Report n. 134

Asset Pricing with Endogenous Aspirations

Emilio Barucci, Maria Elvira Mancino

Pisa, Febbraio 1999

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Emilio Barucci
Dipartimento di Statistica e Matematica applicata all'economia
Università di Pisa.
Maria Elvira Mancino
DIMADEFAS
Università di Firenze.

Abstract

Keywords: Habit, Aspiration, Asset Pricing, Malliavin calculus, Interest rate, Risk premium.

1 Introduction

One of the key point in economic analysis is the modelization of agents' preferences. Under risk, in a dynamic setting, the standard assumption is the time *Additive Expected Utility* (AEU). Given an utility function u related to the istantaneous consumption and a constant discount factor β, the additive expected utility in $t = 0$ associated with an adapted consumption process c_t $(0 \leq t \leq T)$ is given by

$$U(C) = \int_0^T e^{-\beta t} u(c_t) dt, \tag{1}$$

where $C_t = \gamma_0 + \int_0^t c_s ds$, $\gamma_0 > 0$, denotes cumulative consumption.

This utility functional has been extensively criticized in the literature from many points of view. We have theoretical problems related to the decision theory literature (e.g. the Allais and the Ellsberg paradoxes, time separability, etc.) as well as empirical problems related to the asset pricing and to the macroeconomics literature (e.g. the equity premium puzzle, excess volatility, consumption data smoother than the theoretical process, etc.). In part of the literature, these problems have been addressed by changing the utility functional. In this direction, the time additivity is often removed by introducing durable consumption goods [Grossman and Laroque, 1990, Hindy and Huang, 1993], an endogenous discount factor [Uzawa, 1968] or by assuming that the istantaneous utility is a function of the habit which is a smoothed average of past consumption [Constantinides, 1990, Detemple and Zapatero, 1991, Detemple and Zapatero, 1992].

In this paper we relax the time additivity by introducing the classical *habit process* described by a smoothed average of past consumption. The peculiarity of our utility functional is that the istantaneous utility is a function of current consumption and of a process describing the agent's *aspiration*. The agent's aspiration at time t is a linear combination of the current habit (*backward* component) and of the conditional expectation of the habit at the end of the agent's life (T) (*forward* component). Istantaneous utility from current consumption is negatively affected by the agent's aspiration. The aspiration can be interpreted as a *focal point* to evaluate current consumption: the higher is the aspiration, the lower is the utility from current consumption. On this interpretation see also the evolutionary games literature, where agents' decisions are often based on the comparison between the actual payoff and the aspiration which is given by a smoothed average of the experienced payoff (a process similar to the classical habit), e.g. see [Borgers and Sarin, 1996, Karandikar, Mookherjee, Ray, Vega-Redondo, 1997]. Referring to an aspiration process instead of the pure habit-standard of living process, it is plausible to assume the presence of a forward component. With this component we capture the fact that the agent's aspiration and therefore his preferences are affected by what he experienced in the past and by what he expects in the future.

Different specifications of the aspiration process are presented. They depend on the coefficients of the linear combination of the aspiration process. If the two coefficients are positive, then we have a *backward-forward habit*: the standard of living as well as the expectation of the future standard of living negatively affect the istantaneous utility from consumption. If the coefficient associated with the standard of living is negative and the coefficient associated with the expectation of the standard of living at the end of the agent's life is positive then we have a *path dependent forward habit*: preferences are negatively affected by a weighted difference between what I expect to be my future standard of living and my standard of living today. In this case, expecting an increase in the standard of living, I receive a low utility from current consumption. If the sign of the two coefficients is reversed then the utility functional captures a sort of *regret effect*: utility from consumption is small if I expect a bad future, in this case I blame myself because I am consuming the goods today. On the other side, utility is high if I expect a good future.

We propose these utility functionals characterized by different agent's attitudes towards the future and the standard of living. We are not interested in establishing the most plausible from a behavioural point of view. We think they are all plausible, they simply capture different aspects of the agent's attitude towards the future and the standard of living. In particular, the last two specifications describe two different sentiments that an agent can have towards the future status compared to the today status. If the future is better than today then I can be depressed by my actual standard of living or I can be happy being optimistic.

The classical asset pricing equilibrium analysis in the spirit of [Lucas, 1978] is developed. We derive the Arrow-Debreu price process, the interest rate of equilibrium and the assets' risk premia. A habit formation process has been often invoked to address some of the problems encountered in the asset pricing literature assuming the standard AEU, e.g. the equity premium puzzle and the discrepancy between the theoretical and the observed consumption process, e.g. see [Constantinides, 1990, Detemple and Zapatero, 1991]. These topics are addressed in our framework as the specification of the aspiration process changes. We will show that the results obtained with a classical habit are not confirmed with a general aspiration process, it is not easy to evaluate the general equilibrium effects of an aspiration process with a forward component. For a backward-forward habit we are able to establish that the risk premium is higher than the premium obtained with an AEU, in this case the forward component of the aspiration process amplifies the effect of the classical habit process helping us to resolve the equity premium puzzle.

The paper is organized as follows. In Section 2 we present the economy and we discuss related literature. In Section 3 we present the utility functional with an endogenous aspiration process. In Section 4 we solve the optimal consumption problem in a one consumer economy. In Section 5 we develop the classical equilibrium analysis assuming that the representative agent is characterized by an endogenous aspiration process.

2 The Economy and Related Literature

We consider a standard pure exchange one consumer economy with complete markets. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, on which a standard d-dimensional Brownian motion W is defined. The economy has a finite time horizon $[0, T]$. W determines the flow of information through its natural filtration augmented of the P-null sets and made right continuous $(\{\mathcal{F}_t : t \geq 0\})$. We assume $\mathcal{F}_0$ to be trivial.

We denote by

$$\mathcal{L}^2 = \{X : X \text{ is a predictable process such that } E(\int_0^T |X_s|^2 ds) < +\infty\},$$

and by $\mathcal{L}^2_+$ the space of $\mathcal{L}^2$ processes with values in $\mathbb{R}_+$.

There are $d+1$ financial securities, which are continuously traded in frictionless markets. Their equilibrium prices are denoted by S^i $(i = 0, \ldots, d)$. The 0-th security is the risk-free asset, its price is given by $S^0_t = s^0_0 \exp\{\int_0^t r_u du\}$, where r_t is a strictly positive, progressively measurable bounded process and $s^0_0 > 0$. The d-dimensional vector of security prices $S^\top = (S^1, \ldots, S^d)$ (where $\top$ denotes transpose) instead satisfies

$$dS_t = \overline{S}_t[\mu^S_t dt + \sigma^S_t dW_t], \quad S_0 = s_0, \quad \overline{S} = \begin{pmatrix} S^1 & & 0 \\ & \ddots & \\ 0 & & S^d \end{pmatrix},$$

where the d-dimensional vector of mean returns μ^S and the $d \times d$ volatility matrix σ^S are bounded and progressively measurable and $s^i_0 > 0$ for all $i = 1, \ldots, d$.

Each security pays dividends, the cumulative dividends process of security i is denoted by D^i. The vector of cumulative dividends satisfies $dD_t = \mu^D_t dt + \sigma^D_t dW_t$, where $\mu^D \in \mathbb{R}^{d\times 1}$ and

$\sigma^D \in \mathbb{R}^{d\times d}$ are bounded and progressively measurable. Lastly, the gain process is defined as $G = S + D$, where the sum is done component by component and therefore it is an Itô process ($dG_t = \mu_t dt + \sigma_t dW_t$). The gain process can be written in return rates as follows $\overline{S}_t[\mu_t dt + \sigma_t dW_t]$. Let σ_t be invertible.

In a complete markets economy there exists a unique equivalent martingale measure. The so called the *risk-neutral probability measure* given by

$$Q(A) = E[\psi_T 1_A], \qquad A \in \mathcal{F}_T, \tag{2}$$

$$\psi_t = \exp\{-\int_0^t \lambda_s dW_s - \frac{1}{2}\int_0^t ||\lambda_s||^2 ds\} \tag{3}$$

where

$$\lambda_t = \sigma_t^{-1}[\mu_t - r_t\mathbf{1}], \quad \mathbf{1} = \underbrace{(1,\ldots,1)}_{d}$$

denotes the market price of risk. Assuming no arbitrage, the discounted gain from trade is a martingale under Q.

The density ψ can be interpreted as the equilibrium price density of a *one consumer economy.* The agent is described by a pair (U, e), where $U : \mathcal{L}^2_+ \to \mathbb{R}$ is a utility function and $e \in \mathcal{L}^2_+$ is an endowment process. Finally by $c \in \mathcal{L}^2_+$ we denote the consumption process.

A portfolio process or trading strategy, $\pi \equiv (\pi^0, \overline{\pi}) = (\pi^0, \pi^1, \ldots, \pi^d)$, is a measurable, square integrable adapted process, where the i–th component represents the amount of money invested by the agent in the i-th asset.

Definition 2.1 : *We say that a pair of consumption and portfolio process, (c, π), for the representative agent is* **admissible**, *if it satisfies the budget constraint*

$$dX_t = (r_t X_t + e_t - c_t)dt + \overline{\pi}_t(\mu_t - r_t\mathbf{1})dt + \overline{\pi}_t\sigma_t dW_t, \quad X_0 = 0, \quad X_T \geq 0,$$

where X represents the agent's wealth. An admissible pair (c, π) is **optimal** *if there is no other admissible pair (c', π') such that $U(c') > U(c)$.*

Definition 2.2 : *A triple (S, c, π) is called an* **equilibrium** *if (c, π) is optimal, given the price processes S and the market clearing conditions $c_t = e_t$ (consumption good market) and $\pi_t = 0$ (securities market) are satisfied for all $t \in [0, T]$.*

The *Additive Expected Utility* defined in (1) has been challenged in the literature by other utility functionals. Concentrating our attention on the utility functionals removing only the time separability we have the following:

- *Habit formation* [Constantinides, 1990, Detemple and Zapatero, 1991]:

$$U(C) = E(\int_0^T e^{\beta s} u(c_s, y_s) ds),$$

$$y_t = y_0 e^{-\int_0^t \alpha_u du} + \delta_t \int_0^t \mathrm{e}^{-\int_s^t \alpha_u du} c_s ds, \quad y_0 > 0.$$

- *Uzawa utility function* (time varying discount factor):

$$U(C) = E(\int_0^T e^{\int_0^t \beta(c_s) ds} u(c_s) ds),$$

- *Durable consumption goods* [Hindy and Huang, 1993]:

$$U(C) = E(\int_0^T e^{-\beta t} u(\int_0^t k_{t-s} c_s ds) dt),$$

  where k is a progressively measurable bounded process.

The utility obtained from consumption at time t is related to the past through the agent's habit or capital goods durability. In the habit formation setting the istantaneous utility is negatively affected by the agent's habit. The behavioural foundation of this utility functional is that the agent becomes accustomed to a certain standard of living and therefore current consumption is compared to it. As a consequence, the larger is the habit the smaller is the utility obtained from consumption. This interpretation is reinforced by thinking about the linear utility function proposed in [Constantinides, 1990]: $u(c, y) = v(c - y)$ for $c \geq y$ and $u(c, y) = -\infty$ for $y > c$.

The asset pricing analysis with habit formation is provided in many papers, e.g. see [Constantinides, 1990, Detemple and Zapatero, 1991, Detemple and Giannikos, 1996]. Under some conditions and for some specifications of the istantaneous utility function, two main results have been obtained: the equity risk premium is higher than the one obtained with an AEU and the optimal consumption process is smoother than the one obtained with an AEU.

3 Utility with an aspiration process

Let us assume that the agent's preferences are represented by the following system:

$$V_t = E[\int_t^T (u(c_s, z_s) - \beta_s V_s) ds | \mathcal{F}_t] \tag{4}$$

$$y_t = y_0 e^{-\int_0^t \alpha_u du} + \delta_t \int_0^t \mathrm{e}^{-\int_s^t \alpha_u du} c_s ds \tag{5}$$

$$z_t = \eta_t y_t + \mu_t E[y_T | \mathcal{F}_t] = \tag{6}$$

$$\eta_t(y_0 e^{-\int_0^t \alpha_u du} + \delta_t \int_0^t e^{-\int_s^t \alpha_u du} c_s ds) + \mu_t E[y_0 e^{-\int_0^T \alpha_u du} + \delta_T \int_0^T e^{-\int_s^T \alpha_u du} c_s ds | \mathcal{F}_t],$$

where β, α, δ are bounded and positive adapted processes, η, μ are bounded adapted processes. y_0 is a positive constant. The triple of processes (V_t, y_t, z_t) decribes the *utility*, the *habit* and the *aspiration* at time t. V_t is the conditional expected utility at time t associated with the consumption process $(c_{t+s}\ s \geq 0)$. y_t is a smoothed average of past consumption, it represents a proxy of the agent's *standard of living*. z_t is the agent's aspiration affecting istantaneous utility from c_t.

The idea behind this formalization is that the agent's aspiration at time t is influenced by the past and by what he expects to be his future. The dependence of the istantaneous utility gathered from consumption at time t on the standard of living y_t is motivated by the argument that a high standard of living induces a small utitlity from consumption because the agent is accustomed to that standard of living. The same argument can be used to introduce the conditional expectation of the future standard of living: istantaneous utility is affected by what I expect to be my future standard of living. We restrict our attention to an exponentially discounted utility process (β_t does not depend on c or y). Our analysis can be extended to an endogenous discount factor in the spirit of the Uzawa utility function with some computational costs. The analysis can also be extended to utility functionals such that the istantaneous utility is a function of the habit and of the expected habit, independently. Also more compelx habit formation processes can be considered.

The following Assumption is done on the istantaneous utility function.

Assumption 3.1 $u(\cdot,\cdot) : [0,\infty) \times (-\infty,\infty) \to (0,\infty)$ *is twice continuously differentiable, strictly increasing and strictly concave in c, strictly decreasing in z, and concave in the couple. Moreover* $\lim_{c\to 0} u_1(c,z) \leq \infty\ \forall z \in \Re_+$ *and* $\lim_{c\to+\infty} u_1(c,z) = 0\ \forall z \in \Re_+$.

The two processes μ_t and η_t describe the forward-backward characterization of the aspiration. The following cases are of some interest:

- $\mu_t = 0, \eta_t > 0\ \forall t \geq 0$. *Pure backward habit* analyzed in [Constantinides, 1990] and [Detemple and Zapatero, 1991, Detemple and Giannikos, 1996];
- $\eta_t = 0, \mu_t > 0\ \forall t \geq 0$. *Pure forward habit*: the aspiration at time t is simply given by what the agent expects to be his standard of living at the end of his life;
- $\eta_t > 0$ and $\mu_t > 0\ \forall t \geq 0$. *Backward-forward habit*: the aspiration is an increasing and linear function of the two components;

- $\eta_t < 0$ and $\mu_t > 0$ $\forall t \geq 0$. *Path dependent forward habit*: the aspiration is given by the weighted difference between the conditional expectation of the standard of living at the end of the agent's life and of the contemporaneous standard of living;
- $\eta_t \geq 0$ and $\mu_t < 0$ $\forall t \geq 0$. *Regret habit*: the regret effect works as follows, if the conditional expectation of the standard of living is smaller than the standard of living today then I blame myself.

The first habit specification is the classical one. Istantaneous utility from consumption is negatively affected by the standard of living which is a smoothed average of past consumption. The second specification can be interpreted as follows. The agent looks forward to the expected standard of living at the end of his life, this expectation provides the agent with a reference value to evaluate the istantaneous consumption. The expected standard of living plays the role of the classical habit, the difference is that the agent compares consumption to the future standard of living instead of the current standard of living. The third specification is probably the most plausible: the aspiration at time t is a weighted positive linear combination of the standard of living at time t, which is related to consumption before time t, and of the expectation of the standard of living at time T which is related to future consumption. The fourth specification is a variant of the second one: utility at time t is negatively affected by the weighted difference between the expected standard of living at time T and the standard living at time t. As the agent expects a better status in the future than the today status, he is more demanding and therefore he dislikes his actual consumption level. The last specification of the aspiration process captures the opposite effect. The expectation of a better status increases the utility from consumption. The behavioural foundation for this sentiment is a sort of *regret effect* well documented and studied in the decision theory literature: as I expect a worse status in the future, I blame myself because I am consuming the goods today. The interpretation of the last two specifications is straightforward considering two constants with modulus equal to 1 for μ_t and η_t.

Thanks to Assumption 3.1, given a consumption process $c \in \mathcal{L}^2_+$, the existence of a triple (V_t, y_t, z_t) is assured. Following [Duffie and Epstein, 1992], it is easy to show that the utility functional is concave in c. A similar utility functional has been proposed in [Antonelli et al., 1998], where a Backward-Forward Stochastic Differential Utility function in the spirit of [Duffie and Epstein, 1992] has been introduced: the habit is a smoothed convex linear combination of past consumption and of past expected utility. In that setting the forward component is represented by the smoothed average of the conditional expected utility instead of the conditional expected habit.

4 Optimal Consumption

The optimal consumption problem can be handled via dynamic optimization techniques or via the martingale method, see [Cox and Huang, 1989, Cox and Huang, 1991]. Here we follow this second approach which seems the more appropriate for our utility functional.

The optimal consumption problem for the representative agent is equivalent to the following constrained static maximization problem:

$$\max_{C,\pi} \; U(C) \qquad \text{s.t.}$$
$$E^*\left(\int_0^T e^{-\int_o^t r_s ds} c_t dt\right) \leq E^*\left(\int_0^T e^{-\int_o^t r_s ds} e_t dt\right)$$

where E^* denotes expectation under the equivalent martingale measure nested in the financial market model: $E^*(\cdot) = E(\psi_T \cdot)$. ψ is defined by (2) and $U(C) = V_0$, where V_0 is the solution in $t = 0$ of V_t defined in (4)-(6). To simplify the analysis we assume $\beta_t, \alpha_t, \delta_t$ to be constant.

We further specify our setting by assuming that the endowment process is given by

$$de_t = \tilde{\mu}_t^e dt + \tilde{\sigma}_t^e dW_t, \tag{7}$$

with Lipschitz and predictable coefficients $\tilde{\mu}_t^e$ and $\tilde{\sigma}_t^e$ in $\mathbb{R}$ and initial given positive condition ($e_0 > 0$).

First of all we characterize the gradient of the Utility function. In a one consumer economy the Arrow-Debreu equilibrium price process is characterized through the Gateaux derivative of $U(C)$ and its Riesz representation evaluated along the endowment process, see [Duffie and Skiadas, 1994a]. Given a reference pair of cumulative consumption and trading strategy, $(\overline{\pi}, \overline{C})$, and a set F of feasible directions, the Gateaux derivative of $U(C)$ at $(\overline{\pi}, \overline{C})$ is defined as the functional

$$\nabla U(\overline{C}; C) = \lim_{\alpha \to 0} \frac{U(\overline{C} + \alpha C) - U(\overline{C})}{\alpha}, \quad C \in F.$$

Its Riesz representation γ_t is defined as follows:

$$\nabla U(\overline{C}; C) = E\left(\int_0^T (c_t - \overline{c}_t)\gamma_t dt\right).$$

We observe that the Gateaux derivative of z_t is

$$\nabla z_t(\overline{C}; C) = \eta_t \nabla Y_t(\overline{C}; C) + \mu_t E[\nabla Y_T(\overline{C}; C)|\mathcal{F}_t] =$$
$$= \eta_t \delta \int_0^t e^{-\alpha(t-s)} c_s ds + \mu_t \delta E\left[\int_0^T e^{-\alpha(T-s)} c_s ds | \mathcal{F}_t\right].$$

Then the Gateaux derivative of $U(C)$ becomes

$$\nabla U(\overline{C};C) = E[\int_0^T u_1(c_t,z_t)c_t e^{-\beta t}+$$

$$+e^{-\beta t}u_2(c_t,z_t)[\eta_t\delta\int_0^t e^{-\alpha(t-s)}c_s ds + \mu_t\delta E[\int_0^T e^{-\alpha(T-s)}c_s ds|\mathcal{F}_t]]dt].$$

The interpretation of the three terms is straightforward. The first one is related to the istantaneous marginal utility, the second and the third components come from the aspiration process. The second component is due to the backward component of the aspiration, the third component comes from the forward component. After some computations, using the Fubini Theorem we obtain:

$$\nabla U(\overline{C};C) = E[\int_0^T u_1(c_t,z_t)e^{-\beta t}c_t dt] + \delta E[\int_0^T \eta_t E[\int_t^T e^{-\beta s}u_2(c_s,z_s)e^{-\alpha(s-t)}ds|\mathcal{F}_t]c_t dt]+$$

$$+\delta E[\int_0^T \mu_t e^{-\alpha(T-t)}E[\int_t^T e^{-\beta s}u_2(c_s,z_s)ds|\mathcal{F}_t]c_t dt]+\delta E[\int_0^T \mu_t e^{-\alpha(T-t)}\int_0^t e^{-\beta s}u_2(c_s,z_s)ds\ c_t dt].$$

The Riesz representation of the utility gradient evaluated along the consumption process c_t has the following expression

$$\gamma_t = u_1(c_t,z_t)e^{-\beta t} + \eta_t\delta E[\int_t^T e^{-\beta s}u_2(c_s,z_s)e^{-\alpha(s-t)}ds|\mathcal{F}_t]+$$

$$+\mu_t\delta E[\int_t^T e^{-\beta s}e^{-\alpha(T-t)}u_2(c_s,z_s)ds|\mathcal{F}_t] + \mu_t\delta e^{-\alpha(T-t)}\int_0^t e^{-\beta s}u_2(c_s,z_s)ds.$$

The interpretation of the four terms is straightforward. Remember that the agent is characterized by rational expectations, therefore he is able to evaluate the effect of an increase in consumption at time t on the istantaneous utility at every time $s \in [0,T]$. An increase in consumption induces a marginal istantaneous utility $u_1(c_t,z_t)e^{-\beta t}$ and a disutility-utility related to the aspiration process which is made up of a backward (y_t) and of a forward component ($E[y_T|\mathcal{F}_t]$). Because of the backward component of the aspiration process, an increase in consumption at time t causes a disutility in the future associated with the induced effect on the future standard of living ($y_{t+s}\ s \geq 0$). The discounted value of this effect at time t is $E[\int_t^T e^{-\beta s}u_2(c_s,z_s)e^{-\alpha(s-t)}ds|\mathcal{F}_t]$. The forward component of the aspiration process induces a disutility associated with past consumption as well as to future consumption. The first component is $e^{-\alpha(T-t)}\int_0^t e^{-\beta s}u_2(c_s,z_s)ds$, the second is $E[\int_t^T e^{-\beta s}e^{-\alpha(T-t)}u_2(c_s,z_s)ds|\mathcal{F}_t]$. The sign of the last three components is related to η_t, μ_t and therefore to the specification of the aspiration process described in the previous Section. Note that for the first specification of the process z_t we end up with the expression obtained in [Detemple and Zapatero, 1991].

Being $\alpha > 0$, the conditional expectation component of the second term in γ_t is larger in modulus than the conditional expectation component of the third term. Therefore, the effect in the future utility due to the backward component is larger in absolute value than the effect of the forward component.

To prove existence of a solution of the optimal consumption problem for the representative agent we follow the approach in [Detemple and Zapatero, 1992]. Let Assumption 3.1 hold, denote by ζ_s the monetary cost of marginal consumption $(u_1(c_t, z_t)e^{-\beta t})$ augmented by the effects due to the aspiration process:

$$\zeta_t = \rho\gamma_t - \eta_t\delta E[\int_t^T e^{-\beta s}u_2(c_s, z_s)e^{-\alpha(s-t)}ds|\mathcal{F}_t]+$$

$$-\mu_t\delta E[\int_t^T e^{-\beta s}e^{-\alpha(T-t)}u_2(c_s, z_s)ds|\mathcal{F}_t] - \mu_t\delta e^{-\alpha(T-t)}\int_0^t e^{-\beta s}u_2(c_s, z_s)ds$$

where ρ is a positive constant.

Let $I(\zeta_t, z_t)$ denote the inverse of $u_1(c_t, z_t)e^{-\beta t}$ with respect to c_t and $I(\zeta_t, z_t)^+ = max\{0, I(\zeta_t, z_t)\}$ By the concavity of u we say that $c_s^* = I^+(\zeta_s, z_s)$ is optimal if (ζ_s, z_s) is the solution of the following system

$$\zeta_t = \rho^*\gamma_t - \eta_t\delta E[\int_t^T e^{-\beta s}u_2(I^+(\zeta_s, z_s), z_s)e^{-\alpha(s-t)}ds|\mathcal{F}_t]+ \tag{8}$$

$$-\mu_t\delta E[\int_t^T e^{-\beta s}e^{-\alpha(T-t)}u_2(I^+(\zeta_s, z_s), z_s)ds|\mathcal{F}_t] - \mu_t\delta e^{-\alpha(T-t)}\int_0^t e^{-\beta s}u_2(I^+(\zeta_s, z_s), z_s)ds$$

$$(9) z_t = \eta_t(y_0e^{-\alpha t} + \delta\int_0^t e^{-\alpha(t-s)}I^+(\zeta_s, z_s)ds) + \mu_t E[y_0e^{-\alpha T} + \delta\int_0^T e^{-\alpha(T-s)}I^+(\zeta_s, z_s)ds|\mathcal{F}_t]$$

and ρ^* is determined in order to satisfy the budget constraint in the consumption problem.

The approach employed in [Detemple and Zapatero, 1992] is in two steps. First, fixed ρ, we have to show that a solution exists for the system (8)-(9), then we have to show that a value of ρ exists such that the budget constraint is satisfied. The system (8)-(9) is similar to the one obtained with a pure backward habit in [Detemple and Zapatero, 1992], therefore we can adapt their techniques. Restricting our attention to an istantaneous utility function such that $\lim_{c\to 0} u_1(c, z) = \infty$ $\forall z \in \Re_+$ it is enough to assume the following, see [Detemple and Zapatero, 1991].

Assumption 4.1 *$I(\zeta, z)$ and $u_2(I(\zeta, z), z)$ satisfy Lipschitz and growth conditions with respect to z and ζ $\forall\zeta > 0$ and $\forall z \in [\eta_t y_0 e^{-\alpha T}, \infty)$. For $\bar\zeta > 0$ denote by $z(\bar\zeta)_t$ the process solution of (9) with $\zeta_s = \bar\zeta$, $s \in [0, T]$, we require that $\lim_{\bar\zeta\to 0} u_2(I(\bar\zeta, z(\bar\zeta)), z) = 0$.*

This Assumption guarantees existence of a solution of the consumption problem. Considering the more general setting described above with a marginal utility from consumption bounded from above then we can still prove existence of the optimal consumption plan by adapting the conditions in [Detemple and Zapatero, 1992].

5 Equilibrium Analysis

Let z_t^e be the solution of (9) with $c_t = e_t$, i.e.,

$$z_t^e = \eta_t(y_0 e^{-\alpha t} + \delta_t \int_0^t e^{-\alpha(t-s)} e_s ds) + \mu_t E[y_0 e^{-\alpha T} + \delta_T \int_0^T e^{-\alpha(T-s)} e_s ds | \mathcal{F}_t].$$

To ensure that the equilibrium price process is well behaved we impose the following Assumption.

Assumption 5.1 *Let the following conditions be satisfied by the endowment process and the istantaneous utility:*

- $$u_1(e_t, z_t^e) + \eta_t \delta E[\int_t^T e^{-(\beta+\alpha)(s-t)} u_2(e_s, z_s^e) ds | \mathcal{F}_t] +$$
$$\mu_t \delta E[\int_t^T e^{-\beta(s-t)} e^{-\alpha(T-t)} u_2(e_s, z_s^e) ds | \mathcal{F}_t] + \mu_t \delta e^{-\alpha(T-t)} \int_0^t e^{-\beta(s-t)} u_2(e_s, z_s^e) ds > 0 \; \forall t \in [0, T].$$
- $e_t >> 0 \quad \forall t \in [0, T]$.

Assumption 5.1 ensures that the price process is positive and that the uniform properness of preferences holds.

Let $\xi_t = e^{\beta t - \int_0^t r_s ds} \psi_t$ be the Arrow-Debreu price process adjusted by the preference discount factor. Considering a one consumer complete markets economy in equilibrium ($c_t = e_t$) we have that the process γ_t characterizes the Arrow-Debreu price process:

(10)
$$\xi_t = u_1(e_t, z_t^e) + \eta_t \delta E[\int_t^T e^{-(\beta+\alpha)(s-t)} u_2(e_s, z_s^e) ds | \mathcal{F}_t] +$$
$$+ \mu_t \delta E[\int_t^T e^{-\beta(s-t)} e^{-\alpha(T-t)} u_2(e_s, z_s^e) ds | \mathcal{F}_t] + \mu_t \delta e^{-\alpha(T-t)} \int_0^t e^{-\beta(s-t)} u_2(e_s, z_s^e) ds.$$

The interpretation of the four components of the Arrow-Debreu prices process is the one for the process γ_t in the last Section. The Arrow-Debreu price process reflects the different effects of marginal consumption on the utility.

Let us analyze now how the different specifications of the aspiration process affect the Arrow-Debreu price process. We recall that considering an agent characterized by an AEU we have that the price process is simply given by the marginal utility. For a pure backward habit ($\mu_t = 0,\ \eta_t > 0$) we have the characterization of the Arrow-Debreu price process obtained in [Detemple and Zapatero, 1991]. The first three specifications of the aspiration process have similar consequences on the Arrow-Debreu price process: the backward as well as the forward component of the aspiration process associate a disutility to current consumption decisions and therefore the equilibrium price process is smaller than the istantaneous marginal utility from consumption. Ceteris paribus, the price process is smaller than the one obtained with an AEU.

The effect of a Path dependent forward habit and of a Regret habit on the equilibrium price is controversial. It depends on the magnitude of η_t and μ_t, in what follows to evaluate it we restrict our attention to the case $|\eta_t| = |\mu_t|\ \forall t \in [0, T]$. Considering a Path dependent forward habit ($\eta_t < 0,\ \mu_t > 0$) we have that the sum of the second and of the third component is positive, whereas the fourth component is negative. The component regarding consumption in the future increases the price process, whereas the effect due to past consumption reduces the price process. Considering a Regret habit ($\eta_t > 0,\ \mu_t < 0$) we have that the sum of the second and of the third component is negative, whereas the fourth component is positive. The component associated with future consumption reduces the price process, whereas the effect due to past consumption increases the price process.

The interpretation of the last two Arrow-Debreu price process characterizations is as follows. Assuming a path dependent forward habit we have that the istantaneous utility is negatively affected by the weighted difference between the expected future standard of living and the present standard of living, this feature implies that an increase in consumption today reduces (being $\alpha > 0$) the aspiration and therefore it has a positive effect on future utility, whereas the effect through the forward component on the utility of past consumption is negative. Assuming a regret habit we have that the istantaneous utility is positively affected by the weighted difference between the future standard of living and the actual standard of living, this feature implies that an increase in consumption today increases the aspiration and therefore it has a negative effect on future utility and a positive effect on the utility of past consumption.

We have assumed that the istantaneous utility is a decreasing function of the aspiration, in what follows we will assume somewhere that $u_{12} < 0$. Considering the classical habit process, this assumption means that the utility function shows complementarity between habit and consumption, i.e., marginal utility of consumption falls as the standard of living rises. This effect can be explained by assuming complementarity for nearby consumption, i.e., consumption at time t is a good substitute of consumption at time $t + \epsilon$ ($\epsilon > 0$). The same interpretation can be given in our setting with respect to the aspiration process: consumption, past consumption and expected consumption are complementary.

The Arrow-Debreu price process contains many interesting pieces of information. In what follows we assume η_t, μ_t to be constant (η, μ). To simplify the analysis and teh notation we assume that the endowment process satisfies the following equation

$$de_t = e_t(\mu_t^e dt + \sigma_t^e dW_t)$$

with μ_t^e and σ_t^e deterministic.

Denote by $u(t)$ the istantaneous utility evaluated along the endowment process e_t, i.e., $u(e_t, z_t^e)$, and adopt the same notation for the partial derivatives. The following Proposition can be stated about the equilibrium interest rate r_t and the market prices for risk.

Proposition 5.1 *The equilibrium interest rate r_t is*

$$r_t = \beta - (\xi_t)^{-1}[u_{11}(t)e_t\mu_t^e + \frac{1}{2}u_{111}(t)(\sigma_t^e)^2 e_t^2 + u_{12}(t)\eta(\delta e_t - \alpha y_t) - \eta\delta u_2(t) +$$

$$\eta\delta(\alpha+\beta)E[\int_t^T e^{-(\beta+\alpha)(s-t)}u_2(s)ds|\mathcal{F}_t] + \frac{\mu^2}{2}u_{122}(t)\delta^2(\sigma_t^e)^2 E[\int_t^T (e_r - \alpha\int_t^r e^{-\alpha(r-u)}e_u du)dr|\mathcal{F}_t]^2$$

$$+\mu\delta(\alpha+\beta)e^{-\alpha(T-t)}E[\int_0^T e^{-\beta(s-t)}u_2(s)ds|\mathcal{F}_t]].$$

The asset risk premia are

$$\mu_t - r_t = -\sigma_t(\xi_t)^{-1}\sigma_t^e(u_{11}(t)e_t + u_{12}(t)\mu\delta E[\int_t^T (e_r - \alpha\int_t^r e^{-\alpha(r-u)}e_u du)dr|\mathcal{F}_t] +$$

$$+\mu\delta e^{-\alpha(T-t)}e^{\beta t}E[\int_t^T e^{-\beta s}u_{21}(s)e_s + e^{-\beta s}u_{22}(s)(\eta\delta\int_t^s e^{-\alpha(s-u)}e_u du +$$

$$+\mu\delta E[\int_t^T e^{-\alpha(T-u)}e_u du|\mathcal{F}_s])ds|\mathcal{F}_t] + \eta\delta e^{(\beta+\alpha)t}E[\int_t^T e^{-(\beta+\alpha)s}u_{21}(s)e_s +$$

$$+e^{-(\beta+\alpha)s}u_{22}(s)(\eta\delta\int_t^s e^{-\alpha(s-u)}e_u du + \mu\delta E[\int_t^T e^{-\alpha(T-u)}e_u du|\mathcal{F}_s])ds|\mathcal{F}_t]).$$

The general equilibrium analysis of the interest rate and of the market prices of risk is not an easy task. Note that the characterization illustrated in [Detemple and Zapatero, 1991] is obtained with $\mu = 0$.

The first two components of the equilibrium interest rate correspond to those obtained with an AEU, they are related to the istantaneous utility. Ceteris paribus, the interest rate is positively related to the expected growth in consumption ($u_{11} < 0$) and negatively related (when $u_{111} > 0$) to the variance in consumption. The third component associates consumption with the aspiration process, its effect depends on the sign of u_{12} and on the time derivative of the habit. If the utility function exhibits complementarity ($u_{12} \leq 0$) and the habit is going up at time t, then we observe an increase in the interest rate. The explanation of this effect is as follows: an increase in the growth of the aspiration process depresses the Arrow-Debreu prices, this is followed by an increase in the interest rate. The fourth and the fifth components are due to the backward component of the habit, they represent the drift portion of the Arrow-Debreu price process due to the drift in the marginal utility-disutility of future standards of living associated with the backward component of the aspiration process. The last two components come from the forward component of the aspiration process, they represent the drift in the marginal utility-disutility of past and future standards of living associated with the forward component of the aspiration process. The general equilibrium interpretation of the interest rate in equilibrium is similar to the one in [Detemple and Zapatero, 1991].

The sign and the magnitude of the components due to the disutility-utility of standards of living (backward and forward) depend on the aspiration process specification. Considering a backward-forward habit we have that both the two components boost the interest rate. The effect of the classical habit is confirmed with a forward habit and is amplified with a backward-forward habit. Considering a path dependent forward habit or a regret habit we can not establish such a general result, the arguments illustrated above about the Arrow-Debreu prices apply also here.

Being the coefficients of the endowment process deterministic, we have that a single beta consumption CAPM holds as in [Detemple and Zapatero, 1991]:

$$\mu_t - r_t \mathbf{1} = -\beta^e cov(dG_t/S_t, de_t),$$

where the coefficient β^e is easily determined from Proposition 5.1.

It is difficult to assess the general equilibrium effects of an endogenous aspiration process on the asset risk premia. For a general endogenous aspiration process we can not establish a result about the risk premium as in the case of the classical habit process. Assuming a backward-forward habit we can state the following Proposition which is similar to [Detemple and Zapatero, 1991, Proposition 6.2].

Proposition 5.2 *Let $\mu \geq 0$, $\eta \geq 0$. If preferences exhibit complemtarity ($u_{12} \leq 0$) and the* *coefficient of absolute risk aversion is nondecreasing with respect to z then the endogenous aspiration process increases the risk premia.*

Restricting our attention to the class of utility function considered in the above Proposition and assuming a backward-forward habit we are able to establish that the risk premium with an endogenous aspiration process is higher than the risk premium obtained with an AEU. This result reinforces the attempts to solve the equity premium puzzle through the agent's habit. Note that this result is not confirmed by inserting the expected utility in the aspiration process instead of the expected habit, see [Antonelli et al., 1998].

Assuming a more general Itô process as in (7) for the endowment process then we will obtain a two Beta consumption CAPM as pointed out in [Detemple and Zapatero, 1991]. The second Beta will be related to the marginal utility-disutility of standards of living induced by stochastic shifts in the coefficients of the endowment process. Conditions for a single beta consumption CAPM similar to [Detemple and Zapatero, 1991, Proposition 6.1] can be established.

Considering the linear utility function $u(c,z) = v(c-z)$ for $c \geq z$ and $u(c,z) = -\infty$ for $z > c$ we need to stregthen Assumption (5.1) to the following: $e_t >> z_t^e \ \forall t \in [0,T]$. In this setting the interpretation of the equilibrium interest rate and of the market prices of risk is more precise. We have that $u_{12} = -v'' > 0$ (the utility function does not show complementarity) and $u_{122} = v'''$. This fact implies that if $v''' > 0$, then the third and the sixth component boost the equilibrium interest rate.

6 Conclusions

In this paper we have presented a utility functional characterized by the presence of an endogenous aspiration process. The agent's aspiration is a linear combination of the agent's standard of living and of what the agent expects to be the standard of living at the end of his life. Istantaneous utility is negatively affected by the agent's aspiration. This utility functional captures the fact that the agent's preferences in an intertemporal setting can be affected by what he expects for the future and by what he plans to do in the future.

Depending on the coefficients of the linear combination of the aspiration process, different specifications of the aspiration process have been discussed: *Pure backward habit*, *Pure forward habit*, *Backward-forward habit*, *Path dependent forward habit*, *Regret habit*. These specifications reflect different attitudes of the agent towards the future standard of living. In particular, the backward-forward habit assumes that the agent's istantaneous utility at time

t is negatively affected by the standard of living at that time and by the expected standard of living. With this aspiration process we have shown that under some conditions the risk premium is higher than the one obtained with an AEU and therefore a backward-forward habit provides an explanation to the Equity premium puzzle. However, as a conclusion, we can say that the general equilibrium effects of an endogenous aspiration process are not clearly cut as those obtained with the classical habit process.

A Proof of Proposition 5.1

To obtain the equilibrium interest rate and the market price of risk, we have to differentiate both sides in (10).

Denote by

$$M_t = E[\int_0^T e^{-(\beta+\alpha)s} u_2(e_s, z_s) ds | \mathcal{F}_t] \quad , G_t = \int_0^t e^{-(\beta+\alpha)s} u_2(e_s, z_s) ds,$$

then

$$E[\int_t^T e^{-(\beta+\alpha)(s-t)} u_2(e_s, z_s) ds | \mathcal{F}_t] = e^{(\alpha+\beta)t}(M_t - G_t).$$

Moreover let

$$\widetilde{M}_t = E[\int_0^T e^{-\beta s} u_2(e_s, z_s) ds | \mathcal{F}_t],$$

then

$$E[e^{-\alpha(T-t)} \int_0^T e^{-\beta(s-t)} u_2(e_s, z_s) ds | \mathcal{F}_t] = e^{-\alpha(T-t)} e^{\beta t} \widetilde{M}_t.$$

Differentiating the right side in (10) gives

$$du_1(e_t, z_t) + \eta\delta d(e^{(\alpha+\beta)t}(M_t - G_t)) + \mu\delta d(e^{-\alpha(T-t)} e^{\beta t} \widetilde{M}_t) =$$

$$= du_1(e_t, z_t) + \eta\delta(\alpha+\beta) e^{(\alpha+\beta)t}(M_t - G_t)dt + \eta\delta e^{(\alpha+\beta)t}\, d(M_t - G_t) +$$

$$+\mu\delta(\alpha+\beta) e^{-\alpha(T-t)} e^{\beta t} \widetilde{M}_t dt + \mu\delta e^{-\alpha(T-t)} e^{\beta t}\, d\widetilde{M}_t \tag{11}$$

We first compute $du_1(e_t, z_t)$. We have that

$$de_t = e_t(\mu_t^e dt + \sigma_t^e dW_t)$$

$$dz_t = \eta dy_t + \mu\, d(E[y_T | \mathcal{F}_t]).$$

y_t can be written as $y_0 + \int_0^t \delta e_s - \alpha y_s ds$. Then

$$E[y_T | \mathcal{F}_t] = y_0 + E[\int_0^T \delta e_s - \alpha y_s ds | \mathcal{F}_t].$$

In order to differentiate this term we use Clark-Ocone formula, see [Ocone and Karatzas, 1991]. Denote $F = \int_0^T \delta e_s - \alpha y_s ds$, then

$$D_s F = \int_s^T \delta D_s e_r - \alpha D_s y_r \, dr,$$

$$D_s e_r = e_r \sigma_s^e$$

and

$$D_s y_r = \int_s^r \delta D_s e_u - \alpha D_s y_u \, du = \int_s^r \delta e_u \sigma_s^e - \alpha D_s y_u \, du.$$

Therefore

$$D_s y_r = \delta \sigma_s^e \int_s^r e^{-\alpha(r-u)} e_u du.$$

It follows that

$$d(E[y_T|\mathcal{F}_t]) = \delta \sigma_t^e E[\int_t^T (e_r - \alpha \int_t^r e^{-\alpha(r-u)} e_u du) dr|\mathcal{F}_t] dW_t.$$

By Itô formula we have

$$du_1(e_t, z_t) = u_{11}(e_t, z_t) de_t + u_{12}(e_t, z_t) dz_t + \frac{1}{2} u_{111}(e_t, z_t)(\sigma_t^e)^2 e_t^2 dt +$$

$$+\frac{1}{2} u_{122}(e_t, z_t) \mu^2 \delta^2 (\sigma_t^e)^2 E[\int_t^T (e_r - \alpha \int_t^r e^{-\alpha(r-u)} e_u du) dr|\mathcal{F}_t]^2 dt =$$

$$= (u_{11}(e_t, z_t) e_t \mu_t^e + u_{12}(e_t, z_t) \eta(\delta e_t - \alpha y_t) + \frac{1}{2} u_{111}(e_t, z_t)(\sigma_t^e)^2 e_t^2 +$$

$$+\frac{1}{2} u_{122}(e_t, z_t) \mu^2 \delta^2 (\sigma_t^e)^2 (E[\int_t^T (e_r - \int_t^r e^{-\alpha(r-u)} e_u du) dr|\mathcal{F}_t]^2) dt$$

$$+(u_{11}(e_t, z_t) e_t \sigma_t^e + u_{12}(e_t, z_t) \mu \delta \sigma_t^e E[\int_t^T (e_r - \int_t^r e^{-\alpha(r-u)} e_u du) dr|\mathcal{F}_t]) dW_t$$

We compute now $d\widetilde{M}_t$. Let $H = \int_0^T e^{-\beta s} u_2(e_s, z_s) ds$. Then using Clark-Ocone formula we have:

$$d\widetilde{M}_t = E[D_t H|\mathcal{F}_t] dW_t$$

Therefore we have to compute $D_t H$.

$$D_t H = \int_t^T e^{-\beta s}(u_{21}(e_s, z_s) D_t e_s + u_{22}(e_s, z_s) D_t z_s) ds$$

Recall that $D_t e_s = e_s \sigma_t^e$, we compute $D_t z_s$:

$$D_t z_s = \eta D_t y_s + \mu D_t(E[y_T|\mathcal{F}_s])$$

$$= \eta \delta \sigma_t^e \int_t^s e^{-\alpha(s-u)} e_u du + \mu E[D_t y_T|\mathcal{F}_s] I_{t \le s}$$

$$= \eta \delta \sigma_t^e \int_t^s e^{-\alpha(s-u)} e_u du + + \mu \delta E[\sigma_t^e \int_t^T e^{-\alpha(T-u)} e_u du|\mathcal{F}_s] I_{t \le s}$$

Therefore

$$D_t H = \int_t^T e^{-\beta s}(u_{21}(e_s, z_s) e_s \sigma_t^e + u_{22}(e_s, z_s)(\eta \delta \sigma_t^e \int_t^s e^{-\alpha(s-u)} e_u du + \mu \delta E[\sigma_t^e \int_t^T e^{-\alpha(T-u)} e_u du|\mathcal{F}_s]) ds$$

and

$$d\widetilde{M}_t = E(\int_t^T e^{-\beta s} u_{21}(e_s, z_s) e_s \sigma_t^e +$$

$$+ e^{-\beta s} u_{22}(e_s, z_s)(\eta\delta\sigma_t^e \int_t^s e^{-\alpha(s-u)} e_u du + \mu\delta E(\sigma_t^e \int_t^T e^{-\alpha(T-u)} e_u du|\mathcal{F}_s))ds|\mathcal{F}_t)dW_t.$$

Consider now $d(M_t - G_t)$. We have:

$$dG_t = e^{-(\beta+\alpha)t} u_2(e_t, z_t)$$

and using again Clark-Ocone formula

$$dM_t = E(\int_t^T e^{-(\beta+\alpha)s} u_{21}(e_s, z_s) e_s \sigma_t^e$$

$$+ e^{-(\beta+\alpha)s} u_{22}(e_s, z_s)(\eta\delta\sigma_t^e \int_t^s e^{-\alpha(s-u)} e_u du + \mu\delta E(\sigma_t^e \int_t^T e^{-\alpha(T-u)} e_u du|\mathcal{F}_s))ds|\mathcal{F}_t)dW_t.$$

Denote by $u(t)$ the istantaneous utility evaluated along the endowment process e_t, i.e., $u(e_t, z_t)$, and adopt the same notation for the partial derivatives. Equating the drift terms in (10) we obtain the following

$$(\beta - r_t)\xi_t = u_{11}(t) e_t \mu_t^e + u_{12}(t)\eta(\delta e_t - \alpha y_t) + \frac{1}{2} u_{111}(t)(\sigma_t^e)^2 e_t^2 +$$

$$+ \frac{1}{2} u_{122}(t)\mu^2\delta^2(\sigma_t^e)^2 E(\int_t^T (e_r - \alpha \int_t^r e^{-\alpha(r-u)} e_u du) dr|\mathcal{F}_t)^2 + \eta\delta(\alpha+\beta) E(\int_t^T e^{-(\beta+\alpha)(s-t)} u_2(s) ds|\mathcal{F}_t)$$

$$+ \mu\delta(\alpha+\beta) e^{-\alpha(T-t)} E(\int_0^T e^{-\beta(s-t)} u_2(s) ds|\mathcal{F}_t) - \eta\delta u_2(t).$$

Equating the volatility term in (10) we obtain

$$\mu_t - r_t = -\sigma_t(\xi_t)^{-1}(u_{11}(t) e_t \sigma_t^e + u_{12}(t)\mu\delta\sigma_t^e E(\int_t^T (e_r - \alpha \int_t^r e^{-\alpha(r-u)} e_u du) dr|\mathcal{F}_t) +$$

$$\mu\delta e^{-\alpha(T-t)} e^{\beta t} E(\int_t^T e^{-\beta s} u_{21}(e_s, z_s) e_s \sigma_t^e + e^{-\beta s} u_{22}(e_s, z_s)(\eta\delta\sigma_t^e \int_t^s e^{-\alpha(s-u)} e_u du +$$

$$+ \mu\delta E(\sigma_t^e \int_t^T e^{-\alpha(T-u)} e_u du|\mathcal{F}_s) ds|\mathcal{F}_t)) + \eta\delta e^{(\beta+\alpha)t} E(\int_t^T e^{-(\beta+\alpha)s} u_{21}(e_s, z_s) e_s \sigma_t^e +$$

$$+ e^{-(\beta+\alpha)s} u_{22}(e_s, z_s)(\eta\delta\sigma_t^e \int_t^s e^{-\alpha(s-u)} e_u du + \mu\delta E(\sigma_t^e \int_t^T e^{-\alpha(T-u)} e_u du|\mathcal{F}_s)) ds|\mathcal{F}_t)).$$

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