omega(\cdot,\epsilon)$ we now have

$$\rho u_\epsilon(x) - H(x,p) \geq \rho u_\epsilon(x) - H(x, \sum_{i=1}^{n} \lambda_i^n p_i^n) - \sigma_1(n)$$

$$\geq \sum_{i=1}^{n} \lambda_i^n \left(\rho u_\epsilon(x_i^n) - H(x_i^n, p_i^n)\right) - \sigma_2(\epsilon, n) \geq$$

$$\geq -\sum_{i=1}^{n} \lambda_i^n \omega(x_i^n, \epsilon) - \sigma_2(\epsilon,n) \geq -\omega(x,\epsilon) - \sigma_3(\epsilon,n),$$

and letting $n \to \infty$ we obtain

$$\rho u_\epsilon(x) - H(x,p) \geq -\omega(x,\epsilon) \ \text{ for } \ x \in E_{\epsilon_0}. \tag{25}$$

Let

$$\tau_{\epsilon_0} = \min\left\{\inf\{t : x(t;\bar{x},c) \notin E_{\epsilon_0}\}, T\right\}.$$

It follows from (25) that

$$\rho u_\epsilon(x(s)) - < f(x(s;\bar{x},c), c(s)), p_s > - f_0(x(s;\bar{x},c), c(s)) \geq -\sigma(x(s;\bar{x},c),\epsilon), \tag{26}$$

for $0 \leq s \leq \tau_{\epsilon_0}, p_s \in D^+u_\epsilon(x(s;\bar{x},c))$, where $\sigma(x,\epsilon) \to 0$ as $\epsilon \to 0$. Since $\bar{x}$ and c are fixed to simplify notation we will write $x(s)$ for $x(s;\bar{x},c)$. Denote $\psi(s) = u_\epsilon(x(s))$. The function ψ is in $W^{1,1}(0,\tau_{\epsilon_0})$. To see this one notices that if u_ϵ^δ are smooth mollifications of u_ϵ then $u_\epsilon^\delta(x(s))$ converges weakly in $W^{1,1}(0,\tau_{\epsilon_0})$ and also converges in $C[0,\tau_{\epsilon_0}]$ to $\psi(s)$. Let us now choose a point s_0 such that $\psi'(s_0)$ and $x'(s_0)$ exist. The semiconcavity of u_ϵ gives us

$$u_\epsilon(x) \leq u_\epsilon(x(s_0)) + < p_{s_0}, x - x(s_0) > + \frac{|x - x(s_0)|^2}{2\epsilon}$$

Therefore

$$\frac{\psi(s) - \psi(s_0)}{s - s_0} \leq (\geq) < p_s, \frac{x(s) - x(s_0)}{s - s_0} > + \frac{|x(s) - x(s_0)|^2}{2\epsilon(s - s_0)}$$

if $s > s_0$ (respectively $s < s_0$). This, together with the fact that $\psi'(s_0)$ and $x'(s_0)$ exist, yield that $\psi'(s_0) =< p_{s_0}, x'(s_0) >$ for almost all $s_0 \in (0, \tau_{\epsilon_0})$. Hence we can integrate (26) to obtain

$$u_\epsilon(\bar{x}) \geq \int_0^{\tau_{\epsilon_0}} e^{-\rho s} f_0(x(s;\bar{x},c), c(s))ds + e^{-\rho\tau_{\epsilon_0}} u_\epsilon(x(\tau_{\epsilon_0};\bar{x},c)) - \int_0^{\tau_{\epsilon_0}} \sigma(x(\tau_{\epsilon_0};\bar{x},c), c(\tau_{\epsilon_0}), \epsilon)ds.$$

Letting $\epsilon \to 0$ and using the Lebesgue dominated convergence theorem yield

$$u(\bar{x}) \geq \int_0^{\tau_{\epsilon_0}} e^{-\rho s} f_0(x(s;\bar{x},c), c(s))ds + e^{-\rho\tau_{\epsilon_0}} u(x(\tau_{\epsilon_0};\bar{x},c)).$$

We now let $\epsilon_0 \to 0$ to obtain

$$u(\bar{x}) \geq \int_0^{\tau} e^{-\rho s} f_0(x(s;\bar{x},c), c(s))ds + e^{-\rho\tau} u(x(\tau;\bar{x},c)),$$

where $\tau = \min\{\inf\{t : x(t;\bar{x},c) \in \partial E\}, T\}$. If $\tau = T$ we have

$$u(\bar{x}) \geq \int_0^{T} e^{-\rho s} f_0(x(s;\bar{x},c), c(s))ds + e^{-\rho T} u(x(T;\bar{x},c)). \tag{27}$$

If $x(\tau;\bar{x},c) \in \partial E$ then $u(x(\tau;\bar{x},c)) \geq V(x(\tau;\bar{x},c))$ and using the dynamic programming principle we arrive at

$$u(\bar{x}) \geq \int_0^{T} e^{-\rho s} f_0(x(s;\bar{x},c), c(s))ds + e^{-\rho T} V(x(T;\bar{x},c)). \tag{28}$$

If $\tau = T$ for every T then we let $T \to \infty$ in (27), otherwise we use (28). This yields $u \geq V$ in E. ■

Remark 4.5 1. In [Rustichini, 1998a] for the discrete time case a characterisation of the value function as the maximal subsolution of the discrete time Hamilton-Jacobi-Bellman equation is obtained. Such a characterisation does not seem to be possible without further restrictions in the continuous time case, as indicated below in Remark 6.2.

2. When E is open then $E \setminus \text{int}E = \emptyset$ and the value function is then the minimal supersolution. This characterisation is similar to the one obtained in [Soravia, 1997b, Theorem 4.5]. However, the assumptions used in [Soravia, 1997b, Theorem 4.5] are different from ours and moreover we deal with unbounded solutions. In fact, the characterisation provided in Proposition 4.4 seems to be the best possible in the generality stated there. In Remark 6.2 below we have explicit examples of two functions V_1 and V_2 such that $V_1 < V_2$. V_1 is the value function, however they both are viscosity solutions of the associated Hamilton-Jacobi-Bellman equation. To obtain a characterisation on the other side (as the maximal subsolution in a certain family) in this setting one should at least consider subsolutions with a prescribed growth at infinity.

■

Remark 4.6 Suppose that D is independent of c. A sufficient condition for $H = H_{\text{in}}$ in $\text{int}E$ is that $V_* > D$ in $\text{int}E$, i.e. that for every $x \in \text{int}E$ there exist $\delta > 0, \epsilon > 0$ such that

$$V(y) \geq D(y) + \delta \quad \text{for } y \in B(x, \epsilon). \tag{29}$$

To see this we take any $c \in \mathbf{C}$. We need to show that $c \in \mathcal{A}(x)$. Let $c_1(t) = c$ and let r be small enough such that if $y \in B(x, r)$ then $y(t; y, c_1) \in B(x, \epsilon/2)$ for $t \in [0, r]$,

$$\int_0^r e^{-\rho t} |f_0(y(t; y, c_1), c_1(t))| dt < \frac{\delta}{3},$$

and

$$|D(y(t; y, c_1)) - D(y)| < \frac{\delta}{6} \quad \text{for } t \in [0, r].$$

Let $c_2 \in \mathcal{C}(y(r; y, c_1))$ be such that

$$\int_r^{+\infty} e^{-\rho t} f_0(y(t; y(r; y, c_1), c_2), c_2(t)) dt > V(y(r; y, c_1)) - \frac{\delta}{3} \geq D(y(r; y, c_1)) + \frac{2\delta}{3}.$$

Define

$$c(t) = \begin{cases} c_1(t) & t \in [0, r] \\ c_2(t - r) & t > r \end{cases}$$

It is now clear that $c(\cdot) \in \mathcal{C}(y)$ since if $T \in [0, r]$ then

$$\int_T^{+\infty} e^{-\rho t} f_0(y(t; y, c), c(t)) dt > D(y(r; y, c_1)) + \frac{2\delta}{3} - \frac{\delta}{3} > D(y(T; y, c)).$$

Thus $c \in \mathcal{A}(x)$.

A sufficient condition for (29) is for instance that D is a smooth function satisfying $\rho D(x) - H_{\text{in}}(x, DD(x)) < 0$ in $\text{int}E$.

We notice that if V is lower-semicontinuous at every point of $E \setminus \text{int}E$ and $H = H_{\text{in}}$ in intE then V_* is a lower-semicontinuous supersolution of (21), $V = V_*$ on $E \setminus \text{int}E$, and so if the assumptions of Proposition 4.4 are satisfied then $V \leq V_*$ in E, i.e. V is lower-semicontinuous. Therefore, if also the assumptions of Proposition 5.7 are satisfied, the value function V is continuous in E. ∎

5 State Incentive Compatibility Constraints

In this section we will assume that the constraint map D only depends on the state variable x. As we have observed in the Introduction many incentive compatibility constraints for economic problems are of this type.

In the next four subsections we will address the following points:

- in Section 5.1 we will prove that the problem can be viewed as a state constraints problem in a region E to be determined from the data of the problem;

- in Section 5.2 we will give sufficient conditions for closedness and other properties of E;
- in Section 5.3 we will present some regularity properties of the value function;
- in Section 5.4 we will prove an existence result for optimal strategies.

5.1 Equivalence with a state constraints problem

Consider two new control problems in the region E. Given $x_0 \in E$ and $c \in \mathcal{C}$, a number τ such that $x(\tau; x_0, c) \in \partial E$ and $x(t; x_0, c) \in E$ for every $t \leq \tau$ will be called an exit time of the trajectory from E (there can be many τ like this: we will denote by $\Theta(x_0, c)$ the set of all exit times). Define the payoff function

$$J_{ET}(x_0; c, \tau) = \int_0^\tau e^{-\rho t} f_0(x(t), c(t))dt + e^{-\rho\tau} D(x(\tau))$$

and the value function

$$V_{ET}(x_0) = \sup_{c \in \mathcal{C}, \tau \in \Theta(x_0, c)} J_{ET}(x_0; c, \tau).$$

Consider moreover the state constraints problem of maximizing the payoff functional (2) in the narrower region E. For $x_0 \in E$ define the value function

$$V_{SC}(x_0) = \sup_{c \in \mathcal{C}_E(x_0)} J(x_0; c).$$

The following result is stated assuming that Assumption 2.1 holds, but it remains true also if the sets A and C are open.

Theorem 5.1 *Let Assumptions 2.1 and 2.2 hold. Then, for every $x_0 \in E$ we have*

$$V_{SC}(x_0) = V(x_0) \leq V_{ET}(x_0).$$

If E is closed, then

$$V(x_0) = V_{ET}(x_0).$$

Proof. Let us consider the first statement. Given an $x_0 \in E$ it is easy to prove that $V_{SC}(x_0) \geq V(x_0)$, in fact we have that $\mathcal{C}(x_0) \subset \mathcal{C}_E(x_0)$. This follows from the fact that $c \in \mathcal{C}(x_0)$ implies that $x(t; x_0, c) \in E$ for every $t \geq 0$. Indeed by contradiction if $x(t; x_0, c) \notin E$ then $\mathcal{C}(x(t; x_0, c)) = \emptyset$, which is impossible by Lemma 3.1.

To prove that $V_{SC}(x_0) \leq V(x_0)$ we show that for every control strategy $c \in \mathcal{C}_E(x_0)$ there exists another strategy $\bar{c} \in \mathcal{C}(x_0)$ such that $J(x_0; \bar{c}) \geq J(x_0; c)$. If $c \in \mathcal{C}(x_0)$ then there is nothing to prove. Assume that this is not the case. Then there exists $\bar{t} \geq 0$ such that

$$\int_{\bar{t}}^{+\infty} e^{-\rho t} f_0(x(t; x_0, c), c(t))dt < e^{-\rho \bar{t}} D(x(\bar{t}; x_0, c)).$$

Let t_0 be the infimum of the set of such $\bar{t}$. If $t_0 = 0$ then $J(x_0; c) \leq D(x_0) \leq J(x_0; \bar{c})$ for every $\bar{c} \in \mathcal{C}(x_0)$, where $\mathcal{C}(x_0) \neq \emptyset$ because $x_0 \in E$, and there is nothing more to prove. If $t_0 > 0$ then the continuity of f_0 and D implies

$$\int_{t_0}^{+\infty} e^{-\rho s} f_0(x(s; x_0, c), c(s)) ds = e^{-\rho t_0} D(x(t_0; x_0, c)) \tag{30}$$

and, for $t \leq t_0$,

$$\int_{t}^{+\infty} e^{-\rho s} f_0(x(s; x_0, c), c(s)) ds \geq e^{-\rho t} D(x(t_0; x_0, c)).$$

Consider a control strategy $c_{t_0} \in \mathcal{C}(x(t_0; x_0, c))$ that satisfies $J(x(t_0; x_0, c), c_{t_0}) \geq J(x(t_0; x_0, c), c)$. Such a control exists since $x(t_0; x_0, c) \in E$ and (30) holds.

Then, by Lemma 3.2, the control strategy $\bar{c}$ defined as

$$\bar{c}(t) = \begin{cases} c(t) & t \in [0, t_0) \\ c_{t_0}(t - t_0) & t \in [t_0, +\infty) \end{cases}$$

belongs to $\mathcal{C}(x_0)$. Moreover, by (30), the definition of $\bar{c}$, and the fact that $c_{t_0} \in \mathcal{C}(x(t_0; x_0, c))$, it follows that

$$J(x_0; \bar{c}) - J(x_0; c) = \int_{t_0}^{+\infty} e^{-\rho s} f_0(x(s; x_0, \bar{c}), \bar{c}(s)) ds - e^{-\rho t_0} D(x(t_0; x_0, \bar{c})) > 0$$

proving the claim that $V_{SC}(x_0) \leq V(x_0)$.

$V_{ET}(x_0)$ is greater than or equal to the values of the other two value functions. This comes from the observation that both $\mathcal{C}(x_0)$ and $\mathcal{C}_E(x_0)$ are subsets of the set of controls that we consider in the exit time problem.

To prove the second statement, assume that E is closed. Above we have shown that $V_{ET}(x_0) \geq V_{SC}(x_0) = V(x_0)$. It remains to prove that given a control strategy c with exit time $\tau < +\infty$ there exists $\bar{c} \in \mathcal{C}_E(x_0)$ such that

$$J_{ET}(x_0; c) = \int_0^{\tau} e^{-\rho s} f_0(x(s; x_0, c), c(s)) ds + e^{-\rho \tau} D(x(\tau; x_0, c)) \leq J(x_0; \bar{c}).$$

To find such $\bar{c}$ we first consider a control strategy $c_\tau \in \mathcal{C}(x(\tau; x_0, c))$ (such c_τ always exists since, by the closedness of E we have $x(\tau; x_0, c) \in E$). We then define

$$\bar{c}(t) = \begin{cases} c(t) & t \in [0, \tau) \\ c_\tau(t - \tau) & t \in [\tau, +\infty). \end{cases}$$

Since $\mathcal{C}(x(\tau; x_0, c)) \subset \mathcal{C}_E(x(\tau; x_0, c))$ it is clear that $\bar{c} \in \mathcal{C}_E(x_0)$. Moreover, since $c_\tau \in \mathcal{C}(x(\tau; x_0, c))$,

$$J(x(\tau; x_0, c); c_\tau) = \int_0^{+\infty} e^{-\rho s} f_0(x(s; x(\tau; x_0, c), c_\tau), c_\tau(s)) ds \geq D(x(\tau; x_0, c))$$

so that, by the definition of c_τ,

$$J(x_0; \bar{c}) = \int_0^{+\infty} e^{-\rho s} f_0(x(s; x_0, \bar{c}), \bar{c}(s)) ds$$

$$= \int_0^{\tau} e^{-\rho s} f_0(x(s;x_0,c),c(s))ds + \int_{\tau}^{+\infty} e^{-\rho s} e^{-\rho\tau} f_0(x(s-\tau;x(\tau;x_0,c),c_\tau),c_\tau(s-\tau))ds$$

$$= \int_0^{\tau} e^{-\rho s} f_0(x(s;x_0,c),c(s))ds + e^{-\rho\tau} J(x(\tau;x_0,c);c_\tau) \geq J_{ET}(x_0;c,\tau)$$

which completes the proof. ■

Remark 5.2 In light of the above theorem, when D is independent of c, the Hamilton-Jacobi-Bellman equation related to the incentive compatibility constrained problem coincides with the one obtained for the problem constrained in the region E. However this does not help resolve the issue of uniqueness of solutions. We also observe that the above theorem gives a condition under which the state constraints problem and the exit time (or stopping time) problem have the same value function. ■

Remark 5.3 From the proof of Theorem 5.1 above (see the second paragraph of the proof) it follows that, given $x \in E$, if $c^* \in \mathcal{C}_E(x)$ is optimal for the state constraints problem in E, then $c^* \in \mathcal{C}(x)$ and so is optimal also for the incentive constrained problem. This fact will be useful in the analysis of examples in Section 6. ■

5.2 Properties of the set E

We now give a sufficient condition for the closedness of E. Obviously, as recalled in Section 2, $E \subset \{x : V_u(x) \geq D(x)\} \subset A$, and $V_u(x) \geq V(x)$ for $x \in E$.

Proposition 5.4 *If Assumptions 2.1, 2.2 and 2.4, (i)-(ii) are satisfied, then E is closed.*

Proof. We use some arguments related to the well known "Fillipov existence theorem" (see e.g. [Cesari 1983, Ch. 9]).

First we consider a sequence $(x_n)_{n\in\mathbb{N}} \subset E$ converging to a point $x_\infty \in A$. Let $c_n \in \mathcal{C}_E(x_n)$ be an admissible control for x_n provided by Assumption 2.4-(ii). Then we take the equivalent form of our problem by introducing the variable w as in (7) and for every $n \in \mathbb{N}$ we consider the trajectory

$$(x^n, w^n) \stackrel{def}{=} (x,w)(\cdot;x_n,c_n) \in W^{1,1}_{loc}([0,+\infty);\mathbb{R}^{n+1}).$$

Assumptions 2.2, (8) and (13) give equiboundedness and equicontinuity of the sequence $\{(x^n,w^n)\}_{n\in\mathbb{N}}$ and equintegrability of the sequence $\{((x^n)',(w^n)')\}_{n\in\mathbb{N}}$. We now use the Ascoli-Arzelà Theorem to say that, along a suitable subsequence, we have the uniform convergence on bounded subsets of $[0,+\infty)$:

$$x^n \longrightarrow x^\infty(\cdot), \qquad w^n \longrightarrow w^\infty(\cdot),$$

where $x_\infty \in C^0([0,+\infty);\mathbb{R}^n)$ and $w_\infty \in C^0([0,+\infty);\mathbb{R})$. Moreover, by the Dunford-Pettis Theorem (see e.g. [Cesari 1983, p. 329]) we obtain that, along a further subsequence if necessary, we have the L^1-weak convergence on bounded subsets of $[0,+\infty)$:

$$(x^n)' \longrightarrow \xi(\cdot), \qquad (w^n)' \longrightarrow \eta(\cdot),$$

where $\xi_\infty \in L^1_{loc}([0,+\infty);\mathbb{R}^n)$ and $\eta \in L^1_{loc}([0,+\infty);\mathbb{R})$. By integrating we get

$$x^n(t) = x_n + \int_0^t (x^n)'(s)ds$$

so that, passing to the limit we obtain

$$x^\infty(t) = x_\infty + \int_0^t \xi(s)ds$$

so that $x^\infty \in W^{1,1}_{loc}([0,+\infty);\mathbb{R}^n)$.

On the other hand, setting $z^n(t) = e^{-\rho t}w^n(t)$, we have $(z^n)'(t) = e^{-\rho t}f_0(x^n(t), c_n(t))$ and so, by (9), $\int_{t_1}^{+\infty} |(z^n)'(t)|dt \to 0$ as $t_1 \to +\infty$, uniformly for $n \in \mathbb{N}$. We now claim that $(z^n)'$ converges weakly in $L^1([0,+\infty);\mathbb{R})$ to the function $t \to \zeta(t) = e^{-\rho t}\eta(t)$. Indeed, by the weak convergence of w^n above we have that

$$(z^n)' \longrightarrow \zeta,$$

L^1-weakly on bounded subsets of $[0,+\infty)$. Then for any $t_1 > 0$

$$\int_0^{t_1} |\zeta(t)|dt \le \liminf_{n\to+\infty} \int_0^{t_1} |(z^n)'(t)|dt$$

but the right hand side is bounded uniformly for $t_1 > 0$ by (9), so that $\zeta \in L^1([0,+\infty);\mathbb{R})$. Moreover, given $g \in L^\infty([0,+\infty);\mathbb{R})$ we have that for every $t_1 > 0$

$$\left|\int_0^{+\infty} g(t)\left[(z^n)'(t) - \zeta(t)\right]dt\right| \le \left|\int_0^{t_1} g(t)\left[(z^n)'(t) - \zeta(t)\right]dt\right| + \|g\|_\infty \int_{t_1}^{+\infty} \left[|(z^n)'(t)| + |\zeta(t)|\right]dt$$

and the claim follows by taking first t_1 sufficiently big and then letting $n \to +\infty$.

Having the weak convergence of $(z^n)'$ in $L^1([0,+\infty);\mathbb{R})$ allows us to pass to the limit as $n \to +\infty$ in the relation

$$z^n(t) = \int_t^{+\infty} (z^n)'(s)ds$$

obtaining (defining $z^\infty(t) = e^{-\rho t}w^\infty(t)$)

$$z^\infty(t) = \int_t^{+\infty} \zeta(s)ds,$$

so that $z^\infty \in W^{1,1}([0,+\infty);\mathbb{R})$ and $(z^\infty)' = \zeta$ which implies that $w^\infty \in W^{1,1}_{loc}([0,+\infty);\mathbb{R})$, and $(w^\infty)' = \eta$.

At this point we can apply the so-called Closure Theorem (see e.g. [Cesari 1983, p.299 or p.340]) to say that the trajectory (x^∞, w^∞) satisfies the differential inclusion

$$(x^\infty)'(t) \in f(x^\infty(t), \mathbf{C}); \qquad (w^\infty)'(t) \in \rho w^\infty - f_0(x^\infty(t), \mathbf{C}).$$

The hypotheses needed to apply the Closure Theorem of [Cesari 1983, p.299]) require that the set valued map $E \times \mathbb{R} \mapsto E \times \mathbb{R}$

$$(x, w) \to F(x, w) \stackrel{def}{=} (f(x, \mathbf{C}), \rho w - f_0(x, \mathbf{C})) \tag{31}$$

satisfies the so-called property (Q). This property holds when (see again [Cesari 1983, p.293, 8.5.iv]) Assumption 2.4(i) is satisfied and the map in (31) is upper semicontinuous by set inclusion. This property [Cesari 1983, p.291] says that, for every $(x_0, w_0) \in E \times \mathbb{R}$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\bigcup_{(x,w)\in B((x_0,w_0),\delta)} F(x,w) \subset \{(x,w) \text{ such that } d((x,w), F(x_0,w_0)) < \varepsilon\}$$

and this is clearly automatically satisfied when Assumption 2.1 holds. We remark that the result [Cesari 1983, p.293, 8.5.iv] holds only for autonomous set valued maps. This is the reason why we did not use z as a state variable.

Now we can apply a theorem about existence of measurable selectors for solutions of differential inclusions (see [Cesari 1983, p.278, 8.2.iii]) which gives that there exists an admissible strategy c^∞ such that

$$x^\infty(t) = xt; x_\infty, c^\infty) \qquad w^\infty(t) = \int_t^{+\infty} e^{-\rho(s-t)} f_0(x^\infty(s), c^\infty(s))ds.$$

It is now enough to prove that all the constraints are satisfied for (x_∞, w_∞), but this is an immediate consequence of the uniform convergence stated above. ∎

The following proposition about the convexity of E is elementary and will be useful in treating some examples. It also holds when the sets A and C are not closed.

Proposition 5.5 *Let Assumptions 2.1, 2.2 hold. Assume moreover that A is convex, f_0 is concave in (x, c), f is linear, and D is convex. Then E is convex.*

Proof. Let $x_1, x_2 \in E$, and let $c_1 \in \mathcal{C}(x_1)$, $c_2 \in \mathcal{C}(x_2)$. Take $\lambda \in (0,1)$ and set $x_\lambda = \lambda x_1 + (1-\lambda)x_2$, $c_\lambda = \lambda c_1 + (1-\lambda)c_2$. First of all we observe that $x_\lambda \in A$ since A is convex. Then it is enough to prove that $c_\lambda \in \mathcal{C}(x_\lambda)$. By the linearity of f we have that for every $t \geq 0$

$$x(s; x_\lambda, c_\lambda) = \lambda x(s; x_1, c_1) + (1-\lambda)x(s; x_2, c_2)$$

so that, by the concavity of f_0

$$\int_t^{+\infty} e^{-\rho(s-t)} f_0(x(s; x_\lambda, c_\lambda), c_\lambda(s))ds \geq$$

$$\lambda \int_t^{+\infty} e^{-\rho(s-t)} f_0(x(s; x_1, c_1), c_1(s))ds + (1-\lambda) \int_t^{+\infty} e^{-\rho(s-t)} f_0(x(s; x_2, c_2), c_2(s))ds$$

and, by the admissibility of c_1, c_2 and the convexity of D,

$$\geq \lambda D(x(t; x_1, c_1)) + (1-\lambda)D(x(t; x_2, c_2)) \geq D(x(s; x_\lambda, c_\lambda)).$$

which gives the claim. ∎

For the one dimensional case we have also the following result whose proof we omit since the arguments it uses are similar to the ones of the proof of Proposition 5.5.

Proposition 5.6 *Assume that $d = 1$ and that Assumptions 2.1, 2.2 hold.*

1. *Assume moreover that A is convex, f_0 is concave in (x, c) and increasing in x, f is concave, D is convex and decreasing. Then E is convex.*

2. *Assume moreover that f_0 is increasing in x and D is decreasing. Then $x \in E$ implies that $y \in E$ for every $y \geq x$, $y \in A$.*

5.3 Regularity of the value function

Proposition 5.7 *Let Assumptions 2.1, 2.2 and 2.4-(iii) be satisfied. Then V is upper semicontinuous.*

Proof. Let $x_n \to x_0$. We will prove that $\limsup_{n\to+\infty} V(x_n) \leq V(x_0)$ i.e. that $\limsup_{n\to+\infty} V_{SC}(x_n) \leq V_{SC}(x_0)$ thanks to Theorem 5.1. First we observe that we can restrict our attention to the case when $x_n \in E$ for every $n \in \mathbb{N}$ and $x_0 \in E$. We then take a sequence of control strategies $c_n \in \mathcal{C}_E(x_n)$ such that

$$J(x_n, c_n) > V(x_n) - \frac{1}{n}$$

and let $(x^n, w^n) = (x, w)(\cdot; x_n, c_n)$ be the associated trajectories. By applying the same arguments as those used in the proof of Proposition 5.4 we obtain that (x^n, w^n) converges uniformly along a subsequence to an element $(x^\infty, w^\infty) \in W^{1,1}_{loc}([0, +\infty); \mathbb{R}^{n+1})$ which is still associated to an admissible strategy c_∞. Then

$$\limsup_{n\to+\infty} V(x_n) \leq \limsup_{n\to+\infty} w^n(0) + \frac{1}{n} = w^\infty(0) \leq V(x_0),$$

which gives the claim. ■

Remark 5.8 In general we cannot expect continuity of the value function. In fact also in the state constraint case (which is a special case of our problem when $D \leq J$ for all admissible strategies) there are examples where the value function is not lower semicontinuous, even under the assumptions of the above proposition (see e.g. [Soravia, 1997b, Ex. 4.3]). However, using Remark 4.6 above, we can prove continuity in some cases (see Section 6). ■

The following proposition holds under the assumptions of Proposition 5.5.

Proposition 5.9 *Let the Assumptions of Proposition 5.5 be satisfied. Then the value function is concave.*

Proof. Let $x_1, x_2 \in E$ and $\lambda \in (0,1)$ be given. By the definition of the value function we have that, for every $\varepsilon > 0$ there exists $c_{1\varepsilon} \in \mathcal{C}(x_1)$, $c_{2\varepsilon} \in \mathcal{C}(x_2)$ such that

$$\lambda V(x_1) + (1-\lambda)V(x_2) < \lambda J(x_1, c_{1\varepsilon}) + (1-\lambda)J(x_2, c_{2\varepsilon}) + 2\varepsilon,$$

and by concavity of f_0

$$\lambda V(x_1)+(1-\lambda)V(x_2) \leq \int_0^{+\infty} e^{-\rho t} f_0(\lambda x(t;x_1,c_{1\varepsilon})+(1-\lambda)x(t;x_2,c_{2\varepsilon}), \lambda c_{1\varepsilon}(t)+(1-\lambda)c_{2\varepsilon}(t))dt+2\varepsilon.$$

Then, by the linearity of f, setting $x_\lambda = \lambda x_1 + (1-\lambda)x_2$, $c_{\lambda\varepsilon} = \lambda c_{1\varepsilon} + (1-\lambda)c_{2\varepsilon}$ and recalling that $c_{\lambda\varepsilon} \in \mathcal{C}(x_\lambda)$, we have thanks to Proposition 5.5

$$\lambda V(x_1) + (1-\lambda)V(x_2) \leq \int_0^{+\infty} e^{-\rho t} f_0(x(t;x_\lambda, c_{\lambda\varepsilon}), c_{\lambda\varepsilon}(t))dt + 2\varepsilon \leq V_c(x_\lambda) + 2\varepsilon.$$

The claim follows by the arbitrariness of ε. ∎

For the one dimensional case we have the following result whose proof is similar to the last one and we omit it for brevity.

Proposition 5.10 *Assume that $d = 1$ and that Assumptions 2.1 and 2.2 hold.*

1. *Assume moreover that A is convex, f_0 is concave in (x,c) and increasing in x, f is concave, D is convex and decreasing. Then V is concave.*
2. *Assume moreover that f_0 is increasing in x and D is decreasing. Then $x, y \in E$ and $y \geq x$ implies that $V(y) \geq V(x)$.*

5.4 An existence result for optimal strategies

We state and sketch the proof of an existence result for optimal strategies.

Theorem 5.11 *Let Assumptions 2.1, 2.2 and 2.4-(iii) be satisfied. Then for every $x_0 \in E$ there exists an optimal strategy for the constrained problem.*

Proof. We again use a modification of the well-known Fillipov Theorem adapted to our problem in the formulation given at the end of Section 2. We only sketch the main points of the proof since they are based on the arguments used in the proof of Proposition 5.4.

Let $x_0 \in E$, $(c_n)_{n\in\mathbb{N}} \subset \mathcal{C}_E(x_0)$ be a maximizing control sequence for our problem starting at x_0 and satisfying Assumption 2.4-(iii). Let then $(x^n, w^n) = (x,w)(\cdot; x_0, c_n)$ be the associated trajectories. Arguing as in the proof of Proposition 5.4 we obtain that (x^n, w^n) converges uniformly along a subsequence to $(x^\infty, w^\infty) \in W^{1,1}_{loc}([0,+\infty); \mathbb{R}^{n+1})$ for which there is an admissible strategy c_∞. Then

$$V(x_0) = \lim_{n\to+\infty} J(x_0; c_n) = \lim_{n\to+\infty} w^n(0) = w^\infty(0) \leq V(x_0).$$

and so c_∞ is optimal for the state constraints problem in E. The claim follows by applying Remark 5.3. ∎

Remark 5.12 If f_0 is independent of x, strictly concave in c, and the unconstrained problem is closed under convex combinations then the optimal strategy is unique (see Section 6).

∎

6 Examples

In this Section we analyze two simple economic examples. The first one is a consumption problem, the second one is an investment problem with adjustment costs. In both cases we assume that the incentive compatibility constraint is given by a constant. The point of view is the one of a social planner who wants to analyze the optimal capital accumulation in an economy where the private sector can stop the process at any time in the future and go abroad if the future utility/profit is smaller than a given constant. The social planner wants to define the optimal capital accumulation path among the policies which do not lead the agents to break the contract terminating the accumulation process. The incentive compatibility constraint is set equal to a constant to fully carry out the analysis. In a policy perspective we can think of the constant as a control variable (tax) managed by the government at $t = 0$, e.g. a sunk cost to run the firm, etc.. The second best policy defined below can be conceived as a social contract that the government proposes to the private sector taking into account his opportunity to go abroad.

6.1 Consumption with a fixed Positivity Constraint on the Value Function

We consider, for $\alpha \in (0,1)$, the problem:

$$\max J(c) = \max \int_0^{+\infty} e^{-\rho t} \frac{c(t)^\alpha}{\alpha} dt$$

$$\dot{x}(t) = ax(t) - c(t), \quad x(0) = x_0$$

subject to the usual constraints $c \geq 0$, $x \geq 0$ and to the following incentive compatibility constraint on the payoff function:

$$\int_t^{+\infty} e^{-\rho(s-t)} \frac{c^\alpha(s)}{\alpha} ds \geq D(x(t; x_0, c)) \quad \forall t \geq 0. \tag{32}$$

The function D is assumed to be equal to a positive constant $D_0 > 0$.

This is a classical optimal consumption problem, c denotes consumption and x denotes the stock of capital, a is the coefficient of the linear technology, $a > 0$, i.e. the istantaneous interest rate. The constraint (32) says that the continuation value should be greater than a fixed positive level D_0. At every time the agent wants a utility in the future bigger than this constant.

The value function of the unconstrained problem is finite for every starting point $x_0 \geq 0$ if and only if $\alpha a < \rho$ (see e.g. [Fleming and Soner, 1993, p.29]). We will assume that this holds from now on. The HJB equation is

$$\rho V(x) - H\left(x, DV(x)\right) = 0, \quad x \geq 0,$$

where the maximum value Hamiltonian H is given by

$$H(x,p) = \sup_{c \geq 0} \left\{ (ax - c)p + \frac{c^\alpha}{\alpha} \right\} = axp + H_0(p)$$

with

$$H_0(p) = \begin{cases} +\infty & \text{if } p \leq 0 \\ \frac{1-\alpha}{\alpha} p^{\frac{\alpha}{\alpha-1}} & \text{if } p > 0 \end{cases}.$$

The maximum is reached at $c^* = p^{1/(\alpha-1)}$. Thanks to Remark 4.6 (take for instance $D \equiv -1$) we have that $H_{in} = H$ inside the region $A = \mathbb{R}^+$, and $H_{in} = 0$ on the boundary of the region $A = \mathbb{R}^+$ (in fact in this case it is easy to see that $\mathcal{A}(0) = \{0\}$ which implies $H_{in} = 0$ on ∂A).

It is natural for our problem to look for solutions with positive derivative. In this case the HJB equation for the unconstrained problem becomes

$$\rho V(x) - axDV(x) + \frac{1-\alpha}{\alpha}[DV(x)]^{\frac{\alpha}{\alpha-1}} = 0,$$

and the value function satisfies $V(0) = 0$, which in light of the above remarks about the Hamiltonians is a necessary condition for it to solve the equation in the viscosity sense.

It can be easily checked that the value function is a classical solution of the above equation and is given by $V_u(x) = bx^\alpha$ where $b = \frac{1}{\alpha}\left[\frac{\rho - a\alpha}{1-\alpha}\right]^{\alpha-1}$. In feedback form, the optimal policy is $c^*(t) = b_1 x^*(t)$, where $b_1 = \frac{\rho - a\alpha}{1-\alpha}$. The optimal trajectory x^* starting at a given point x_0 is given by $x^*(t) = e^{(a-b_1)t}x_0$ so that $c^*(t) = b_1 e^{(a-b_1)t}x_0$.

Let us now consider the constrained case. First we observe that V_u is greater than D_0 when

$$x \geq x^0 = \left[\frac{D_0}{b}\right]^{\frac{1}{\alpha}} > 0.$$

Therefore we have $E \subset \{x_0 : V_u(x_0) \geq D_0\} = [x^0, +\infty)$. We have the following result, depending on the value of ρ/a.

Proposition 6.1

(i) If $a \geq \rho$ (which is equivalent to $a \geq b_1$) we have that $E = [x^0, +\infty)$ and $V_u = V$ in E (otherwise we have $V = -\infty$). The optimal policy for the unconstrained problem is also optimal for the constrained one.

(ii) If $\rho > a$ (which is equivalent to $a < b_1$) then the value function V is smaller than V_u. In this case $E = [\overline{x}, +\infty)$, where

$$\overline{x} = \frac{(\rho\alpha D_0)^{1/\alpha}}{a} > x^0$$

and there exists a unique optimal policy given by $\overline{c}^*(t) = c(x_0)e^{(a-b_1)t}$ *(where* $c(x_0) < b_1 x_0$*) until the point* $x = \overline{x}$ *is reached and then by* $\overline{c}^*(t) = (\rho\alpha D_0)^{1/\alpha}$.

Proof. If $a \geq \rho$, then the optimal policy for the unconstrained problem is admissible for the constrained one for every initial datum $x_0 \geq x^0$ since

$$\int_t^{+\infty} e^{-\rho(s-t)}\frac{[c^*(s)]^\alpha}{\alpha}ds = \frac{b_1^\alpha x_0^\alpha e^{\alpha(a-b_1)t}}{\alpha[\rho-\alpha(a-b_1)]} \geq \frac{b_1^\alpha x_0^\alpha}{\alpha[\rho-\alpha(a-b_1)]} = \frac{b_1^{\alpha-1}}{\alpha}x_0^\alpha = V_u(x_0) \geq D_0.$$

If $\rho > a$ then it can easily be shown by looking at the above calculations that the optimal policy for the unconstrained problem $c(t) = b_1 x(t)$ violates the constraint (32) for large $t > 0$, so that the value function V is smaller than V_u. From Proposition 5.4 it follows that E is closed (Assumption 2.4 (ii) is satisfied by simply taking the constant control $c \equiv ay$ at every starting point $y \in E$). From Proposition 5.6 it follows that E is convex and that if $x \in E$ then also any $y \geq x$ belongs to E. This means that E is either empty or $E = [\overline{x}, +\infty)$ for some $\overline{x} \in (0, +\infty)$. Now it is clear that the point $(\rho\alpha D_0)^{1/\alpha}/a$ belongs to E since for this point the constant control $c \equiv (\rho\alpha D_0)^{1/\alpha}$ satisfies the incentive constraint. Indeed, for every $t \geq 0$ we have

$$\int_t^{+\infty} e^{-\rho(s-t)}\frac{\rho\alpha D_0}{\alpha}ds = D_0.$$

This fact implies that E is not empty and that $\overline{x} \in [x^0, (\rho\alpha D_0)^{1/\alpha}/a]$. To determine $\overline{x}$ we use the Maximum Principle for state constraints optimal control problems. From Theorem 5.1 we know that the problem

$$\max J(c) = \max \int_0^{+\infty} e^{-\rho t}\frac{c(t)^\alpha}{\alpha}dt,$$

$$\dot{x}(t) = ax(t) - c(t), \quad x(0) = x_0$$

$$\int_t^{+\infty} e^{-\rho(s-t)}\frac{c(s)^\alpha}{\alpha}ds \geq D_0 \quad \forall t > 0 \tag{33}$$

corresponds to the following state constraints control problem in the half line $E = [\overline{x}, +\infty)$:

$$\max J(c) = \int_0^{+\infty} e^{-\rho t}\frac{c(t)^\alpha}{\alpha}dt$$

$$\dot{x}(t) = ax(t) - c(t), \quad x(0) = x_0, \quad x(t) \geq \overline{x}.$$

It can be shown that this problem has a unique solution by applying a slight modification of the arguments used to prove Theorem 5.11 (we omit the technicalities here) and Remark 5.12. However the existence of an optimal strategy can also be proved by checking the sufficient conditions for optimality (see e.g. [Hartl, Sethi and Vickson, 1995]). The current value Hamiltonian and the Lagrangean are, respectively

$$H(x, c, p) = \frac{c^\alpha}{\alpha} + p(ax - c), \qquad L(x, c, p; \mu) = H(x, c, p) + \mu(x - \overline{x}),$$

the maximum value Hamiltonian is

$$H_0(x,p) = axp + \frac{1-\alpha}{\alpha} p^{\alpha/(\alpha-1)}$$

A sufficient condition for (x,c) to be an optimal pair starting at a given point x_0 is that there exists a continuous, piecewise differentiable function $p : \mathbb{R}^+ \mapsto \mathbb{R}^+$, and a piecewise continuous function $\mu : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that

$$\dot{p}(t) = (\rho - a)p(t) - \mu(t),$$

$$\dot{x}(t) = ax(t) - c(t), \quad x(0) = x_0,$$

$$x(t) \geq \overline{x}, \quad \mu(t) \geq 0, \quad \mu(t)(x(t) - \overline{x}) = 0,$$

with the transversality condition $\lim_{t\to+\infty} e^{-\rho t} p(t)x(t) = 0$, where

$$c(t) = p(t)^{\frac{1}{\alpha-1}},$$

see [Hartl, Sethi and Vickson, 1995].

Assume by contradiction that $\overline{x} < (\rho\alpha D_0)^{1/\alpha}/a$. Setting the starting point $x_0 = \overline{x}$, it is easy to check that the control $c \equiv a\overline{x}$ satisfies the above conditions if we have $p(t) \equiv (a\overline{x})^{\alpha-1}$ and $\mu(t) \equiv (\rho - a)p(t) > 0$. This control is then optimal for the state constrained problem in E. But from Remark 5.3 such a control is also optimal for the incentive constrained problem. Since it gives a value of the functional strictly less than D_0 we get a contradiction. Therefore $\overline{x} = (\rho\alpha D_0)^{1/\alpha}/a$.

The optimal strategy for a given starting point $x_0 > \overline{x}$ can be found implicitly by solving the above system by straightforward arguments that we outline below. From the equation for the costate p we get that, before touching the boundary we have $p(t) = p(0)e^{(\rho-a)t}$. It then follows that, before touching the boundary, the optimal control strategy has the form $\overline{c}^*(t) = c(x_0)e^{(a-b_1)t}$ where $c(x_0) = p(0)^{1/(\alpha-1)}$. The optimal state trajectory is given by

$$\overline{x}^*(t) = \left[x_0 - \frac{c(x_0)}{b_1}\right] e^{at} + \frac{c(x_0)}{b_1} e^{(a-b_1)t}.$$

It can be shown that the state-costate pair (x,p) satisfies the transversality condition and the continuity of p only when $c(x_0) < b_1 x_0$ and

$$c(x_0)\left[b_1 x_0 - c(x_0)\right]^{-1+b_1/a} = a\overline{x}\left[(b_1 - a)\overline{x}\right]^{-1+b_1/a}$$

(it is enough to impose that x decreases and that at the time $\overline{t}$ when $x(\overline{t}) = \overline{x}$, p is continuous). From the uniqueness it follows that this is the optimal trajectory. ∎

The incentive compatibility constraint affects the agent's saving process. If the interest rate is higher than the discount rate then the first best solution satisfies the incentive comaptibility constraint, but a large initial stock of capital is needed to have a solution. So, taking into account the fact that the consumer can go abroad in the presence of a fixed cost, then the second best solution gives the first best solution if the interest rate is higher than the discount rate provided that the initial stock of capital is large enough, otherwise a solution does not exist. In this case the economy is so rewarding that the agents do not want to go away. If the discount rate is higher than the interest rate then the second best solution foresees a slower rate of consumption than the first best solution and the constrained value function is smaller then the unconstrained value function. Moreover a solution for the constrained problem exists provided that the initial stock of capital is larger than a minimal stock of capital which is higher than the one needed in the first case. If this does not happen then there is no second best solution.

The Hamilton Jacobi-Bellman equation for the constrained problem is

$$\rho V(x) - H_0(x, DV(x)) = 0, \tag{34}$$

where, similarly to the unconstrained case we have that $H_{in} = H$ inside the region E and $H_{in} = \rho D_0$ on ∂E, see Remark 4.6. The value function satisfies $V(\overline{x}) = D_0$ which again is necessary for V to be a viscosity supersolution. The equation is therefore the same as the one for the unconstrained case but it is defined on a different set. However, if $\rho \leq a$ we have $V_u = V$, whereas for $\rho > a$ we have $V_u > V$.

Remark 6.2 The above HJB equation for the constrained problem does not even have in general a unique classical solution. Indeed, if we allow for linearly growing solutions, then in the case $\rho = a = 1$, $\alpha = 1/2$ and $D_0 = 2$, the HJB equation becomes (since $x^0 = 1$ here)

$$V(x) - xDV(x) - \frac{1}{DV(x)} = 0 \tag{35}$$

and $V(1) = 2$. It is easy to check that the functions $V_1(x) = 2x^{1/2}$ and $V_2(x) = x + 1$ are both solutions of (35). The function V_1 is the value function. See e.g. [Soravia, 1997a, Soravia, 1997b] for an analysis of the nonuniqueness of solutions of Hamilton-Jacobi equations arising in state constraints optimal control problems. ∎

Remark 6.3 Despite the lack of uniqueness of solutions of the HJB equation (34), we have that the value function is always a viscosity solution of (34) and that it is characterized as in Proposition 4.4. Moreover, by Remark 4.6, V is continuous. The sufficient conditions for the optimal control allow us in some cases to compute the value function. ∎

6.2 Optimal Investment with a fixed cost

Let us consider the classical optimal investment problem with quadratic adjustment costs and a linear technology:

$$\max J(k_0; u) = \max \int_0^{+\infty} e^{-\rho t}[ak(t) - bu(s) - \frac{c}{2}u^2(t)]dt,$$

$$\dot{k}(t) = u(t) - \mu k(t), \quad k(0) = k_0,$$

$a > 0,\ b > 0,\ c > 0$, subject to the usual constraint $k \geq 0$ and to the incentive constraint

$$\int_t^{+\infty} e^{-\rho(s-t)}[ak(s) - bu(s) - \frac{c}{2}u^2(s)]ds \geq \overline{D} \quad \forall t > 0. \tag{36}$$

u denotes investments and k is the stock of capital. The constant $\overline{D} > 0$ represents a fixed cost to run the firm.

Set $\overline{a} = a/(\rho + \mu)$, the expected return from a unit of capital. Assuming that $\overline{a} \geq b$ (which means that investments are profitable) and choosing measurable control strategies u such that $t \mapsto e^{-\rho(t)}u^2(t)$ are square integrable we get that the optimal control-state trajectory for the unconstrained problem is

$$u^*(t) \equiv \frac{1}{c}\left[\overline{a} - b\right], \quad k^*(t) = \frac{u^*}{\mu} + e^{-\mu t}\left[k_0 - \frac{u^*}{\mu}\right], \tag{37}$$

and the unconstrained value function is

$$V_u(k_0) = \frac{ak_0}{\rho + \mu} + \frac{1}{2c\rho}\left[\overline{a} - b\right]^2.$$

Recalling that the current value Hamiltonian is defined as

$$F_0(k,p,u) = (-\mu k + u)p + ak - bu - \frac{c}{2}u^2 = [-\mu kp + ak] + \left[up - bu - \frac{c}{2}u^2\right] \stackrel{def}{=} F_{01}(k,p) + F_{02}(p;u)$$

and the maximum value Hamiltonian as

$$H_0(k,p) = \sup_{u \in \mathbb{R}} F_0(k,p;u) = [-\mu kp + ak] + \left[\frac{(p-b)^2}{2c}\right] \stackrel{def}{=} H_{01}(k,p) + H_{02}(p),$$

where the maximum point is reached at $u = (p-b)/c$, we observe that V_u can be written as

$$V_u(k_0) = \overline{a}k_0 + \frac{1}{\rho}H_{02}\left(\overline{a}\right).$$

V_u is a viscosity solution (here also classical) of the HJB equation

$$\rho V(k) = -\mu k DV(k) + ak + H_{02}(DV(k)); \qquad k \geq 0.$$

The set $\left\{k : V_u(k) \geq \overline{D}\right\}$ surely contains the admissible region E for the constrained problem. The set is given by

$$[\widehat{k}, +\infty), \quad \text{where} \quad \widehat{k} = \max\left\{0, \frac{1}{\overline{a}}\left[\overline{D} - \frac{1}{\rho}H_{02}\left(\overline{a}\right)\right]\right\}.$$

We now describe the behavior of the optimal trajectories for the constrained problem. We will not discuss the issue of the HJB equation for the constrained problem, mentioning only that all of the results of Sections 4 and 5.3 can be applied.

Proposition 6.4 *Depending on the value of* $\overline{D}$ *we have* *(i) If*

$$\overline{D} \le \frac{1}{\rho} H_{02}\left(\overline{a}\right)$$

then for every $k_0 \ge 0$ *the optimal strategy* u^* *for the unconstrained problem is still admissible and optimal for the constrained one. Therefore* $E = A = [0, +\infty)$ *and* $V_u = V$.

(ii) If

$$\frac{1}{\rho} H_{02}\left(\overline{a}\right) < \overline{D} \le \frac{\overline{a}u^*}{\mu} + \frac{1}{\rho} H_{02}\left(\overline{a}\right)$$

then for every $k_0 \ge \widehat{k} > 0$ *the optimal strategy* u^* *for the unconstrained problem is still admissible and optimal for the constrained one. For* $k_0 < \widehat{k}$ *we have* $V_u(k_0) < \overline{D}$ *and* $V(k_0) = -\infty$. *Hence* $E = [\widehat{k}, +\infty) \subset A$ *and* $V_u = V$ *on* E.

(iii) If

$$\frac{\overline{a}u^*}{\mu} + \frac{1}{\rho} H_{02}\left(\overline{a}\right) < \overline{D} \le \frac{\overline{a}u^*}{\mu} + \frac{1}{\rho} H_{02}\left(\overline{a}\right) + \frac{\overline{a}^2 \rho}{2c\mu^2}$$

then the optimal policy for the unconstrained problem is no more admissible for the constrained one, no matter what the starting point k_0 *is. In this case* $E = [\overline{k}, +\infty)$ *where* $\overline{k}$ *increases with* $\overline{D}$, $\overline{k} > \hat{k} > u^*/\mu$ *(the equality holds when* $\overline{D} \to (\overline{a}u^*/\mu) + H_{02}(\overline{a})/\rho$*),* $\overline{k} \le (u^*/\mu) + \overline{a}\rho/(c\mu^2)$ *(the equality holds when* $\overline{D} = (\overline{a}u^*/\mu) + [H_{02}(\overline{a})/\rho] + \overline{a}^2\rho/(2c\mu^2)$*). Moreover* $\overline{k}$ *is the smallest solution of the equation*

$$\left[\overline{a} + \frac{c\mu u^*}{\rho}\right] - H_{02}\left(\mu\overline{k}\right) = \overline{D}.$$

The optimal strategy $\overline{u}^*$ *for the constrained problem is greater than the one for the unconstrained problem since we have*

$$\overline{u}^*(t) = u^* + c(k_0)e^{(\rho+\mu)t}$$

for a given $c(k_0) > 0$. *The optimal state trajectory is decreasing till it hits the point* $\overline{k}$ *and then it is constant. It is always greater (with its derivative, too) than the optimal trajectory for the unconstrained problem.*

(iv) If

$$\overline{D} > \frac{\overline{a}u^*}{\mu} + \frac{1}{\rho} H_{02}\left(\overline{a}\right) + \frac{\overline{a}^2 \rho}{2c\mu^2}$$

then E *is empty and there is no solution for the constrained problem.*

Proof. We only sketch the proof since the main arguments are similar to the ones used in the proof of Proposition 6.1. Setting for $t \ge 0$

$$J(t, k_0; u) = \int_t^{+\infty} e^{-\rho(s-t)}[ak(s) - bu(s) - \frac{c}{2}u^2(s)]ds,$$

by easy calculations we obtain

$$J(t,k_0;u) = \overline{a}k(t) + \int_t^{+\infty} e^{-\rho(s-t)}\left[(\overline{a}-b)\,u(s) - \frac{c}{2}u^2(s)\right] ds =$$

$$\overline{a}k(t) + \int_t^{+\infty} e^{-\rho(s-t)} F_{02}\left(\overline{a}, u(s)\right) ds$$

so that

$$J(t,k_0;u^*) = \frac{\overline{a}u^*}{\mu} + \frac{1}{\rho}H_{02}\left(\overline{a}\right) + e^{-\mu t}\overline{a}\left[k_0 - \frac{u^*}{\mu}\right]$$

$$\geq \frac{1}{\rho}H_{02}\left(\frac{a}{\rho+\mu}\right) \qquad \forall t \geq 0$$

and

$$\lim_{t\to+\infty} J(t,k_0;u^*) = \frac{au^*}{\mu(\rho+\mu)} + \frac{1}{\rho}H_{02}\left(\frac{a}{\rho+\mu}\right).$$

Then, by imposing the incentive constraint (36) we easily obtain (i) and (ii) and that, for

$$\overline{D} > \frac{\overline{a}u^*}{\mu} + \frac{1}{\rho}H_{02}\left(\overline{a}\right),$$

the strategy u^* is not admissible for the constrained problem, no matter what the starting point k_0 is. We now analyze the form of the region E and the optimal state-control trajectories for the constrained problem by the techniques used in the proof of the Proposition 6.1. We first apply Proposition 5.4 (Assumption 2.4 (ii) is satisfied here by simply taking the constant control $c \equiv -\mu k$ at every starting point $k \in E$) and Proposition 5.6 to get that the region E is a half-line contained in $[\widehat{k}, +\infty)$ or is empty.

Assume that E is nonempty. Then it must be $E = [\overline{k}, +\infty)$ for some $\overline{k} \geq \hat{k}$. By Theorem 5.1 we know that the constrained problem is equivalent to the state constraints problem in the region E. Then we can use (as in Proposition 6.1) the sufficient conditions for optimality for the state constraints problem in the region E (see [Hartl, Sethi and Vickson, 1995]) to get that at $k_0 = \overline{k}$ the constant control strategy $\overline{u}^* \equiv \mu\overline{k}$ is optimal for the constrained problem. Still arguing as in Proposition 6.1 we then get that $\overline{k}$ must be characterized by the property that

$$J(\overline{k};\overline{u}^*) \equiv \overline{D}. \tag{38}$$

By easy calculations we can see that $J(\overline{k};\overline{u}^*)$ is a quadratic concave function of $\overline{k}$ and has the maximum at the point $(u^*/\mu) + \overline{a}\rho/(c\mu^2)$ whose value is $(\overline{a}u^*/\mu) + [H_{02}(\overline{a})/\rho] + \overline{a}^2\rho/(2c\mu^2)$. This means that for $\overline{D}$ strictly greater than this value we get a contradiction and so we must have $E = \emptyset$ (case iv). For $\overline{D}$ less than or equal to this value we compute $\overline{k}$ by solving (38). To find the optimal policy in the case (iii) and to show that $c(k_0) > 0$ we use, again as in Proposition 6.1, the sufficient conditions for optimality. ∎

As the fixed cost goes up we observe an interesting scenario:

- If it is low enough then neither the optimal policy nor the state region allowing existence of the optimal solution change with respect to the unconstrained case.
- If the fixed cost increases then the optimal policy is the first best solution but a large enough stock of capital is required to run the firm. The fixed cost does not affect the optimal investment policy, but the state region allowing existence of the optimal solution becomes smaller.
- If the fixed cost is furthermore increased then the state region allowing existence of the optimal solution is futhermore restricted and a second best optimal control strategy is obtained. The investment rate is higher than the first best solution. In this parameters region the fixed cost affects both the optimal investment strategy and the state region.
- Finally, if the fixed cost goes beyond a certain level then there is no possibility to recover it, no matter what the initial stock of capital is.

Summing up, the incentive compatibility constraint has two effects with respect to the unconstrained problem: it restricts the state region for which a solution exists, it induces a higher rate of capital accumulation and a smaller value function. The optimal policy foresees a stationary level of the state variable when the incentive constraint becomes binding.

7 Conclusions

In this paper we have analyzed dynamic incentive compatibility constrained problems in continuous time. The incentive constraint is a constraint on the continuation value of the payoff function. More precisely, at every time the residual payoff is supposed to be greater than or equal to a certain function of the state and/or of the control. We have characterized the value function associated with the constrained problem by proving that the Dynamic Programming Principle holds and that it is a viscosity solution of the Hamilton-Jacobi-Bellman equation. Restricting our attention to an incentive compatibility constraint which only depends on the value of the state we have shown the equivalence of the constrained problem with a state constrained problem in an endogenous region. This equivalence is useful to define the optimal strategy by means of the Pontryagin Maximum Principle.

Two simple economic problems have been analyzed where the incentive compatibility constraint is given by a positive constant. We have shown that the constrained problem coincides with the unconstrained problem only in some cases. In general as the constraint becomes more binding we have three effects: the state region allowing existence for the constrained problem shrinks, the rate of capital accumulation becomes higher than the first best rate and the value function becomes smaller than the one obtained in the unconstrained problem.

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